Face Numbers of Poset Associahedra: Results and Conjectures

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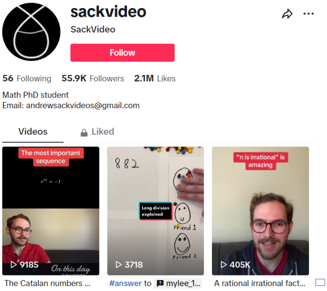
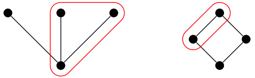
Readers: Gregg Musiker, Pavlo Pylyavskyy
Joint work with Andrew Sack

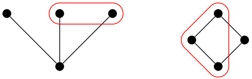


Tube

Given a poset P, a tube τ is a connected convex subposet of P such that $1 < |\tau| < |P|$.



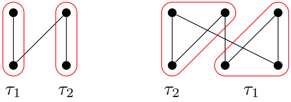
Example



Non-example

Tube

Given two tubes τ_1, τ_2, we say $\tau_1 \prec \tau_2$ if $\tau_1 \cap \tau_2 = \emptyset$, and there exists $v_1 \in \tau_1$ and $v_2 \in \tau_2$ such that $v_1 <_P v_2$.



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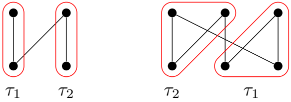
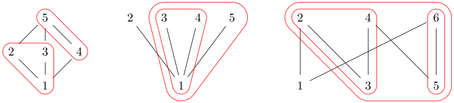


Potential problem: may/will have $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_k \prec \tau_1$.

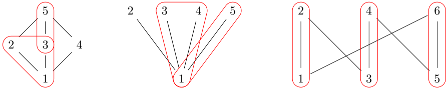
Tubing

A *tubing* T of P is a set of tubes such that

- any pair of tubes in T is either nested or disjoint, and
- there is no potential problem $\{\tau_1, \tau_2, \ldots, \tau_k\} \subseteq T$ such that $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_k \prec \tau_1$.



Examples

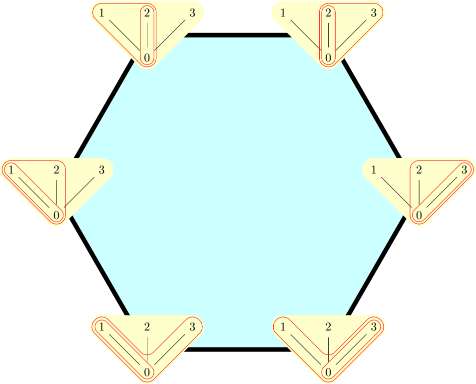


Non-examples

Poset Associahedra

Definition (Galashin '21)

For a finite poset P, there exists a simple, convex polytope $\mathscr{A}(P)$ whose face lattice is isomorphic to the set of tubings ordered by reverse inclusion. This polytope is called the **poset associahedron** of P.



f and h-vector

- f-vector: $(f_0, f_1, \ldots, f_d)$ where

$$f_i = \#i\text{-dimensional faces}$$

Eg: $(6, 6, 1)$

- f-polynomial:

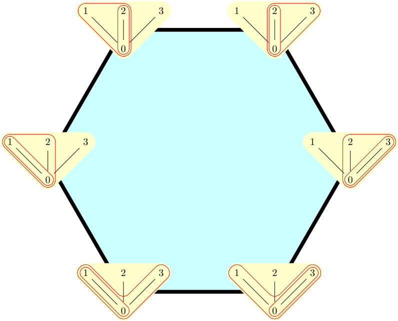
$$f(t) = 6 + 6t + t^2$$

- h-vector and h-polynomial:

$$f(t) = h(t+1)$$

$$6+6t+t^2 = 1+4(t+1)+(t+1)^2$$

$\rightarrow$ (1,4,1)



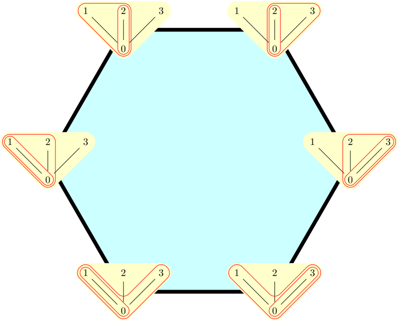
Permutohedra

- When the h-polynomial is symmetric, we have the γ-*vector* and γ-*polynomial*:

  $$1 + 4t + t^2 = (1 + t)^2 + 2t$$

  $\rightarrow$ (1,2)

