The Möbius Function of the Rook Monoid

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Abstract. The rook monoid R_n can be viewed as a partially ordered set (poset) with the Bruhat order. A result due to Can gives that the Möbius function $\mu(a, b)$ of elements $a \leq b$ in R_n is either 0 or $(-1)^{\text{rank}(b)-\text{rank}(a)}$. This motivates us to seek a description of the set $I(a) = \{b \in R_n \mid \mu(a, b) \neq 0\}$. We show that $I(a)$ is a closed interval of R_n that is isomorphic to a closed interval of the Bruhat order on a symmetric group, S_m for some m.

Submitted under the supervision of Professor Victor Reiner to the University Honors Program at the University of Minnesota-Twin Cities in partial fulfillment of the requirements for the degree of Bachelor of Science, *magna cum laude* in Mathematics.

Wednesday May 15, 2019

1 Introduction

The rook monoid R_n is a submonoid of all $n \times n$ matrices whose elements have entries in $\{0,1\}$ and correspond to non-attacking rook placements in an $n \times n$ board. With the Bruhat order defined in Section 2 below, R_n becomes a poset. We represent elements of R_n by a vector $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. where a_i gives the row index of the rook in the i^{th} column, or $a_i = 0$ if there is no such rook. Figure 1.1 shows the Hasse diagram of R_2 together with the corresponding non-attacking rook placements in a 2×2 board.

We distinguish every component of $a \in R_n$ by placing subscripts on the repeated zero components according their relative positions and how many times they appear in a. For example, in R_4, we write $(1, 0_1, 0_2, 3)$ to indicate that 0_1 is the first zero component of a and that 0_2 is the second zero component of a. As components of a, 0_1 and 0_2 are distinct but share the same value (both are equal to zero).

In [2, Theorem 5.2], Can has shown that for $a \leq b$ in R_n

$$\mu(a,b) = \begin{cases} (-1)^{\text{rank}(b)-\text{rank}(a)} & \text{if every length 2 interval in } [a,b] \text{ has 4 elements} \\ 0 & \text{otherwise} \end{cases}.$$

Thus, for $a \leq b$ in R_n, $\mu(a,b) \in \{-1, 0, 1\}$.

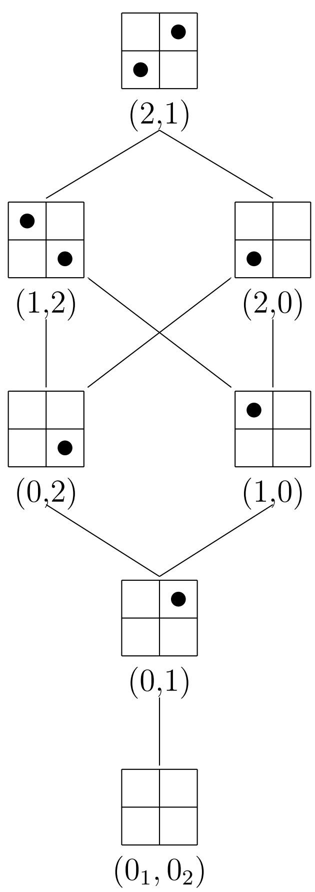


Figure 1.1: The Hasse diagram of R_2 with the Bruhat order

Motivated by this result, we want to describe, for each $a \in R_n$, the elements of the set $I(a) = \{b \in R_n \,|\, \mu(a,b) \neq 0\}$. Before stating our main results, we present some definitions.

Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. We write $\overline{a}$ to denote the rearrangement of the components $a_1, \ldots, a_n$ of a in a non-increasing fashion. That is, $\overline{a} = (a_{\sigma_1}, a_{\sigma_2}, \ldots, a_{\sigma_n}) \in \mathbb{R}^n$ where $a_{\sigma_1}, a_{\sigma_2}, \ldots, a_{\sigma_n}$ is a permutation of $a_1, \ldots, a_n$ such that $a_{\sigma_1} \geq a_{\sigma_2} \geq \cdots \geq a_{\sigma_n}$.

For $i \in \{1, \ldots, n\}$, we write $[a]_i = a_i$, the $i-$component of a. For $x \in \mathbb{R}$, and $k \in \mathbb{Z}_+$, we define

$$a[x_k] = \begin{cases} i, & \text{if the } [a]_i \text{ is the } k^{\text{th}}\text{component of } a \text{ that is equal to } x \\ 0, & \text{otherwise} \end{cases}.$$

For example, by letting $a = (0_1, 4, 6, 3, 0_2, 1) \in R_6$, we have that $\overline{a} = (6, 4, 3, 1, 0_1, 0_2)$, $[a]_1 = 0_1, [a]_3 = 6, a[1_1] = 6, a[0_2] = 5,$ and $a[1_2] = 0$.

We are now reader to state our main results. The first result gives that $I(a)$ is a closed interval of R_n that is also isomorphic to a closed interval of the Bruhat order on a symmetric group S_m for some m.

Theorem 1.1. *For $a \in R_n$, let $I(a) = \{b \in R_n \,|\, \mu(a,b) \neq 0\}$. Then*

(i) there exists a map $R_n \xrightarrow{f} R_n$ such that $I(a)$ is the interval $[a, f(a)] \subseteq R_n$, and

(ii) there are elements $a_{\text{co}}, f(a)_{\text{co}} \in S_m$ for some $m \geq n$ such that $I(a)$ is isomorphic to the interval $[a_{\text{co}}, f(a)_{\text{co}}] \subseteq S_m$.

That is, $I(a) = [a, f(a)] \cong [a_{\text{co}}, f(a)_{\text{co}}]$.

The descriptions of $f(a), a_{\text{co}}$, and $f(a)_{\text{co}}$ will be given later in Section 6 and Section 7. In R_2, we have

$$f(a) = \begin{cases} (0,1) & \text{if } a = (0_1, 0_2) \\ (2,1) & \text{otherwise} \end{cases}.$$

Thus, $I(0_1, 0_2)) = \{(0_1, 0_2), (0, 1)\}$, and $I(a) = [a, (2, 1)]$ for all $a \neq (0_1, 0_2)$.

The second result provides a criteria to check whether an element $b \in R_n$ satisfies $\mu(a,b) \neq 0$. By Theorem 1.1, this is equivalent to showing $b \in [a, f(a)]$. We found that when the number of zero components of a is either 0 or 1, $f(a)$ is the maximal element of R_n. Thus, in this case, b satisfies $\mu(a,b) \neq 0$ if $a \leq b$.

Theorem 1.2. *Let a, b be elements of R_n such that $a \leq b$. Suppose that the number of zero components of a is $p > 1$. Then $b \in [a, f(x)]$ if and only if all of the following two conditions hold.*

(i) $[\overline{b}]_i \leq [\overline{a}]_i + 1$ for all $i = 1, \ldots, n$, and

(ii) $b[0_j] \leq a[0_{j+1}] - 1$ for all $j = 1, \ldots, p-1$.

2 Rook Monoid and Bruhat Order

For a positive integer n, the rook monoid R_n is the set, together with multiplication, of all $n \times n$ matrices M such that each entry of M is either 1 or 0 and each row and column of M contains at most one nonzero entry. We can identify each element $M \in R_n$ with an element in $\mathbb{R}^n$. This identification is depicted below.

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Leftrightarrow (2, 0, 1) \in R_3; M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Leftrightarrow (1, 3, 4, 0) \in R_4$$

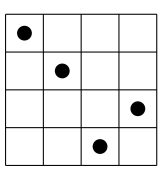
Thus, as a set, we can equivalently describe the rook monoid R_n as the subset of $\mathbb{R}^n$ defined by

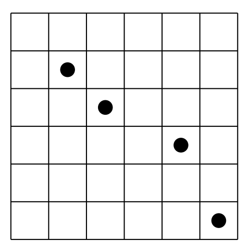
$$R_n = \{(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \mid a_i \in \{0, 1, \ldots, n\} \text{ and if } a_i > 0 \text{ then } a_i \neq a_j \text{ for } i \neq j\}.$$

In this paper, we follow the latter definition of R_n, i.e., $R_n \subseteq \mathbb{R}^n$.

Example 2.1. Some elements of R_4 are $(1, 2, 3, 0), (0, 1, 0, 2)$, and $(1, 2, 4, 3)$. Because the element $(1, 1, 3, 2) \in \mathbb{R}^4$ has repeated positive components (the components that are equal to 1), it is not an element of R_4. Because the element $(1, 3, 5, 0) \in \mathbb{R}^4$ has a component that is greater than 4, it is not an element of R_4.

Each element $a \in R_n$ corresponds to a non-attacking rook placement in an $n \times n$ board as depicted below.


$\Leftrightarrow (1, 2, 4, 3) \in R_4$


$\Leftrightarrow (0, 2, 3, 0, 4, 6) \in R_6$

With the Bruhat order, R_n becomes a poset (partially order set). The first concrete description of the Bruhat order on R_n is given by Pennell, Putcha, and Renner in [5, Theorem 3.8]. We state it here.

