Face Numbers of Poset Associahedra: Results and Conjectures

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Given a poset $P$, a tube $\tau$ is a connected convex subposet of $P$ such that $1 < |\tau| < |P|$.

Example

Non-example
Given two tubes $\tau_1, \tau_2$, we say $\tau_1 \prec \tau_2$ if $\tau_1 \cap \tau_2 = \emptyset$, and there exists $v_1 \in \tau_1$ and $v_2 \in \tau_2$ such that $v_1 <_P v_2$. 

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Potential problem: may/will have $\tau_1 \prec \tau_2 \prec \tau_3 \prec \ldots \prec \tau_k \prec \tau_1$. 

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Potential problem: may/will have $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_k \prec \tau_1$. 
A tubing $T$ of $P$ is a set of tubes such that
- any pair of tubes in $T$ is either nested or disjoint, and
- there is no potential problem $\{\tau_1, \tau_2, \ldots, \tau_k\} \subseteq T$ such that $\tau_1 \prec \tau_2 \prec \ldots \prec \tau_k \prec \tau_1$.

Examples

Non-examples
Definition (Galashin ’21)
For a finite poset $P$, there exists a simple, convex polytope $\mathcal{A}(P)$ whose face lattice is isomorphic to the set of tubings ordered by reverse inclusion. This polytope is called the **poset associahedron** of $P$. 
**f** and **h**-vector

- **f-vector:** \((f_0, f_1, \ldots, f_d)\) where
  \[ f_i = \#i\text{-dimensional faces} \]
  Eg: \((6, 6, 1)\)

- **f-polynomial:**
  \[ f(t) = 6 + 6t + t^2 \]

- **h-vector and h-polynomial:**
  \[ f(t) = h(t + 1) \]
  \[ 6 + 6t + t^2 = 1 + 4(t + 1) + (t + 1)^2 \]
  \[ \rightarrow (1, 4, 1) \]

Permutohedra
When the $h$-polynomial is symmetric, we have the $\gamma$-vector and $\gamma$-polynomial:

$$1 + 4t + t^2 = (1 + t)^2 + 2t$$

$\rightarrow (1,2)$

Note: Not necessarily nonnegative
$\gamma$-nonnegativity??? (BIG)

For our polytopes: real-rooted $\Rightarrow \gamma$-nonnegative, log-concave, unimodal
Big Questions

γ-nonnegativity??? (BIG)

real-rootedness??? (BIGGER)

For our polytopes: real-rooted $\Rightarrow$ γ-nonnegative, log-concave, unimodal
Vic’s Favorite Examples

Permutohedra
Vic’s Favorite Examples

Associahedra
Broom posets: $A_{n,k} = C_{n+1} \oplus A_k$
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Question: What do their face numbers count?
Permutohedra

- *h*-vector: Eulerian number

\[ h_i = |\{ w \in S_n \mid \text{des}(w) = i \}| \]

\# vertices = \( n! \)
Associahedra

- **$h$-vector:** Narayana number
  
  \[ h_i = ? \]

- **# vertices = Catalan number**

Associahedra
Stack-sorting, denoted $s$, is an algorithm that “sorts” a permutation in linear time.

$$s(3142)$$
Stack-sorting, denoted $s$, is an algorithm that “sorts” a permutation in linear time.

Stack-sortable permutations are counted by Catalan numbers!!!
Associahedra

- **$h$-vector: Narayana number**
  \[ h_i = \left| \{ w \in s^{-1}(12 \ldots n) \mid \text{des}(w) = i \} \right| \]

- **# vertices** = \( |s^{-1}(12 \ldots n)| \) = Catalan number
Permutohedra

- **h-vector: Eulerian number**
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Associahedra

- $h$-vector: Narayana number
  $$h_i = |\{w \in s^{-1}(12 \ldots n) \mid \text{des}(w) = i\}|$$

- Number of vertices = $|s^{-1}(12 \ldots n)|$ = Catalan number
Define $\mathcal{G}_{n,k} = \{ w \mid w \in \mathcal{G}_{n+k}, w_i = i \text{ for all } i > k \}$. Eg. $\mathcal{G}_{3,2} = \{12345, 21345\}$. 
Broom Posets

Define $\mathcal{S}_{n,k} = \{w \mid w \in \mathcal{S}_{n+k}, w_i = i \text{ for all } i > k\}$.
Eg. $\mathcal{S}_{3,2} = \{12345, 21345\}$.

**Theorem (N., Sack ’23)**

Let $h = (h_0, h_1, \ldots, h_{n+k-1})$ be the $h$-vector of $\mathcal{A}(A_{n,k})$. Then

\[h_i = |\{w \in s^{-1}(\mathcal{S}_{n,k}) \mid \text{des}(w) = i\}|\]
Theorem (Brändén ’08)

For \( A \subseteq \mathcal{S}_n \), we have

\[
\sum_{\sigma \in s^{-1}(A)} x^\text{des}(\sigma)
\]

is \( \gamma \)-nonnegative.

Corollary

The \( \gamma \)-vector of \( \mathcal{A}(A_{n,k}) \) is nonnegative.
Proposition (N., Sack '23)

\[
\left| \left\{ w \in s^{-1}(2134 \ldots n) \mid \text{des}(w) = i \right\} \right| = \\
\left| \left\{ w \in s^{-1}(1234 \ldots n) \mid \text{des}(w) = i, w_1 < n, w_n < n \right\} \right|
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Proposition (N., Sack '23)

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Proposition (N., Sack '23)

\[ h_{A_{n,2}}(x) = 2h_{A_{n+2,0}}(x) - (1 + x)h_{A_{n+1,0}}(x). \]
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Theorem (N., Sack ’23)

\[
h_{A_{n,2}}(x) \text{ is real-rooted.}
\]
Conjectures

Conjecture (Hard)

The $h$-polynomials of $\mathcal{A}(A_{n,k})$ are real-rooted.

Conjecture (Harder)

$A(P)$ are $\gamma$-positive for all $P$.

Conjecture (Very Hard)

The $h$-polynomials of $\mathcal{A}(P)$ are real-rooted for all $P$. 
Conjecture (Hard)

*The* $h$*-polynomials of* $\mathcal{A}(A_{n,k})$ *are real-rooted.*

Conjecture (Harder)

*\mathcal{A}(P)* *are* $\gamma$*-positive for all* $P$. 
**Conjecture (Hard)**

\[ \text{The } h\text{-polynomials of } A(A_{n,k}) \text{ are real-rooted.} \]

**Conjecture (Harder)**

\[ A(P) \text{ are } \gamma\text{-positive for all } P. \]

**Conjecture (Very Hard)**

\[ \text{The } h\text{-polynomials of } A(P) \text{ are real-rooted for all } P. \]
Question (Another Direction)

*Find more stack-sorting preimages that give $h$-polynomials of $\mathcal{A}(P)$.*

Eg: Known for two-leg broom posets

How about many-leg broom posets?
Questions

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*Find more stack-sorting preimages that give h-polynomials of \( \mathcal{A}(P) \).*

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**Question (A Rabbit Hole)**

*Real-rootedness of descent generating polynomials of stack-sorting preimages.*
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