CRITICAL GROUPS FOR COMPLETE MULTIPARTITE GRAPHS AND CARTESIAN PRODUCTS OF COMPLETE GRAPHS

BRIAN JACOBSON, ANDREW NIEDERMAIER, AND VICTOR REINER

ABSTRACT. The critical group of a connected graph is a finite abelian group, whose order is the number of spanning trees in the graph, and which is closely related to the graph Laplacian. Its group structure has been determined for relatively few classes of graphs, e.g. complete graphs, and complete bipartite graphs.

For complete multipartite graphs $K_{n_1,\ldots,n_k}$, we describe the critical group structure completely. For Cartesian products of complete graphs $K_{n_1} \times \cdots \times K_{n_k}$, we generalize results of H. Bai on the k-dimensional cube, by bounding the number of invariant factors in the critical group, and describing completely its p-primary structure for all primes p that divide none of $n_1, \ldots, n_k$.

1. Introduction and background

The critical group of a connected graph is a finite abelian group whose structure is a subtle isomorphism invariant of the graph. It is closely connected with the graph Laplacian, as we now explain.

If $G = (V, E)$ is a finite graph without self-loops, but with multiple edges allowed, then its *Laplacian* $L(G)$ is the $|V| \times |V|$ *Laplacian matrix* $L(G)$ defined by

$$L(G)_{v,v'} = \begin{cases} \deg_G(v) & \text{if } \ v = v' \\ -m_{v,v'} & \text{else} \end{cases}$$

where $m_{v,v'}$ denotes the multiplicity of the edge $\{v, v'\}$ in E. When G is connected, the kernel of $L(G)$ is spanned by the vectors in $\mathbb{R}^V$ which are constant on the vertices. Thinking of $L(G)$ as a map $\mathbb{Z}^{|V|} \to \mathbb{Z}^{|V|}$, its cokernel has the form

$$\mathbb{Z}^{|V|}/\mathrm{im}L(G) \;\cong\; \mathbb{Z} \oplus K(G).$$

where $K(G)$ is defined to be the *critical group*. It follows from Kirchoff's Matrix-Tree Theorem that the order $|K(G)|$ is the number $\kappa(G)$ of spanning trees in G.

Kirchoff's Matrix-Tree Theorem.(see e.g. [2, Chapter 6])

(i)

$$\kappa(G) = (-1)^{i+j} \det \overline{L(G)}$$

Key words and phrases. Graph Laplacian, critical group, abelian sandpile, chip-firing, Smith normal form, matrix-tree theorem, complete multipartite graph, Cartesian product graph.

This paper arose from work supported by the Summer 2002 University of Minnesota Research Experiences for Undergraduates (REU) program, and partly supported by NSF grant DMS–9877047.

where $\overline{L(G)}$ is a reduced Laplacian matrix obtained from $L(G)$ by striking out any row i and column j.

(ii) *If the eigenvalues of $L(G)$ are indexed $\lambda_1, \ldots, \lambda_{n-1}, \lambda_n$, where $n = |V|$ and $\lambda_n = 0$, then*

$$\kappa(G) = \frac{\lambda_1 \cdots \lambda_{n-1}}{n}.$$

The *critical group* $K(G)$ has also been called the *Picard group*, the *Jacobian group*, the *tree group*, the *sandpile group*, and has a close connection with the *critical configurations* in a certain *chip-firing* game on G (known as *abelian sandpiles* in the physics literature)—see e.g. [3], [4], [5], [6], [7], [9, §14.13], [12].

Compared to the number of results on the spanning tree number $\kappa(G)$, there are relatively few results describing the group structure of $K(G)$ in terms of structure of G. There are also very few interesting infinite families of graphs for which the group structure has been completely determined. For example, Cayley's celebrated formula $\kappa(K_n) = n^{n-2}$ is suggestive of the structure of the critical group for the complete graph K_n; it turns out that

$$K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}.$$

This paper studies the critical group for two families of graphs generalizing the complete graphs K_n, and for which the spanning tree numbers are known: complete multipartite graphs $K_{n_1,\ldots,n_k}$ and Cartesian products $K_{n_1} \times \cdots \times K_{n_k}$.