Lemma 2.2. *Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in R_n$. The Bruhat order on R_n is the smallest partial order on R_n generated by declaring $a \leq b$ if either*

1. *there exists an $1 \leq i \leq n$ such that $b_i > a_i$ and $b_i = a_j$ for all $j \neq i$, or*

2. *there exists and $1 \leq i < j \leq n$ such that $b_i = a_j, b_j = a_i$ with $b_i > b_j$, and for all $k \notin \{i, j\}$, $b_k = a_k$.*

A more recent paper [3, Lemmas 2.11, 2.13] by Can and Renner analyzed the covering relations in R_n. It was found that there was a missing detail in Lemma 2.11. We state the corrected versions of them below.

Lemma 2.3 (Covering Relation Of Type-1)**.** *Let $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in R_n$. Suppose that there exists an index i such that $a_i < b_i$ and $a_k = b_k$ for all $k \neq i$. Then b is a cover of a if and only if they satisfy all of the following two conditions.*

1. *If $a_i = 0$, then $a_j > 0$ for all $j > i$, and*
2. *There exists a sequence of indices $1 \leq j_1 < \cdots < j_s \leq i$ such that the set $\{a_{j_1}, \dots, a_{j_s}\}$ is equal to $\{a_i, a_i + 1, \dots, a_i + s - 1\}$ and $b_i = a_i + s$.*

Lemma 2.4 (Covering Relation Of Type-2)**.** *Let $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in R_n$. Suppose that that there exist indices $i < j$ such that $a_j = b_i$, $a_i = b_j$ with $b_j < b_i$, and $a_k = b_k$ for all $k \notin \{i, j\}$. Then b is a cover of a if and only if for each $s \in \{i + 1, \dots, j - 1\}$ either $a_j < a_s$ or $a_s < a_i$.*

Definition 2.5. We say that b is a *type-1 cover* of a, denoted by $a \lessdot^1 b$, if and only if b is a cover of a and a, b satisfy Lemma 2.3. Similarly, we say that b is a *type-2 cover* of a, denoted by $a \lessdot^2 b$, if and only if b is a cover of a and a, b satisfy Lemma 2.4.

In many instances, we want to distinguish the components of an element $a = (a_1, a_2, \dots, a_n) \in R_n$. To do so, we put subscripts on the components of a according to their relative positions and the number of times they appear in a. For instance, instead of writing $a = (1, 0, 2, 4, 0) \in R_5$, we write $a = (1_1, 0_1, 2_1, 4_1, 0_2)$ to indicate that 0_1 is the first component of a that is equal to 0, 0_2 is the second component of a that is equal to 0, and so on. We can omit putting subscripts on the components that do not repeat to write $a = (1, 0_1, 2, 4, 0_2)$, i.e., only put subscripts on the repeated zero components. As a component of a, 0_1 is different from 0_2, although their values are the same (equal to 0).

Example 2.6. In R_7, we have

$$(1, 5, 3, 7, 4, 0, 6) \lessdot^1 (2, 5, 3, 7, 4, 0, 6), (1, 0_1, 3, 4, 7, 0_2, 2) \lessdot^1 (1, 0_1, 3, 4, 7, 0_2, 5),$$
$$(0, 2, 6, 5, 3, 2, 1) \lessdot^2 (0, 2, 6, 5, 3, 2, 1), (0_1, 1, 7, 5, 3, 4, 0_2) \lessdot^2 (0_1, 4, 7, 5, 3, 1, 0_2).$$

We note that the symmetric group S_n is a subset of R_n and thus is a subposet of R_n under the Bruhat order.

3 Deodhar Order and Its Equivalence to the Bruhat Order

In this section, we introduce the Deodhar order on $\mathbb{R}^n$, the set of all n-tuples of real numbers. This order turns $\mathbb{R}^n$ into a partially ordered set. The main point of introducing the Deodhar order is that it restricts to the Bruhat order on the rook monoid.

Similarly to R_n, for $a \in \mathbb{R}^n$, we put subscripts on the components of a according to their relative positions and how many times they appear in a in order to distinguish them from one another. We can omit putting subscripts on the components that do not repeat.

Definition 3.1. Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. We write $\mathrm{Comp}(a) = \{a_1, a_2, \dots, a_n\}$.

Example 3.2. Let $a = (2_1, -3.2, 2_2, 1, 0, 5), b = (1, 2_1, 0_1, 2_2, 2_3, 0_2) \in \mathbb{R}^6$. Then we have that

$$\begin{aligned}
[a]_3 &= 2_2 \\
\{[a]_1\} &= \{[b]_2\} = \{2_1\} \\
\mathrm{Comp}(a) &= \{-3.2_1, 0_1, 1_1, 2_1, 2_2, 5_1\} \\
\mathrm{Comp}(b) &= \{0_1, 0_2, 1_1, 2_1, 2_2, 2_3\} \\
\mathrm{Comp}(a) \cup \mathrm{Comp}(b) &= \{-3.2_1, 0_1, 0_2, 1, 2_1, 2_2, 2_3, 5_1\}.
\end{aligned}$$

Definition 3.3. Let $k \in \{1, \dots, n\}$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. The $k-$*truncation* of a is $a(k) = (a_1, \dots, a_k) \in \mathbb{R}^k$.

Definition 3.4 (Deodhar Order)**.** Let a and b be elements of $\mathbb{R}^n$. We say that a is less than or equal to b if and only if for each $k \in \{1, 2, \dots, n\}$

$$\left[\overline{a(k)}\right]_i \leq \left[\overline{b(k)}\right]_i$$

for all $i = 1, 2, \dots, k$. We write $a \leq b$ to denote that a is less than or equal to b.

Example 3.5. In $\mathbb{R}^4$, let $a = (1, 0, 2, 4), b = (4, 3, e, 1)$, $c = (0, 2_1, 2_2, 8)$, and $d = (1, 0, \pi, -2)$. We show that $a \leq b$ while $c \not\leq d$ and $d \not\leq c$. We check for each $k \in \{1, 2, 3, 4\}$ that

$$\begin{aligned}
&k = 1; && \left[\overline{a(1)}\right]_i = [(1)]_i \leq [(4)]_i = \left[\overline{b(1)}\right]_i \\
&k = 2; && \left[\overline{a(2)}\right]_i = [(1, 0)]_i \leq [(4, 3)]_i = \left[\overline{b(2)}\right]_i \\
&k = 3; && \left[\overline{a(3)}\right]_i = [(2, 1, 0)]_i \leq [(4, 3, e)]_i = \left[\overline{b(3)}\right]_i \\
&k = 3; && \left[\overline{a(4)}\right]_i = [(4, 2, 1, 0)]_i \leq [(4, 3, e, 1)]_i = \left[\overline{b(4)}\right]_i.
\end{aligned}$$

Thus, $a \leq b$.

We found that when $k = 2$,

$$\left[\overline{c(2)}\right]_1 = [(2, 1)]_1 = 2 > 1 = [(1, 0)]_1 = \left[\overline{d(2)}\right]_1.$$

Thus, $c \not\leq d$. We also found that when $k = 3$,

$$\left[\overline{d(3)}\right]_1 = [(\pi, 2, 1)]_1 = \pi > 2 = [(2_1, 2_2, 0)]_1 = \left[\overline{c(3)}\right]_1.$$

Thus, $d \not\leq c$. Note we say in this case that c and d are *incomparable*.

The order in Definition 3.4 is called Deodhar order. It turns $\mathbb{R}^n$ into a poset.

Remark 3.6. If $a \leq b$, then $a(k) \leq b(k)$ for all $k = 1, 2, \ldots, n$.

It is apparent that R_n is a subposet of $\mathbb{R}^n$ with the Deodhar order (and so is S_n). For $x, y \in R_n$, it might seem ambiguous whether the symbol "$\leq$" is referring to the Deodhar order or the Bruhat order when we write $x \leq y$. However, this will not be an issue as the next lemma states that the Deodhar order and the Bruhat order are equivalent on R_n. The proof of the lemma can be found in Section 5 of [3].

Lemma 3.7. *The Deodhar order is the same as the Bruhat order on R_n.*

The Deodhar order provides a combinatorial description of the Bruhat order on R_n that will be useful for proving our main result. From now on, we will only describe the order in R_n as the Deodhar order. In preparing for the discussion in Section 7 and Section 6, we devote the rest of this section to establishing some essential properties of this order.

Proposition 3.8. *Let $a \in \mathbb{R}^n$. Then $a \leq \overline{a}$.*

Proof. For each k, we have

$$\left[\overline{a(k)}\right]_i \leq [\overline{a}(k)]_i = \left[\overline{\overline{a}(k)}\right]_i$$

for all $i = 1, \ldots, k$. Thus, $a \leq \overline{a}$. □

Definition 3.9. Let $a \in \mathbb{R}^n$ and $x \in \mathbb{R}$. We define $\eta(a, x) =$ the number of components of a that are greater than or equal to x.