  Note: Not necessarily nonnegative



Permutohedra

Big Questions

γ-nonnegativity??? (BIG)

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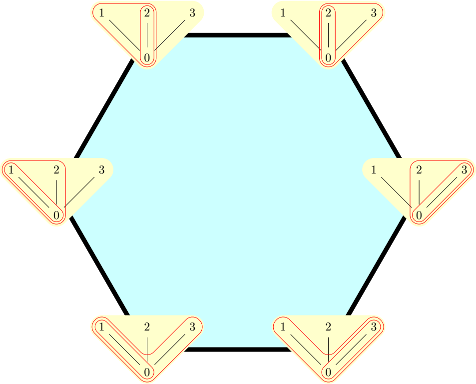
real-rootedness??? (BIGGER)

For our polytopes: real-rooted $\Rightarrow$ γ-nonnegative, log-concave, unimodal

Vic's Favorite Examples



Permutohedra

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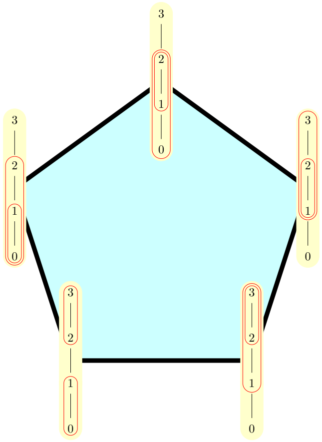
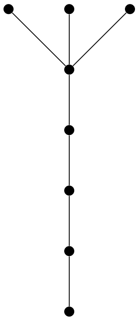


Associahedra

Broom Posets

Broom posets: $A_{n,k} = C_{n+1} \oplus A_k$



$A_{4,3}$

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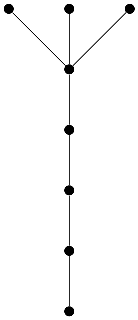


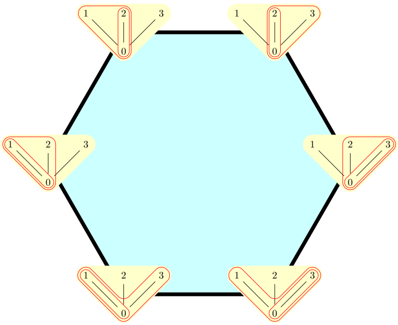
$A_{4,3}$

Question: What do their face numbers count?

Permutohedra

- *h*-vector: Eulerian number
  $h_i = |\{w \in \mathfrak{S}_n \mid \mathrm{des}(w) = i\}|$

  $\#$ vertices $= n!$

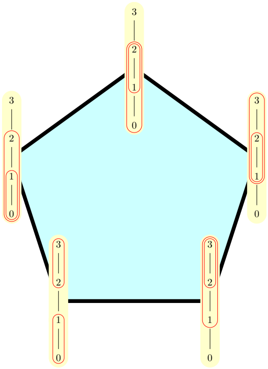


Permutohedra

Associahedra

- *h*-vector: Narayana number
  $h_i = ?$

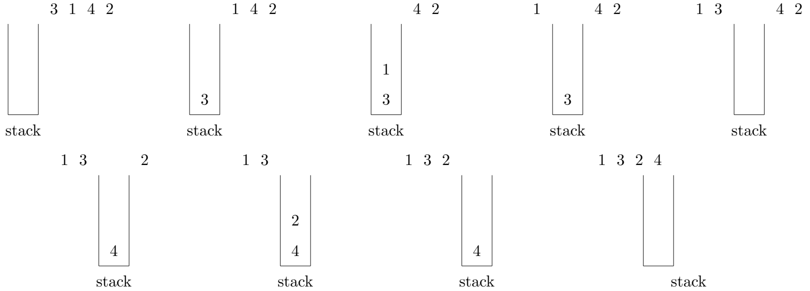
- $\#$ vertices $=$ Catalan number



Associahedra

Stack-sorting

Stack-sorting, denoted s, is an algorithm that “sorts” a permutation in linear time



$s(3142)$

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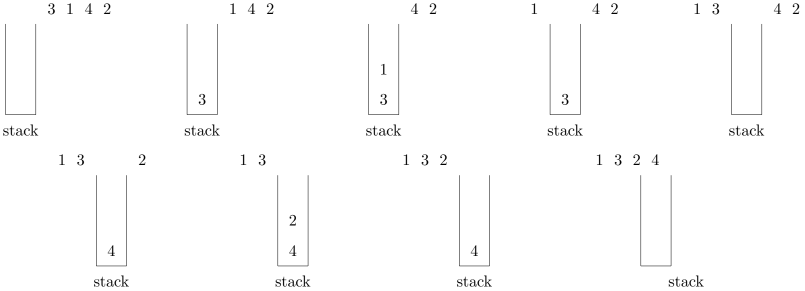

$s(3142)$

Stack-sortable permutations are counted by Catalan numbers!!!