For the complete multipartite graph $K_{n_1,\ldots,n_k}$, one knows from simple eigenvalue calculations (see Section 2 below, or [10]) that if we set

$$\begin{aligned} N &:= n_1 + \cdots + n_k (= |V|) \\ N_i &:= N - n_i, \end{aligned}$$

then

$$\kappa(K_{n_1,\ldots,n_k}) = N^{k-2} \prod_{i=1}^{k} N_i^{n_i - 1}. \tag{1}$$

In the bipartite case $k = 2$, Lorenzini [11] calculated that

$$K(K_{n_1,n_2}) \cong (\mathbb{Z}/n_1\mathbb{Z})^{n_2-2} \oplus (\mathbb{Z}/n_2\mathbb{Z})^{n_1-2} \oplus \mathbb{Z}/n_1 n_2\mathbb{Z}, \tag{2}$$

which one can check has order agreeing with the case $k = 2$ in (1). Our first main result shows that for $k > 2$, the answer is a bit more subtle, involving arithmetic properties of the integers N_i and $k-1$. Let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be the *invariant factors* of $\oplus_{i=1}^{k} \mathbb{Z}/N_i\mathbb{Z}$ that is, the unique sequence of positive integers having σ_i divide σ_{i+1} for each $i \leq k-1$ and

$$\bigoplus_{i=1}^{k} \mathbb{Z}/N_i\mathbb{Z} \cong \bigoplus_{i=1}^{k} \mathbb{Z}/\sigma_i\mathbb{Z}.$$

Also define

$$\begin{aligned} g &:= \gcd(k-1, N_1, N_2, \ldots, N_k) \\ h &:= \sigma_1\sigma_2/g \end{aligned}.$$

Theorem 1. *For $k > 2$ one has*

$$K(K_{n_1,\ldots,n_k}) \cong \bigoplus_{i=1}^{k} (\mathbb{Z}/N_i\mathbb{Z})^{n_i-2} \;\oplus\; \mathbb{Z}/g\mathbb{Z} \;\oplus\; \mathbb{Z}/h\mathbb{Z} \;\oplus\; \bigoplus_{i=3}^{k} \mathbb{Z}/\sigma_i N\mathbb{Z}.$$

Note that this agrees with (1) because

$$gh\prod_{i=3}^{k}(\sigma_i N) = N^{k-2}\prod_{i=1}^{k}\sigma_i = N^{k-2}\prod_{i=1}^{k}N_i.$$

For the Cartesian product $K_{n_1}\times\cdots\times K_{n_k}$, we will henceforth assume (without loss of generality) that $n_i \geq 2$ for each i. One knows from simple eigenvalue calculations (see Section 2) that if we set $N_S := \sum_{i\in S} n_i$, then

$$\kappa(K_{n_1}\times\cdots\times K_{n_k}) = \frac{1}{\prod_{i=1}^{k} n_i}\prod_{\emptyset\neq S\subseteq[k]} N_S^{\prod_{i\in S}(n_i-1)} \tag{3}$$

Describing the exact structure of the critical group $K(K_{n_1}\times\cdots\times K_{n_k})$ appears to be difficult, due to bad behavior of its p-primary structure for primes p that divide some of $n_1,\ldots,n_k$. However, for other primes, things are as simple as one could hope, based on (3).

Theorem 2. *For every prime p that divides none of $n_1,\ldots,n_k$, the Sylow p-subgroup (or p-primary component) of the critical group $K(K_{n_1}\times\cdots\times K_{n_k})$ has the following description:*

$$\mathrm{Syl}_p\, K(K_{n_1}\times\cdots\times K_{n_k}) \cong \bigoplus_{\emptyset\neq S\subseteq[k]} \mathrm{Syl}_p\left(\mathbb{Z}/N_S\mathbb{Z}\right)^{\prod_{i\in S}(n_i-1)}.$$

We also have the following result in general.