Remark 3.10. Because $\overline{a}$ is a rearrangement of the components of a, it follows that $\eta(a, x) = \eta(\overline{a}, x)$.

Remark 3.11. We have that $[\overline{a}]_i < x$ if and only if $\eta(a, x) = \eta(\overline{a}, x) < i$, since all of the components of a that are greater than or equal to x are placed on the left of the $i-$component in $\overline{a}$. Equivalently, $[\overline{a}]_i \geq x$ if and only if $\eta(a, x) = \eta(\overline{a}, x) \geq i$.

Proposition 3.12. *Let a and b be elements of $\mathbb{R}^n$. Then*

$$[\overline{a}]_i \leq \left[\overline{b}\right]_i$$

for all $i = 1, \ldots, n$ if and only if $\eta(a, [a]_j) \leq \eta(b, [a]_j)$ for all $j = 1, \ldots, n$.

Proof. Suppose that $[\overline{a}]_i \leq \left[\overline{b}\right]_i$ for all $i = 1, \ldots, n$. Assume to the contrary that there exists an index j such that $\eta(a, [a]_j) > \eta(b, [a]_j)$. Then, by *Remark* 3.11, the $\eta(a, [a]_j)-$component of $\overline{b}$ is less than $[a]_j$. Note that the $\eta(a, [a]_j)-$component of $\overline{a}$ is equal to $[a]_j$. Thus, the $\eta(a, [a]_j)-$component of $\overline{a}$ is greater than the $\eta(a, [a]_j)-$component of $\overline{b}$, a contradiction. Therefore, $\eta(a, [a]_j) \leq \eta(b, [a]_j)$ for all $j = 1, \ldots, n$.

Conversely, suppose that $\eta(a, [a]_j) \leq \eta(b, [a]_j)$ for all $j = 1, \ldots, n$. Assume to the contrary that there exists an index i such that $[\overline{a}]_i > \left[\overline{b}\right]_i$. Then, by *Remark* 3.11, $\eta\left(b, [\overline{a}]_i\right) < i \leq \eta\left(a, [\overline{a}]_i\right)$. Since $[\overline{a}]_i = [a]_j$ for some $j \in \{1, \ldots, n\}$, we must have $\eta(b, [a]_j) < \eta(a, [a]_j)$, a contradiction. Therefore, $[\overline{a}]_i \leq \left[\overline{b}\right]_i$ for all $i = 1, \ldots, n$. □

The next corollary characterizes the Deodhar order.

Corollary 3.13 (Deodhar Order)**.** *Let a and b be elements of $\mathbb{R}^n$. Then $a \leq b$ if and only if for each $k \in \{1, \ldots, n\}$*

$$\eta(a(k), [a]_j) \leq \eta(b(k), [a]_j)$$

for all $i = 1, \ldots, k$.

Proof. By Proposition 3.12, for each k, $\left[\overline{a(k)}\right]_i \leq \left[\overline{b(k)}\right]_i$ for all $i = 1, \ldots, k$ if and only if $\eta(a(k), [a(k)]_j) \leq \eta(b(k), [a(k)]_j)$ for all $j = 1, \ldots, k$. Since for $j \in \{1, \ldots, k\}$ we have that $\eta(a(k), [a(k)]_j) = \eta(a(k), [a]_j)$ and $\eta(b(k), [a(k)]_j) = \eta(b(k), [a]_j)$, the assertion follows. □

Proposition 3.14. *Let $a, b \in \mathbb{R}^n$. If $[a]_i \leq [b]_i$ for all $i = 1, \ldots, n$, then $a \leq b$.*

Proof. Suppose that $[a]_i \leq [b]_i$ for all $i = 1, \ldots, n$. Then for each k we have

$$\eta(a(k), [a]_j) \leq \eta(b(k), [a]_j)$$

for all $j = 1, \ldots, k$. Thus, $a \leq b$ by Corollary 3.13. □

Proposition 3.15 (The First Kind of Component Insertion, CI1)**.** *Let $x, y \in \mathbb{R}$ and $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n$. We construct $\alpha, \beta \in \mathbb{R}^{n+1}$ by letting*

$$\alpha = (a_1, \ldots, a_p, x, a_{p+1}, \ldots, a_n), \beta = (b_1, \ldots, b_q, y, a_{q+1}, \ldots, b_n).$$

That is, α is obtained from a by inserting x to a right after its $p-$component, and β is obtained from b by inserting y to b right after its $q-$component. If all of the following four conditions hold

1. *$a \leq b$,*
2. *$p \leq q$,*
3. *$x \leq \min\{\mathrm{Comp}(a)\}$, and*
4. *$x \leq y$,*

then $\alpha \leq \beta$.

Proof. Suppose that α and β satisfy the four conditions stated in Proposition 3.15. Then for $k < p$, we have that

$$\eta\left(\alpha(k), [\alpha]_j\right) = \eta\left(a(k), [a]_j\right) \leq \eta\left(b(k), [a]_j\right) = \eta\left(\beta(k), [\alpha]_j\right)$$

for all $j = 1, \ldots, k$. For $k \geq p$ and $[\alpha]_j = x$,

$$\eta\left(\alpha(k), [\alpha]_j\right) = \eta\left(\beta(k), [\alpha]_j\right) = k,$$

since $x \leq \min\{\text{Comp}(a)\} \leq \min\{\text{Comp}(b)\}$ and $x \leq y$. Finally, for $k \geq p$ and $[\alpha]_j > x$, we have that

$$\eta\left(\alpha(k), [\alpha]_j\right) = \eta\left(a(k-1), [\alpha]_j\right) \leq \eta\left(b(k-1), [\alpha]_j\right) \leq \eta\left(\beta(k), [\alpha]_j\right).$$

Thus, $\eta\left(\alpha(k), [\alpha]_j\right) \leq \eta\left(\beta(k), [\alpha]_j\right)$ for all $i = 1, \ldots, k$. Therefore, $\alpha \leq \beta$ by Corollary 3.13. □

In the following example, we demonstrate how to apply our established results to compare some elements in R_n.

Example 3.16. Let $\alpha = (1, 3, 0_1, 0_2, 5, 2, 0_3), \beta = (6, 4, 0_1, 3, 2, 0_2, 1) \in \mathbb{R}^7$ (In fact, $\alpha, \beta \in R_7$). We use Proposition 3.8, Proposition 3.14, and Proposition 3.15 to show that $\alpha < \beta$.

Firstly, by Proposition 3.8, $a = (1, 3, 5, 2, 0) \leq \overline{a} = (5, 3, 2, 1, 0)$. Secondly, by Proposition 3.14, $\overline{a} \leq b = (6, 4, 3, 2, 1)$. Thus, $a \leq b$.

We then construct $a' = (1, 3, 0_1, 5, 2, 0_2)$ by adding 0 right after the 2−component of a and construct $b' = (6, 4, 0, 3, 2, 1)$ by adding 0 right after the 2−component of b. Thus, by Proposition 3.15, $a' \leq b'$.

We now see that α is constructed by adding 0 right after the 3−component of a' and β is constructed by adding 0 right after the 5−component of b'. Therefore, by Proposition 3.15, $\alpha \leq \beta$.

4 Möbius Function

For a poset P, the Möbius function μ is the integer valued function defined on $P \times P$ by

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu(x, z) & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}.$$

There is a formula for computing the Möbius function of the rook monoid in the literature. A result by Can [2, Theorem 5.2] allow us to compute all possible values of the Möbius function for the rook monoid. We state it as a lemma for later reference here.

Lemma 4.1. *Let $a, b \in R_n$ such that $a \leq b$. Then $\mu(a, b) \in \{-1, 0, 1\}$.*

Verma [8] computed all possible values of the Möbius function for the symmetric group S_n. This computation was later simplified by Stembridge in [7]. We also state as a lemma here.

Lemma 4.2. *Let $a, b \in S_n$ such that $a \leq b$. Then $\mu(a, b) \in \{-1, 1\}$.*

5 EL-Shellability of the Rook Monoid

We next introduce another tool for proving our main result. It is the *EL-shellability* of the rook monoid. This result is due to Can [2].

Let P be a graded poset with maximal and minimal elements. This condition gives that all maximal chains in P have equal length n. We denote $C(P) = \{(a, b) \in P \times P \,|\, a \lessdot b\}$. An *edge-labeling* of P is the map of the form $f : C(P) \to \Gamma$, where Γ is a poset.