Associahedra

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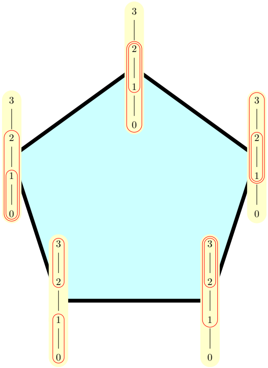
  $h_i = |\{w \in s^{-1}(12 \ldots n) \mid \operatorname{des}(w) = i\}|$

- $\#$ vertices $= |s^{-1}(12 \ldots n)| =$ Catalan number



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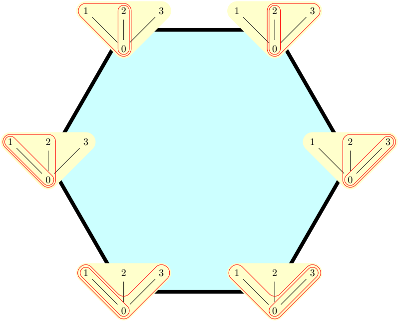

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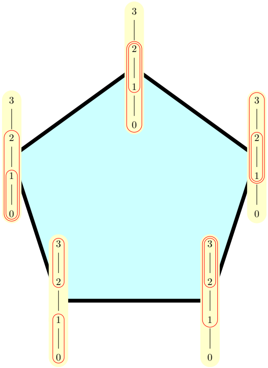

Associahedra

Broom Posets

Define $\mathfrak{S}_{n,k} = \{w \mid w \in \mathfrak{S}_{n+k}, w_i = i \text{ for all } i > k\}$.
Eg. $\mathfrak{S}_{3,2} = \{12345, 21345\}$.

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Theorem (N., Sack '23)

Let $h = (h_0, h_1, \ldots, h_{n+k-1})$ be the h-vector of $\mathscr{A}(A_{n,k})$. Then

$$h_i = \left|\{w \in s^{-1}(\mathfrak{S}_{n,k}) \mid \operatorname{des}(w) = i\}\right|$$

Broom Posets

Theorem (Brändén '08)

For $A \subseteq \mathfrak{S}_n$, we have

$$\sum_{\sigma \in s^{-1}(A)} x^{\mathrm{des}(\sigma)}$$

is γ-nonnegative.

Corollary

The γ-vector of $\mathscr{A}(A_{n,k})$ is nonnegative.

Happy Coincidence

Proposition (N., Sack '23)

$$|\{w \in s^{-1}(2134\ldots n) \mid \operatorname{des}(w) = i\}| =$$
$$|\{w \in s^{-1}(1234\ldots n) \mid \operatorname{des}(w) = i, w_1 < n, w_n < n\}|$$

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Proposition (N., Sack '23)

$$h_{A_{n,2}}(x) = 2h_{A_{n+2,0}}(x) - (1+x)h_{A_{n+1,0}}(x).$$

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Theorem (N., Sack '23)

$h_{A_{n,2}}(x)$ *is real-rooted.*

Conjectures

Conjecture (Hard)

The h-polynomials of $\mathscr{A}(A_{n,k})$ *are real-rooted.*

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$\mathscr{A}(P)$ *are* γ*-positive for all* P*.*

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The h-polynomials of $\mathscr{A}(A_{n,k})$ *are real-rooted.*

Conjecture (Harder)

$\mathscr{A}(P)$ *are* γ*-positive for all* P.

Conjecture (Very Hard)

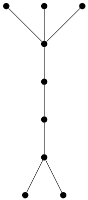
The h-polynomials of $\mathscr{A}(P)$ *are real-rooted for all* P.

Questions

Question (Another Direction)

Find more stack-sorting preimages that give h-polynomials of $\mathscr{A}(P)$.

Eg: Known for two-leg broom posets



$A_{2,3,3}$

How about many-leg broom posets?

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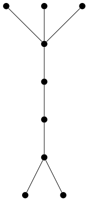

$A_{2,3,3}$

How about many-leg broom posets?

Question (A Rabbit Hole)

Real-rootedness of descent generating polynomials of stack-sorting preimages.

Acknowledgement

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