Theorem 3. *The critical group $K(K_{n_1}\times\cdots\times K_{n_k})$ has at most*

$$\frac{1}{2}\left(\prod_{i=1}^{k} n_i + \prod_{i=1}^{k}(n_i-2)\right) - 1$$

invariant factors.

In the special case where $n_1 = \cdots = n_k = 2$, so that $K_{n_1}\times\cdots\times K_{n_k} = (K_2)^n$ is the 1-skeleton of the k-dimensional cube, this critical group structure was studied by H. Bai [1]. He was able to show a result (Corollary 17 below) equivalent to the specialization of Theorem 2, and also that Theorem 3 is tight in this special case.

In the remainder of the paper, we prove these results. Although not strictly necessary for what follows, Section 2 reviews the results on Laplacian eigenvalues needed to compute the spanning tree numbers given by (1) and (3). Section 3 proves Theorem 1, while Section 4 proves Theorems 2 and 3. It turns out that both Theorem 2 and 3 follow from a somewhat surprising result (Theorem 13), which greatly simplifies the Laplacian presentation for the critical group. This result says that $L(K_{n_1}\times\cdots\times K_{n_k})$, which we know to have integer eigenvalues and hence be diagonalizable over $\mathbb{Q}$, is actually triangularizable over $\mathbb{Z}$ (that is, it is triangularizable by a *unimodular* change-of-basis). Note that this need not happen in general for Laplacian matrices, e.g. one can check that a path with 4 vertices has Laplacian matrix which cannot be triangularized even over $\mathbb{Q}$.

Our main tools will be the use of Smith normal form (see e.g. [8, Chapter 12]) or other diagonal forms for an integer matrix, which can be achieved by row and column operations that are invertible over $\mathbb{Z}$. Say that two matrices $A, B \in \mathbb{Z}^{m\times n}$ are *equivalent* (written $A \sim B$) if there exist matrices $P \in GL(m,\mathbb{Z}), Q \in GL(n,\mathbb{Z})$ such that $B = PAQ$. Equivalently, B is obtainable from A by a sequence of row and column operations in which one is allowed to

- permute rows or columns,
- scale any row or column by -1, or
- add any integer multiple of one row (resp. column) to another row (resp. column).

It is easily seen that $A \sim B$ implies $\mathrm{coker}A \cong \mathrm{coker}B$. Our methodology will often be to bring the Laplacian matrix $L(G)$ into a simpler (hopefully diagonal) form via row and column operations. The *Smith normal form* is a diagonal canonical form for our equivalence relation: every $A \in \mathbb{Z}^{m \times n}$ is equivalent to a unique diagonal matrix S (i.e. $S_{i,j} = 0$ for $i \neq j$) whose diagonal entries $s_1, \ldots, s_t$ (where $t = \min(m, n)$) have s_i dividing s_{i+1} for $i = 1, 2, \ldots, t-1$. We will also use the fact that the values s_i can also be interpreted as follows: for each i, the product $s_1 s_2 \cdots s_i$ is the greatest common divisor of all i-by-i minor subdeterminants of A.

2. Spanning tree numbers

In this section we recall how certain natural constructions on graphs behave well with respect to Laplacian eigenvalues, and use this to deduce the formulae (1) and (3). Given a graph G with n vertices, we will number the eigenvalues of its Laplacian matrix $L(G)$ in weakly decreasing order $(\lambda_1(G), \ldots, \lambda_n(G))$, so that $\lambda_n(G) = 0$.

For example, trivially the *disjoint union* $G_1 + G_2$ of two graphs has

$$L(G_1 + G_2) = L(G_1) \oplus L(G_2)$$

and hence its eigenvalues are the (multiset) union of the eigenvalues for each G_i.