Definition 5.1. Let $f : C(P) \to \Gamma$ be an edge-labeling of P. The *Jordan-Hölder sequence* of a maximal chain $\mathbf{c} : a = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = b$ of P is the $n-$tuple

$$f(\mathbf{c}) = (f(x_0, x_1), f(x_1, x_2), \ldots, f(x_{k-1}, x_k)) \in \Gamma^k.$$

With respect to an edge-labeling f of P, we call the maximal chain $\mathbf{c}$ *increasing* if

$$f(x_0, x_1) \leq f(x_1, x_2) \leq \cdots \leq f(x_{k-1}, x_k).$$

Definition 5.2. Let $f : C(P) \to \Gamma$ be an edge-labeling of P. Then f is said to be an *EL-labeling* of P if and only if

1. for each interval $[a, b] \subseteq P$, there exists a unique maximal chain $\mathbf{c}$ such that $f(\mathbf{c})$ is increasing.
2. the Jordan-Hölder sequence $f(\mathbf{c}) \in \Gamma^k$ of the unique chain $\mathbf{c}$ is the smallest of the Jordan-Hölder sequences of maximal chains $a = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = b$, where the order in Γ^k is the *lexicographic order.*

The poset P is said to be *EL-shellable* if and only if there exists an *EL-labeling* of P.

Note that R_n is a graded poset that contains maximal and minimal elements. In [2], an EL-labeling of R_n is described. We state its description below.

Description of an EL-Labelling of R_n: *Let the poset* $\Gamma = \{0, 1, \ldots, n\} \times \{1, 2, \ldots, n\}$ *with the lexicographic order. We define* $F : C(R_n) \to \Gamma$ *by*

$$F(a, b) = \begin{cases} (a_i, b_i) & \text{if } a \lessdot^1 b \\ (a_i, a_j) & \text{if } a \lessdot^2 b \end{cases}, \tag{1}$$

where a_i, b_i, a_j *are described in Lemma 2.3 and Lemma 2.4. This edge-labeling* F *is an EL-labeling of* R_n.

Lemma 5.3. *The rook monoid* R_n *is an EL-shellable poset.*

Figure 5.1 shows the EL-labeling F defined in (1) in the interval $[(0_1, 0_2, 0_2), (1, 2, 3)] \subseteq R_3$. Note that, for simplicity of the figure, we omit writing subscripts in the components of the elements in $[(0_1, 0_2, 0_2), (1, 2, 3)]$.

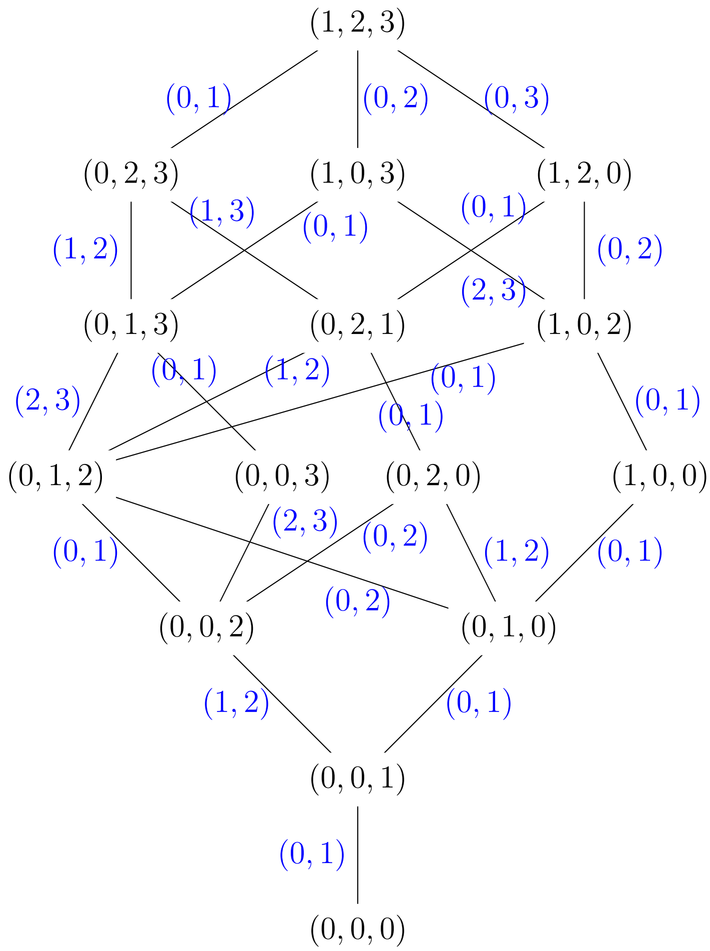


Figure 5.1: The EL-labelling F in $[(0,0,0),(1,2,3)] \subseteq R_3$

The EL-shellability together with Lemma 4.1 and Theorem 2.7 in [1] lead to the following result.

Lemma 5.4. *Let $a, b \in R_n$. If $\mu(a,b) \neq 0$, then there exists a unique maximal chain $\mathbf{c} : a = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = b$ such that the Jordan-Hölder sequence $F(\mathbf{c})$ is strictly decreasing, i.e., $F(x_0, x_1) > F(x_1, x_2) > \cdots > F(x_{k-1}, x_k)$.*

Definition 5.5. Let $a, b \in R_n$ and $\mathbf{c} : a = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = b$ be a maximal chain. With respect to the EL-labeling F defined in (1), the *Type-1 subsequence*

$$F_1(\mathbf{c}) = (F(x_{i_0}, x_{i_0+1}), F(x_{i_1}, x_{i_1+1}), \ldots, F(x_{i_q}, x_{i_q+1}))$$

is the subsequence of the Jordan-Hölder sequence $F(\mathbf{c})$ that contains all $F(x_i, x_{i+1})$ such that $x_i \lessdot^1 x_{i+1}$.

Similarly, the *Type-2 subsequence*

$$F_2(\mathbf{c}) = (F(x_{i_0}, x_{i_0+1}), F(x_{i_1}, x_{i_1+1}), \ldots, F(x_{i_r}, x_{i_r+1}))$$

is the subsequence of the Jordan-Hölder sequence $F(\mathbf{c})$ that contains all $F(x_i, x_{i+1})$ such that $x_i \lessdot^2 x_{i+1}$.

Lemma 5.6. *Let $a, b \in R_n$ such that $\mu(a,b) \neq 0$ and $\mathbf{c}$ be the unique maximal chain described in Lemma 5.4, i.e., $\mathbf{c}$ is strictly decreasing. Write*

$$F_1(\mathbf{c}) = (F(x_{i_0}, x_{i_0+1}), F(x_{i_1}, x_{i_1+1}), \ldots, F(x_{i_q}, x_{i_q+1})) = ((y_0, z_0), (y_1, z_1), \ldots, (y_q, z_q)).$$

Then $z_0 > y_0 \geq z_1 > y_1 \geq z_2 > y_2 \geq \cdots \geq z_q > y_q$.

Proof. It's clear from the definition that $z_s > y_s$ for all s. Since $\mathbf{c}$ is strictly decreasing, $y_0 \geq y_1 \geq \cdots \geq y_q$. Note that for $F(x_i, x_{i+1}) = (y, z) \in F(\mathbf{c})$, we have by Lemma 2.3 that $\{y, y+1, \ldots, z-1\} \subseteq \mathrm{Comp}(x_i)$. In particular, $F(x_i, x_{i+1}) = (y, z)$ implies $y \in \mathrm{Comp}(x_i)$. We show $y_s \geq z_{s+1}$. Let's assume to the contrary that $y_s < z_{s+1}$. Since $\{y_{s+1}, y_{s+1}+1, \ldots, z_{s+1}-1\} \subseteq \mathrm{Comp}(x_{s+1})$ and $y_s \geq y_{s+1}$, we deduce $y_s \in \{y_{s+1}, y_{s+1}+1, \ldots, z_{s+1}-1\} \subseteq \mathrm{Comp}(x_{s+1})$. But because the component y_s of x_s is increased to z_s to produce x_{s+1} in the chain $\mathbf{c}$, we must also have that $y_s \notin \mathrm{Comp}(x_{s+1})$, a contradiction. Therefore, $y_s \geq z_{s+1}$ for all s. Consequently, we obtain $z_0 > y_0 \geq z_1 > y_1 \geq z_2 > y_2 \geq \cdots \geq z_q > y_q$ as desired. □

6 Proof of Theorem 1.2

We begin with defining terminologies and establishing necessary results.

Definition 6.1. Let $a \in R_n$. The element $b \in R_n$ is said to be a *unit increment* of a, denoted by $a \lessdot_1 b$, if and only if there exists an index i such that $[a]_i + 1 = [b]_i$ and $a \lessdot^1 b$. To emphasize the fact that $[a]_i + 1 = [b]_i$, we will sometimes say that b is a unit increment of a by component i. If we can find a unit increment of a by component i, we then say that this component is *incrementable*.

Example 6.2. For $a = (3, 0_1, 2, 0_2, 5, 6, 0_3) \in R_7$, we have that

$$b = (4, 0_1, 2, 0_2, 5, 6, 0_3), c = (3, 0_1, 2, 0_2, 5, 7, 0_3), d = (3, 0_1, 2, 0_2, 5, 6, 1)$$

are all of the unit increments of a. The element $e = (3, 0_1, 2, 0_2, 7, 6, 0_3)$ is not a unit increment of a because e is obtained from a by adding 2 to 5 in a. The element $f = (3, 0_1, 2, 1, 5, 6, 0_2)$ is also not a unit increment of a because f is not a type-1 cover of a (since we add 1 to the zero component that is not the last zero component of a).

Definition 6.3. Let $a \in R_n$. We define $\hat{a}$ to be the element of R_n that is obtained from a by rearranging the nonzero components of a in a non-increasing fashion.

Definition 6.4. Let $a \in R_n$. We define $\langle a \rangle$ to be the element of R_n that is obtained from a by moving the first zero component of a to its last component, if a contains a zero component. If a does not contain any zero component, then we let $\langle a \rangle = a$.

Definition 6.5. Let $a \in R_n$. We define $\lceil a \rceil$ to be the element of R_n that satisfies the chain of unit increments $a = c_0 \lessdot_1 c_1 \lessdot_1 c_2 \lessdot_1 \cdots \lessdot_1 c_k = \lceil a \rceil$ where c_1 is the unit increment of c_0 by its greatest incrementable component, for all $2 \le i \le k$, c_i is the unit increment of c_{i-1} by its greatest incrementable component that is less than the incremented component of c_{i-2}, and $c_k = \lceil a \rceil$ is the unit increment of c_{k-1} by its least incrementable component.

Remark 6.6. Because $\lceil a \rceil$ satisfies the chain of unit increments described in Definition 6.5, we have that

$$[\overline{a}]_i \le \left[\overline{\lceil a \rceil}\right]_i \le [\overline{a}]_i + 1$$

for all $i = 1, \dots, n$.

Example 6.7. Let $a = (1, 2, 0_1, 0_2, 5), b = (1, 3, 2, 5, 4) \in R_5$. Then

$$\hat{a} = (5, 2, 0_1, 0_2, 1), \langle a \rangle = (1, 2, 0_1, 5, 0_2), \lceil a \rceil = (2, 3, 0, 1, 5),$$
$$\hat{b} = (5, 4, 3, 2, 1), \langle a \rangle = (1, 3, 2, 5, 4), \lceil b \rceil = (1, 3, 2, 5, 4).$$

Proposition 6.8. *Let $a \in R_n$. Then a is less than or equal to $\hat{a}, \langle a \rangle$, and $\lceil a \rceil$.*

Proof. Firstly, we show $a \le \hat{a}$. Because $\hat{a}$ is obtained from a by rearranging the nonzero components of a in a non-increasing fashion, we have that for each k

$$\left[\overline{a(k)}\right]_j \le \left[\overline{\hat{a}(k)}\right]_j$$

for all $j = 1, \dots, k$. Thus, $a \le \hat{a}$.

Secondly, we show $a \le \langle a \rangle$. By Lemma 2.3, awapping a zero component of a with a nonzero component on the right of it creates a type-1 cover of a. Since $\langle a \rangle$ can be obtained from a by such sequences of swapping, it follows that $a \le \langle a \rangle$.

Finally, we show $a \le \lceil a \rceil$. Since $\lceil a \rceil$ satisfies the chain $a = c_0 \lessdot_1 c_1 \lessdot_1 c_2 \lessdot_1 \cdots \lessdot_1 c_k = \lceil a \rceil$, we must have that $a = c_0 \lessdot^1 c_1 \lessdot^1 c_2 \lessdot^1 \cdots \lessdot^1 c_k = \lceil a \rceil$. □

Definition 6.9. Let a be an element of R_n. We define $f : R_n \longrightarrow R_n$ by $f(a) = \lceil \langle \hat{a} \rangle \rceil$.

Example 6.10. As in the previous example, let $a = (1, 2, 0_1, 0_2, 5), b = (1, 3, 2, 5, 4) \in R_5$. Then

$$\hat{a} = (5, 2, 0_1, 0_2, 1), \langle \hat{a} \rangle = (5, 2, 0_1, 1, 0_2), \lceil \langle \hat{a} \rangle \rceil = (5, 3, 0, 2, 1) = f(a),$$
$$\hat{b} = (5, 4, 3, 2, 1), \langle \hat{b} \rangle = (5, 4, 3, 2, 1), \lceil \langle \hat{b} \rangle \rceil = (5, 4, 3, 2, 1) = f(b).$$

Remark 6.11. If the number of zero components of $a \in R_n$ is either 1 or 0, then $f(a) = (n, n-1, \dots, 1)$, the maximum element of R_n.

Proposition 6.12. *Let $a \in R_n$. Then $a \le f(a)$.*

Proof. By Proposition 1, we have $f(a) = \lceil \langle \hat{a} \rangle \rceil \ge \langle \hat{a} \rangle \ge \hat{a} \ge a$. □

Remark 6.13. Since

$$\left[\overline{f(a)}\right]_i = \left[\overline{\lceil\langle\hat{a}\rangle\rceil}\right]_i = \left[\overline{\lceil a\rceil}\right]_i,$$

it follows from *Remark* 6.6 that $[\overline{a}]_i \leq \left[\overline{f(a)}\right]_i \leq [\overline{a}]_i + 1$ for all $i = 1, \ldots, n$.

Proposition 6.14. *Let a be an element of R_n such that its the number of zero components is $p \geq 1$. Then the number of zero components of $f(a)$ is $p-1$. Furthermore, $f(a)[0_j] = a[0_{j+1}] - 1$ for all $j = 1, \ldots, p-1$.*

Proof. This is because $f(a)$ is obtained from a by moving the first zero component to the last zero component, making the other zero components move to the left by one position. Then this last zero component is incremented to 1 in the chain of unit increments, making $f(a)$ have the number of zero components one less than of a. □

By *Remark* 6.11, if the number of zero components of a is either 1 or 0, then $b \in [a, f(a)]$ whenever $a \leq b$. Theorem 1.2 gives a characterization of the interval $[a, f(a)]$ when the number of zero components of a is greater than one. We present the proof here.

Proof of Theorem 1.2: Let $a < b$ and the number of zero components of a be $p > 1$. Suppose that $b \in [a, f(a)]$. Assume to the contrary that the first condition fails. That is, there exists an index $i \in \{1, \ldots, n\}$ such that $\left[\overline{b}\right]_i > [\overline{a}]_i + 1$. Then, by *Remark* 6.13,

$$\left[\overline{b}\right]_i > [\overline{a}]_i + 1 \geq \left[\overline{f(a)}\right]_i.$$

Thus, b is neither less than or equal to $f(a)$, a contradiction. Therefore, the first condition must hold.

Now assume to the contrary that the second condition fails. That is, there exists an index $j \in \{1, \ldots, p-1\}$ such that $b[0_j] > a[0_{j+1}] - 1 \geq 0$. By letting $a[0_{j+1}] - 1 = k$, we deduce from Proposition 6.14 that

$$\left[\overline{b(k)}\right]_j > 0 = \left[\overline{f(a)(k)}\right]_j.$$

Thus, b is neither less than or equal to $f(a)$, a contradiction. Therefore, the second condition must also hold.

Conversely, suppose that all of the two conditions hold. Note that the first condition implies $\left[\overline{b}\right]_i \leq \left[\overline{f(a)}\right]_i$ for all $i = 1, \ldots, n$ and that the second condition together with Proposition 6.14 implies $b[0_j] \leq a[0_{j+1}] - 1 = f(a)[0_j]$ for all $j = 1, \ldots, p-1$.

Let u, v be elements of R_{n-p+1} such that u is obtained from b by removing its first $p-1$ zero components and v is obtained from $f(a)$ by removing its first $p-1$ zero components. To see that $b \leq f(a)$, we construct b from v and construct $f(a)$ from u. The argument that we are about to give is similar to the argument given in Example 3.16.

Note that $v = \overline{v}$. Since $\left[\overline{b}\right]_i \le \left[\overline{f(a)}\right]_i$ for all $i = 1, \dots, n$, if follows that

$$[\overline{u}]_i \le [\overline{v}]_i = [v]_i$$

for all $i = i, \dots, n$. Thus, $u \le \overline{u} \le v$ by Proposition 3.8 and Proposition 3.14.

We construct u_1 from u by adding 0 to u right before its $b[0_1]-$ component. Similarly, we construct v_1 from v by adding 0 to v right before its $f(x)[0_1]-$ component. Since $b[0_1] \le f(x)[0_1]$, it follows from Proposition 3.15 that $u_1 \le v_1$. We then keep on constructing u_2 from u_1 by adding 0 to u_1 right before its $b[0_2]-$ component and construct v_2 from v_1 by adding 0 to v_1 right before its $f(x)[0_2]-$ component. With the same argument, we deduce $u_2 \le v_2$. By constructing u_i from u_{i-1} and v_i from v_{i-1} this way, we eventually obtain $u_{p-1} = b$ and $v_{p-1} = f(a)$. Thus, $b \le f(a)$. Therefore, $b \in [a, f(a)]$. □

7 Proof of Theorem 1.1

As usual, we start with defining necessary terminologies.