It is also well-known [9, Lemma 13.1.3] that if G has no multiple edges, then its *complement graph* $\overline{G}$ (having same vertex set V, and edge set equal to the pairs in V which are *not* edges of G) satisfies

$$\lambda_i(\overline{G}) = n - \lambda_{n-i}(G) \qquad \text{for} \quad i = 1, 2, \ldots, n-1.$$

Note that the complete multipartite graph $K_{n_1,\ldots,n_k}$ is the complement of the disjoint union $K_{n_1} + \cdots + K_{n_k}$. Since $L(K_n)$ is well-known and easily seen to have eigenvalues $(\underbrace{n, \ldots, n}_{n-1}, 0)$, one concludes that $L(K_{n_1} + \cdots + K_{n_k})$ has eigenvalues

$$(\underbrace{n_1, \ldots, n_1}_{n_1 - 1}, \ldots, \underbrace{n_k, \ldots, n_k}_{n_k - 1}, \underbrace{0, \ldots, 0}_{k}).$$

Hence if we recall the notation $N := \sum_{i=1}^k n_i$, then $L(K_{n_1,\ldots,n_k})$ has eigenvalues

$$(\underbrace{N - n_1, \ldots, N - n_1}_{n_1 - 1}, \ldots, \underbrace{N - n_k, \ldots, N - n_k}_{n_k - 1}, \underbrace{N, \ldots, N}_{k-1}, 0).$$

Applying Kirchoff's Theorem Part (ii) and recalling $N_i := N - n_i$ then gives (1):

$$\begin{aligned} \kappa(K_{n_1,\ldots,n_k}) &= \frac{N^{k-1} \prod_{i=1}^k (N - n_i)^{n_i - 1}}{N} \\ &= N^{k-2} \prod_{i=1}^k N_i^{n_i - 1}. \end{aligned}$$

Another well-behaved operation is the *Cartesian product* of two graphs

$$G_1 = (V_1, E_1), \qquad G_2 = (V_2, E_2),$$

defined by

$$G_1 \times G_2 := (V, E) = (V_1 \times V_2,\ (V_1 \times E_2) \amalg (E_1 \times V_2)).$$

In other words, $(x_1, x_2), (y_1, y_2)$ spans is an edge of $G_1 \times G_2$ if either $x_1 = y_1$ and $\{x_2, y_2\}$ is an edge of G_2, or $x_2 = y_2$ and $\{x_1, y_1\}$ is an edge of G_1.

When one identifies the space $\mathbb{R}^{V_1 \times V_2}$ with the tensor product $\mathbb{R}^{V_1} \otimes \mathbb{R}^{V_2}$ in the obvious way, one can check that this identifies

$$L(G_1 \times G_2) = L(G_1) \otimes 1 + 1 \otimes L(G_2).$$

Consequently, a complete set of eigenvectors for $L(G_1 \times G_2)$ are obtained by tensoring any $\lambda^{(1)}$-eigenvector v_1 for $L(G_1)$ with any $\lambda^{(2)}$-eigenvector v_2 for $L(G_2)$ to obtain the $(\lambda^{(1)} + \lambda^{(2)})$-eigenvector $v_1 \otimes v_2$ for $L(G_1 \times G_2)$.

If one iterates this and uses the known eigenvalues for $L(K_n)$, one concludes that $L(K_{n_1} \times \cdots \times K_{n_k})$ has the eigenvalue $N_S := \sum_{i \in S} n_i$ with multiplicity $\prod_{i \in S}(n_i - 1)$ for each subset $S \subseteq [k]$. Noting that the 0 eigenvalue occurs exactly once for $S = \emptyset$, and that $K_{n_1} \times \cdots \times K_{n_k}$ has $\prod_{i=1}^{k} n_i$ vertices, Kirchoff's Theorem Part (ii) then immediately yields (3).

3. Complete multipartite graphs

We recall the statement of Theorem 1.