Definition 7.1. Let $a \in R_n$. We write $\mathcal{R}(a) = \mathrm{Comp}(a) \cup \mathrm{Comp}(f(a))$.

Definition 7.2. Let $a \in R_n$. For $m = |\mathcal{R}(a)|$, let $c_1, c_2, \dots, c_m$ be all of the elements of $\mathcal{R}(a)$ that are written in non-decreasing fashion (we arrange 0_p before 0_q if $p < q$). We define the *canonical assignment* of the elements in $\mathcal{R}(a)$ to be the bijective map $\sigma : \mathcal{R}(a) \to \{1, 2, \dots, m\}$ defined by

$$\sigma = \begin{pmatrix} c_1 & c_2 & \cdots & c_m \\ 1 & 2 & \cdots & m \end{pmatrix}.$$

Example 7.3. Let $a = (1, 5, 0_1, 0_2, 2, 0_3), b = (1, 3, 2, 5, 4, 0) \in R_6$. Then $f(a) = (6, 3, 0_1, 2, 0_2, 1)$, and $f(b) = (6, 5, 4, 3, 2, 1)$. Thus, $\mathcal{R}(a) = \{0_1, 0_2, 0_3, 1, 2, 3, 5, 6\}$ and $\mathcal{R}(b) = \{0, 1, 2, 3, 4, 5, 6\}$. Therefore, the canonical assignment of the elements in $\mathcal{R}(a)$ is

$$\begin{pmatrix} 0_1 & 0_2 & 0_3 & 1 & 2 & 3 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

and the canonical assigment of the elements in $\mathcal{R}(b)$ is

$$\begin{pmatrix} 0_1 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

Definition 7.4. Let $a \in R_n$ and σ be its canonical assignment. With respect to σ, we define the *correspondent* of a to be the element

$$a_{\mathrm{co}} = (\sigma([a]_1), \sigma([a]_2), \dots, \sigma([a]_n), t_{n+1}, t_{n+2}, \dots, t_m)$$

of the symmetric group S_m, $m = |\mathcal{R}(a)|$, where $t_j \le t_k$ for all $n + 1 \le j \le k \le m$.

The correspondent of $f(a)$ with respect to the canonical assignment σ is defined in the same way. That is, its correspondent is the element

$$f(a)_{\mathrm{co}} = \left(\sigma\left([f(a)]_1\right), \sigma\left([f(a)]_2\right), \ldots, \sigma\left([f(a)]_n\right), t_{n+1}, t_{n+2}, \ldots, t_m\right)$$

of the symmetric group S_m, $m = |\mathcal{R}(a)|$, where $t_j \leq t_k$ for all $n+1 \leq j \leq k \leq m$.

Example 7.5. As in the previous example, let $a = (1,5,0,0,2,0), b = (1,3,2,5,4,0) \in R_6$. We computed that the canonical assignment of the elements in $\mathcal{R}(a)$ is

$$\sigma_a = \begin{pmatrix} 0_1 & 0_2 & 0_3 & 1 & 2 & 3 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix},$$

and the canonical assigment of the elements in $\mathcal{R}(b)$ is

$$\sigma_b = \begin{pmatrix} 0_1 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}.$$

Then, the correspondent of a is $a_{\mathrm{co}} = (4,7,1,2,5,3,6,8)$ and the correspondent of b is $b_{\mathrm{co}} = (2,4,3,6,5,1,7)$. Note that $a_{\mathrm{co}} \in S_8$ while $b_{\mathrm{co}} \in S_7$.

Remark 7.6. For $a \in R_n$ and $i, j \in \{1, 2, \ldots, n\}$, we have that $[a]_i \leq [a]_j$ if and only if $\sigma([a]_i) \leq \sigma([a]_j)$.

We now temporarily turn to establish some properties of the symmetric group.

Lemma 7.7. *Let $a \in R_n$. Suppose that a_{co} and $f(a)_{\mathrm{co}}$ are elements of S_m. Then every $b \in [a_{\mathrm{co}}, f(a)_{\mathrm{co}}]$ satisfies $[b]_{n+1} < [b]_{n+2} < \cdots < [b]_m$.*

Proof. By Definition 7.4, we have that

$$[a_{\mathrm{co}}]_j < [a_{\mathrm{co}}]_{j+1} \text{ and } [f(a)_{\mathrm{co}}]_j < [f(a)_{\mathrm{co}}]_{j+1}$$

for all $j = n+1, n+2, \ldots, m-1$. Thus, for each $j \in \{n+1, n+2, \ldots, m-1\}$,

$$\{1, 2, \ldots, [a_{\mathrm{co}}]_j\} \subseteq \mathrm{Comp}(a_{\mathrm{co}}(j)) \text{ and } \{1, 2, \ldots, [f(a)_{\mathrm{co}}]_j\} \subseteq \mathrm{Comp}\left(f(a)_{\mathrm{co}}(j)\right).$$

Let $b \in [a, f(a)]$. Define

$$x_n = \text{the least positive integer such that } \{1, 2, \ldots, x_n\} \not\subseteq \mathrm{Comp}\left(\overline{a_{\mathrm{co}}(n)}\right),$$

$$y_n = \text{the least positive integer such that } \{1, 2, \ldots, y_n\} \not\subseteq \mathrm{Comp}\left(\overline{f(a)_{\mathrm{co}}(n)}\right),$$

$$z_n = \text{the least positive integer such that } \{1, 2, \ldots, z_n\} \not\subseteq \mathrm{Comp}\left(\overline{b(n)}\right).$$

Then $x_n = [a_{\mathrm{co}}]_{n+1}$ and $y_n = [f(a)_{\mathrm{co}}]_{n+1}$. We first show that $z_n = [b]_{n+1}$. To see this, we note that since for each $k \in \{1, 2, \ldots, n\}$

$$\left[\overline{a(k)}\right]_i \leq \left[\overline{f(a)(k)}\right]_i \leq \left[\overline{a(k)}\right]_i + 1,$$

for all $i = 1, \ldots, k$, it follows from *Remark* 7.6 that for each $k \in \{1, 2, \ldots, n\}$

$$\left[\overline{a_{\text{co}}(k)}\right]_i \leq \left[\overline{f(a)_{\text{co}}(k)}\right]_i \leq \left[\overline{a_{\text{co}}(k)}\right]_i + 1 \tag{2}$$

for all $i = 1, \ldots, k$. This gives $x_n \geq y_n$. For if they were to satisfy $x_n \leq y_n - 1$, then $x_n \in \text{Comp}\left(\overline{f(a)_{\text{co}}(n)}\right)$. Consequently, the $(x_n)^{\text{th}}$ component of $\overline{f(a)_{\text{co}}(n)}$ from the right (its $(n - x_n + 1)-$component) would be x_n which is less than the $(x_n)^{\text{th}}$ component of $\overline{a_{\text{co}}(n)}$ from the right. This would then contradict (2). Thus, for the same reason, we also have that $x_n \geq z_n$ and $z_n \geq y_n$.

Applying $x_n \geq y_n$ together with (2), we deduce

$$\begin{aligned}
\left[\overline{f(a)_{\text{co}}(n)}\right]_{m-i+1} &= \left[\overline{a_{\text{co}}(n)}\right]_{m-i+1} && \text{if } i \leq y_n, \\
\left[\overline{f(a)_{\text{co}}(n)}\right]_{m-i+1} &= \left[\overline{a_{\text{co}}(n)}\right]_{m-i+1} + 1 && \text{if } y_n < i \leq x_n, \\
\left[\overline{a_{\text{co}}(n)}\right]_{m-i+1} \leq \left[\overline{f(a)_{\text{co}}(n)}\right]_{m-i+1} &\leq \left[\overline{a_{\text{co}}(n)}\right]_{m-i+1} + 1 && \text{else.}
\end{aligned}$$

Thus, $\{1, 2, \ldots, y_n - 1, y_n + 1, \ldots, x_n\} \subseteq \text{Comp}\left(\overline{f(a)_{\text{co}}(n)}\right)$. Consequently, we have that $\{1, 2, \ldots, x_n\} \subseteq \text{Comp}\left(\overline{f(a)_{\text{co}}(n+1)}\right)$ and

$$\left[\overline{a_{\text{co}}(n+1)}\right]_i \leq \left[\overline{f(a)_{\text{co}}(n+1)}\right]_i \leq \left[\overline{a_{\text{co}}(n+1)}\right]_i + 1 \tag{3}$$

for all $i = 1, \ldots, n+1$. Because $b(n+1) \leq f(a)_{\text{co}}(n+1)$, it follows that $\{1, 2, \ldots, x_n\} \subseteq \text{Comp}\left(\overline{b(n+1)}\right)$. Since $z_n \leq x_n$, this implies $z_n \in \text{Comp}\left(\overline{b(n+1)}\right)$. Because $z_n \notin \text{Comp}\left(\overline{b(n)}\right)$, we deduce $[b]_{n+1} = z_n$ as desired.

Define

$$\begin{aligned}
x_{n+1} &= \text{the least positive integer such that } \{1, 2, \ldots, x_{n+1}\} \not\subseteq \text{Comp}\left(\overline{a_{\text{co}}(n+1)}\right), \\
y_{n+1} &= \text{the least positive integer such that } \{1, 2, \ldots, y_{n+1}\} \not\subseteq \text{Comp}\left(\overline{f(a)_{\text{co}}(n+1)}\right), \\
z_{n+1} &= \text{the least positive integer such that } \{1, 2, \ldots, z_{n+1}\} \not\subseteq \text{Comp}\left(\overline{b(n+1)}\right).
\end{aligned}$$

With (3) and the same argument as before, we get $[b]_{n+2} = z_{n+1}$. Therefore, $[b]_{n+1} < [b]_{n+2}$.

By continuing this same argument with $[b]_{n+3}, [b]_{n+4}, \ldots, [b]_m$, we obtain

$$[b]_{n+1} < [b]_{n+2} < \cdots < [b]_m$$

as desired ∎

Lemma 7.8. *Let $a \in R_n$. Suppose that a_{co} and $f(a)_{\text{co}}$ are elements of S_m. Let b, c be elements of $[a_{\text{co}}, f(a)_{\text{co}}]$. Then $b \leq c$ if and only if $b(n) \leq c(n)$.*

Proof. If $b \leq c$, then it is clear that $b(n) \leq c(n)$.

Conversely, suppose that $b(n) \leq c(n)$. We show by induction on i that $b(n+i) \leq c(n+i)$ for $i = 0, 1, \ldots, m-n$. The assertion holds when $i = 0$, since we have $b(n) \leq c(n)$. Let's suppose that $b(n+i) \leq c(n+i)$ for all $i = 0, 1, \ldots, k$ for some $k \leq m-n$. As in the proof of Lemma 7.7, we have that

$$\begin{aligned}
[b]_{n+k+1} &= x_{n+k} \\
&= \text{the least positive integer such that } \{1, 2, \ldots, x_{n+k}\} \not\subseteq \operatorname{Comp}\left(\overline{b(n+k)}\right) \\
[c]_{n+k+1} &= y_{n+k} \\
&= \text{the least positive integer such that } \{1, 2, \ldots, y_{n+k}\} \not\subseteq \operatorname{Comp}\left(\overline{c(n+k)}\right).
\end{aligned}$$

Because $b(n) \leq c(n)$, we must have $x_{n+k} \geq y_{n+k}$. Moreover, for $z \in \operatorname{Comp}(c(n+k))$ such that $y_{n+k} \leq z \leq x_{n+k}$, we get

$$\begin{aligned}
&\eta(c(n+k), z) = \eta(b(n+k), z) && \text{if } x_{n+k} = y_{n+k} \\
&\eta(c(n+k), z) \geq \eta(b(n+k), z) + 1 && \text{otherwise.}
\end{aligned}$$

Thus,

$$\eta\left(c(n+k+1), [b(n+k+1)]_j\right) = \eta\left(b(n+k+1), [b(n+k+1)]_j\right)$$

if $[b(n+k+1)]_j \leq x_{n+k}$, and

$$\eta\left(c(n+k+1), [b(n+k+1)]_j\right) \geq \eta\left(b(n+k+1), [b(n+k+1)]_j\right)$$

otherwise.

Corollary 3.13 together with the assumption that $b(n+i) \leq c(n+i)$ for all $i = 0, 1, \ldots, k$ imply $b(n+k+1) \leq c(n+k+1)$. Therefore, $b \leq c$ by induction. □

We are now ready to prove our first main result.

Proof of Theorem 1.1: Firstly, we show that $[a, f(a)] \cong [a_{\text{co}}, f(a)_{\text{co}}]$. To see this, we show the following map is an isomorphism: $\varphi : [a, f(a)] \to [a_{\text{co}}, f(a)_{\text{co}}]$ defined by

$$\varphi(b) = b_{\text{co}} = (\sigma([b]_1), \sigma([b]_2), \ldots, \sigma([b]_n), t_{n+1}, t_{n+2}, \ldots, t_m) \in S_m$$

where σ is the canonical assignment of a and $t_j \leq t_k$ for all $n+1 \leq j \leq k \leq m$.

The map φ is well-defined by Lemma 7.7. It is then clear that φ is a bijection. We only need to show that φ is order preserving, i.e., $a \leq b$ if an only if $\varphi(a) \leq \varphi(b)$. If $a \leq b$, then $\varphi(a)(n) \leq \varphi(b)(n)$ by *Remark* 7.6. Since $\varphi(a), \varphi(b) \in [a_{\text{co}}, f(a)_{\text{co}}]$, it follows from Lemma 7.8 that $\varphi(a) \leq \varphi(b)$. Conversely, if $a, b \in [a, f(a)]$ be elements such that $\varphi(a) \leq \varphi(b)$, then

$$\begin{aligned}
a &= \left(\sigma^{-1}\left([\varphi(a)]_1\right), \ldots, \sigma^{-1}\left([\varphi(a)]_n\right)\right) \\
b &= \left(\sigma^{-1}\left([\varphi(b)]_1\right), \ldots, \sigma^{-1}\left([\varphi(b)]_n\right)\right).
\end{aligned}$$

By *Remark* 7.6, the Deodhar order preserves under the inverse map φ^{-1}. That is, $a \leq b$. Thus, φ is an isomorphism. Therefore, $[a, f(a)] \cong [a_{\text{co}}, f(a)_{\text{co}}]$.

Secondly, we show that $I(a) = [a, f(a)]$. Let $b \in [a, f(a)]$. Since $[a, f(a)] \cong [a_{\text{co}}, f(a)_{\text{co}}]$ and, by Lemma 4.2, $\mu(a_{\text{co}}, b_{\text{co}}) \neq 0$ for any $b_{\text{co}} \in [a_{\text{co}}, f(a)_{\text{co}}]$, it follows that every b satisfies $\mu(a, b) \neq 0$.

Conversely, if $b \in R_n$ satisfies $\mu(a, b) \neq 0$, then there exists, by Lemma 5.4, a maximal chain $\mathbf{c} : a = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = b$ such that the Jordan-Hölder sequence $F(\mathbf{c})$ is *strictly* decreasing. We show that $\left[\overline{b}\right]_j \leq [\overline{a}]_j + 1$ for all $i = 1, \ldots, n$. To see this, we note that since $\text{Comp}(x_{i-1}) = \text{Comp}(x_i)$ if $x_{i-1} \lessdot^2 x_i$, it suffices to only consider x_i such that $x_{i-1} \lessdot^1 x_i$. This observation suggests us to look at the type-1 subsequence $F_1(\mathbf{c})$ of the Jordan-Hölder sequence $F(\mathbf{c})$. By writing

$$F_1(\mathbf{c}) = (F(x_{i_0}, x_{i_0+1}), F(x_{i_1}, x_{i_1+1}), \ldots, F(x_{i_q}, x_{i_q+1})) = ((y_0, z_0), (y_1, z_1), \ldots, (y_q, z_q)),$$

we get $z_0 > y_0 \geq z_1 > y_1 \geq z_2 > y_2 \geq \cdots \geq z_q > y_q$ by Lemma 5.6. Because $x_{i_0} \lessdot^1 x_{i_0+1}$, it follows that

$$[\overline{x_{i_0+1}}]_j = \begin{cases} [\overline{x_{i_0}}]_j + 1 & \text{if } j \in A_0 \\ [\overline{x_{i_0}}]_j & \text{otherwise} \end{cases} \tag{I}$$

where $A_0 = \{\overline{x_{i_0}}[z_0 - 1], \overline{x_{i_0}}[z_0 - 1] + 1, \ldots, \overline{x_{i_0}}[y_0]\}$. Since $\text{Comp}(x_{i_0}) = \text{Comp}(a)$, we have $[\overline{x_{i_0}}]_j = [\overline{a}]_j$. Thus, equation (I) becomes

$$[\overline{x_{i_0+1}}]_j = \begin{cases} [\overline{a}]_j + 1 & \text{if } j \in B_0 \\ [\overline{a}]_j & \text{otherwise} \end{cases} \tag{II}$$

where $B_0 = A_0 = \{\overline{a}[z_0 - 1], \overline{a}[z_0 - 1] + 1, \ldots, \overline{a}[y_0]\}$. Similarly,

$$[\overline{x_{i_1+1}}]_j = \begin{cases} [\overline{x_{i_1}}]_j + 1 & \text{if } j \in A_1 \\ [\overline{x_{i_1}}]_j & \text{otherwise} \end{cases} \tag{III}$$

where $A_1 = \{\overline{x_{i_1}}[z_1 - 1], \overline{x_{i_1}}[z_1 - 1] + 1, \ldots, \overline{x_{i_1}}[y_1]\}$. Because $z_0 - 1 \geq y_0 > z_0 - 1 \geq y_1$ and $[\overline{x_{i_1}}]_j = [\overline{x_{i_0+1}}]_j$, we deduce

$$\begin{aligned} A_1 &= \{\overline{x_{i_0+1}}[z_1 - 1], \overline{x_{i_0+1}}[z_1 - 1] + 1, \ldots, \overline{x_{i_0+1}}[y_1]\} \\ &= \{\overline{x_{i_0}}[z_1 - 1], \overline{x_{i_0}}[z_1 - 1] + 1, \ldots, \overline{x_{i_0}}[y_1]\} \\ &= \{\overline{a}[z_1 - 1], \overline{a}[z_1 - 1] + 1, \ldots, \overline{a}[y_1]\}. \end{aligned}$$