Theorem 1. *For $k > 2$ one has*

$$K(K_{n_1,\ldots,n_k}) \cong \bigoplus_{i=1}^{k} (\mathbb{Z}/N_i\mathbb{Z})^{n_i - 2} \ \oplus\ \mathbb{Z}/g\mathbb{Z} \ \oplus\ \mathbb{Z}/h\mathbb{Z} \ \oplus\ \bigoplus_{i=3}^{k} \mathbb{Z}/\sigma_i N\mathbb{Z}.$$

where $\sigma_1, \sigma_2, \ldots, \sigma_k$ are the invariant factors of $\oplus_{i=1}^{k} \mathbb{Z}/N_i\mathbb{Z}$, and

$$\begin{aligned} g &:= \gcd(k-1, N_1, N_2, \ldots, N_k) \\ h &:= \sigma_1\sigma_2/g \end{aligned}.$$

Proof. We start with a description of the Laplacian matrix $L(K_{n_1,\ldots,n_k})$, and then perform some easy row and column operations to simplify it, up to equivalence.

For the sake of notation, let I_m denote an $m \times m$ identity matrix, and $J_{m \times n}$ an $m \times n$ matrix having all entries equal to 1. Then it is easily seen that by ordering the vertices of $K_{n_1,\ldots,n_k}$ in their groups of size $n_1, n_2, \ldots, n_k$, one has

$$L(K_{n_1,\ldots,K_{n_k}}) = \begin{bmatrix} N_1 I_{n_1} & -J_{n_1 \times n_2} & \cdots & -J_{n_1 \times n_k} \\ -J_{n_1 \times n_2} & N_2 I_{n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -J_{n_{k-1} \times n_k} \\ -J_{n_k \times n_1} & \cdots & -J_{n_k \times n_{k-1}} & N_k I_{n_k} \end{bmatrix}.$$

In the first stage of reduction, one can perform row and column operations on $L(K_{n_1,\ldots,n_k})$ to make the on- and off-diagonal blocks look as follows:

$$\begin{bmatrix} N_i & 0 & 0 & \cdots & 0 & -n_iN_i \\ 0 & N_i & 0 & \cdots & 0 & 0 \\ 0 & 0 & N_i & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & N_i & 0 \\ 0 & 0 & \cdots & 0 & 0 & N_i \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

The matrices P and Q such that $P \cdot L(K_{n_1,\dots,K_{n_k}}) \cdot Q$ has this form are block diagonal $P = \mathrm{diag}(P_1, \dots, P_k), Q = \mathrm{diag}(Q_1, \dots, Q_k)$, where P_i and Q_i are $n_i \times n_i$ matrices given as:

$$P_i = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1-n_i \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & -1 \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}, \; Q_i = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 & 1-n_i \\ 0 & 1 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 1 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

Looking at the form of the blocks in (4), one sees that each integer N_i occurs as the unique non-zero entry in its row and column $n_i - 2$ times, and hence

$$\mathbb{Z} \oplus K(G) \cong \left(\bigoplus_{i=1}^{k} (\mathbb{Z}/N_i\mathbb{Z})^{\oplus(n_i-2)} \right) \oplus \mathrm{coker} L \tag{5}$$

where L is the $2k \times 2k$ matrix obtained by removing these rows and columns:

$$L = \begin{bmatrix} N_1 & -n_1N_1 & 0 & 0 & \cdots & & 0 & 0 \\ 0 & N_1 & -1 & 0 & & & -1 & 0 \\ 0 & 0 & N_2 & -n_2N_2 & \ddots & & \vdots & \\ -1 & 0 & 0 & N_2 & & & & \\ \vdots & & \ddots & & \ddots & & 0 & 0 \\ & & & & & & -1 & 0 \\ 0 & 0 & \cdots & & 0 & 0 & N_k & -n_kN_k \\ -1 & 0 & & & -1 & 0 & 0 & N_k \end{bmatrix}$$