Together with equation (II), equation (III) becomes

$$[\overline{x_{i_1+1}}]_j = \begin{cases} [\overline{a}]_j + 1 & \text{if } j \in B_1 \\ [\overline{a}]_j & \text{otherwise} \end{cases} \tag{IX}$$

where $B_1 = B_0 \cup A_1 = \{\overline{a}[z_0 - 1], \overline{a}[z_0 - 1] + 1, \ldots, \overline{a}[y_0]\} \cup \{\overline{a}[z_1 - 1], \overline{a}[z_1 - 1] + 1, \ldots, \overline{a}[y_1]\}$. By keep on arguing in the same fashion for $[\overline{x_{i_2+1}}]_j, \ldots, \left[\overline{x_{i_q+1}}\right]_j$, we eventually obtain

$$\left[\overline{x_{i_q+1}}\right]_j = \begin{cases} [\overline{a}]_j + 1 & \text{if } j \in B_q \\ [\overline{a}]_j & \text{otherwise} \end{cases}$$

where

$$B_q = B_0 \cup \cdots \cup B_{q-1} \cup A_q = \{\overline{a}[z_0 - 1], \overline{a}[z_0 - 1] + 1, \ldots, \overline{a}[y_0]\} \cup \cdots \cup \{\overline{a}[z_q - 1], \overline{a}[z_q - 1] + 1, \ldots, \overline{a}[y_q]\}.$$

Because $\mathrm{Comp}(b) = \mathrm{Comp}(x_{i_q})$, we therefore obtain $\left[\overline{b}\right]_j \leq [\overline{a}]_j + 1$ for all $i = 1, \ldots, n$ as desired.

Now we show that $b[0_j] \leq a[0_{j+1}] - 1$. We only need to consider a such that its number of zero components is $p > 1$. Let's write $F(x_i, x_{i+1}) = (y_i, z_i)$. Consider the following two cases.

Case 1: There doesn't exists s such that $y_s = 0$.

Then throughout the chain $\mathbf{c} : a = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = b$, the zero components of all x_i stay in the same positions as in a. Thus, $b[0_j] \leq a[0_{j+1}] - 1$ for all $j = 1, 2, \ldots, p-1$.

Case 2: There exists s such that $y_s = 0$.

Assume to the that there exists an index $j \in \{1, 2, \ldots, p-1\}$ such that $b[0_j] > a[0_{j+1}] - 1$. Let t be the least integer such that $x_t[0_j] > a[0_{j+1}] - 1$. Then there exists the greatest integer r such that $x_r[0_j] \leq a[0_{j+1}] - 1$. Thus, $F(x_r, x_{r+1}) = (0, z_r)$ and either the component 0_{j+1} in x_r is increased to z_r or the the component 0_{j+1} in x_r is swapped with z_r in x_r to produce x_{r+1}.

If $x_r \lessdot^1 x_{r+1}$, then the component 0_{j+1} in x_r is increased to z_r to produce x_{r+1}. Moreover, by Lemma 2.3, this is the last zero component of x_r and so $j = p - 1$. That is, we have

$$\begin{aligned} x_r &= (\ldots, 0_{p-1}, \ldots, 0_p, \ldots\ldots\ldots\ldots) \\ x_{r+1} &= (\ldots, 0_{p-1}, \ldots, z_r, \ldots\ldots\ldots\ldots) \\ x_t &= (\ldots\ldots\ldots, \ldots, z_r, \ldots, 0_{p-1}, \ldots). \end{aligned}$$

Note that $x_t \lessdot^2 x_{t-1}$ and the component 0_{p-1} of x_{t-1} is swapped with the component z_t in x_{t-1} to produce x_t. For this instance, Lemma 2.3 also gives that, in x_{r+1}, the components on the right of z_r are greater than or equal to z_r. Moreover, for $r < i < t$, the $x_{r+1}[z_r]-$component of x_i and after are placed in the exact same positions as in x_{r+1}. As the component z_t is on the right of z_r, we then have $z_t > z_r$. Thus, $F(x_r, x_{r+1}) = (0, z_r) \leq (0, z_t) = F(x_{t-1}, x_t)$, a contradiction. Therefore, $b[0_j] \leq a[0_{j+1}] - 1$ for all $j = 1, 2, \ldots, p-1$.

If $x_r \lessdot^2 x_{r+1}$, then the component 0_{j+1} in x_r is swapped with z_r to produce x_{r+1}. That is, we have

$$\begin{aligned} x_r &= (\ldots, 0_j, \ldots, 0_{j+1}, \ldots\ldots\ldots\ldots, \; z_r \;, \ldots) \\ x_{r+1} &= (\ldots, 0_j, \ldots, \; z_r \;, \ldots\ldots\ldots\ldots, 0_{j+1}, \ldots) \\ x_t &= (\ldots\ldots\ldots\ldots, \; z_r \;, \ldots, 0_j, \ldots, 0_{p-1}, \ldots). \end{aligned}$$

For this instance, Lemma 2.4 gives that, in x_{r+1}, the components on the right of z_r and on the left of 0_{j+1} are greater than or equal to z_r. Moreover, for $r < i < t$, the components between the $x_{r+1}[z_r]-$component and the $x_{r+1}[0_{j+1}]-$componentof x_i are placed in the exact same positions as in x_{r+1}. As the component z_t is on the right of z_r and on the left of 0_{j+1}, we then have $z_t \geq z_r$. Thus, $F(x_r, x_{r+1}) = (0, z_r) \leq (0, z_t) = F(x_{t-1}, x_t)$, a contradiction. Therefore, $b[0_j] \leq a[0_{j+1}] - 1$ for all $j = 1, 2, \ldots, p-1$.

By Theorem 1.2, we conclude that $b \in [a, f(a)]$. Therefore, $I(a) = [a, f(a)]$. This completes the proof. □

Note that we can also deduce the result given in Lemma 4.1 by employing Theorem 1.1 together with Lemma 4.2.

For $a, b \in R_n$, Theorem 1.1 states that $\mu(a, b) \neq 0$ if and only if $b \in [a, f(a)]$. We then employ Theorem 1.2 to see if $b \in [a, f(a)]$ by checking whether b satisfies the two conditions given in Theorem 1.2. We demonstrate this application in the following example.

Example 7.9. In R_8, we check that $a = (1, 2, 0_1, 5, 7, 3, 0_2, 0_3)$ is less than or equal to all of $b = (2, 3, 1, 6, 7, 5, 0_1, 0_2)$, $c = (2, 3, 1, 5, 8, 4, 0_1, 0_2)$, and $d = (2, 6, 8, 3, 4, 0_1, 0_2, 1)$. We show that $\mu(a, b) = \mu(a, c) = 0$ and $\mu(a, d) \neq 0$.

To see this, we check that

$$\left[\overline{b}\right]_3 = [(7, 6, 5, 3, 2, 1, 0_1, 0_2)]_3 = 5 > [\overline{a}]_3 + 1 = [(7, 5, 3, 2, 1, 0_1, 0_2, 0_3)]_3 + 1 = 4 \text{ and}$$

$$c[0_1] = 7 > a[0_2] - 1 = 7 - 1 = 6.$$

Thus, b and c do not satisfy the two conditions given in Theorem 1.2. Hence, both b and c are not elements of $[a, f(a)]$. Therefore, by Theorem 1.1, $\mu(a, b) = \mu(a, c) = 0$.

We check that

$$\left[\overline{d}\right]_i = [(8, 6, 4, 3, 2, 1, 0_1, 0_2)]_i \leq [\overline{a}]_i + 1 = [(7, 5, 3, 2, 1, 0_1, 0_2, 0_3)]_i + 1$$

for all $i = 1, 2, \ldots, n$, and that $d[0_j] \leq a[0_{j+1}] - 1$ for all $j = 1, 2$. Thus, $d \in [a, f(a)]$ by Theorem 1.2. Therefore, $\mu(a, d) \neq 0$ by Theorem 1.1.

8 Open Questions

The results presented in this paper lead us to wonder if there are their analogues in related algebraic structures. In particular, we would like to answer the following questions.

1. Are there analogues of these results for other reductive monoids as described by Solomon in [6]?

2. Are there analogues of Stembridge's results in [7] relating Bruhat order on S_n to the 0-Hecke monoid, using Halverson's q-rook monoid as described in [4] with parameters set to zero?

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