Further reduction of L can be achieved by re-ordering rows and columns to obtain

$$L' := \begin{bmatrix} 0 & -1 & \cdots & -1 & N_1 & 0 & \cdots & 0 \\ -1 & 0 & \ddots & \vdots & 0 & N_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & 0 \\ -1 & \cdots & -1 & 0 & 0 & \cdots & 0 & N_k \\ N_1 & 0 & \cdots & 0 & -n_1N_1 & 0 & \cdots & 0 \\ 0 & N_2 & \ddots & \vdots & 0 & -n_2N_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & N_k & 0 & \cdots & 0 & -n_kN_k \end{bmatrix}.$$

One can further reduce L' to obtain

$$\begin{bmatrix} I_{k-1} & 0 \\ 0 & L'' \end{bmatrix}, \tag{6}$$

where

$$L'' := \begin{bmatrix} NN_1 & 0 & \cdots & 0 & N_1 \\ 0 & NN_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & N_{k-1} \\ 0 & \cdots & 0 & NN_k & N_k \\ N_1 & \cdots & N_{k-1} & N_k & k-1 \end{bmatrix}.$$

The $P, Q \in GL_{2k}(\mathbb{Z})$ achieving this reduction of L' are

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & & & \vdots \\ 0 & & \ddots & \ddots & \ddots & & & \vdots \\ -N_1 & \ddots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & -N_2 & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -N_k & \ddots & & \ddots & 1 \\ -1 & \cdots & -1 & -1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} -1 & \cdots & -1 & 0 & -N_2 & -N_3 & \cdots & -N_k & 2-k \\ 1 & 0 & \cdots & 0 & N_2 & 0 & \cdots & 0 & 1 \\ 0 & \ddots & \ddots & & \ddots & N_3 & \ddots & \vdots & \vdots \\ \vdots & \ddots & 1 & \ddots & & \ddots & \ddots & 0 & 1 \\ \vdots & & \ddots & -1 & \ddots & & \ddots & N_k & 1 \\ \vdots & & & \ddots & \ddots & \ddots & & \ddots & 0 \\ \vdots & & & & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 \end{bmatrix}$$

Since (6) implies that $\text{coker}L' \cong \text{coker}L''$, comparing (5) with Theorem 1 shows that it only remains to prove

$$\text{coker}L'' \cong \mathbb{Z} \oplus \mathbb{Z}/g\mathbb{Z} \; \oplus \; \mathbb{Z}/h\mathbb{Z} \; \oplus \; \bigoplus_{i=3}^{k} \mathbb{Z}/\sigma_i N\mathbb{Z}.$$

Note that the column vector $[1\ 1\ \ldots\ 1\ -N]^T$ spans the kernel of L'', which is therefore a $(k+1)\times(k+1)$ matrix of rank k. Also note that the gcd of the entries of L'' coincides with g. Bearing in mind that $\sigma_1\sigma_2\cdots\sigma_j$ is the gcd of all j-fold products $N_{i_1}\cdots N_{i_j}$, Theorem 1 follows from this claim:

for each $j = 2, 3, \ldots, k$, the gcd of all $j \times j$ minors of L'' equals the gcd of all products of the form $N^{j-2}N_{i_1} \cdots N_{i_j}$.

To this end, note that each $N^{j-2}N_{i_1} \cdots N_{i_j}$ does occur as such a $j \times j$ minor of L'', namely by choosing

$$\begin{aligned} \text{rows } &\{i_1, \ldots, i_{j-2}, i_{j-1}, k+1\} \\ \text{columns } &\{i_1, \ldots, i_{j-2}, i_j, k+1\} \end{aligned}$$

from L''.

Thus it only remains to show that each $j \times j$ minor of L'', say the one indexed by rows R and columns C, is divisible by the greatest common divisor of the products $N^{j-2}N_{i_1} \cdots N_{i_j}$. This is easily verified in cases, based on whether R, C contain $k+1$, and how they intersect:

- When neither R nor C contains $k+1$, one is looking at the $j \times j$ minors of the matrix $\mathrm{diag}(NN_1, \ldots, NN_k)$, which clearly are either 0 or of the form $\pm N^j N_{i_1} \cdots N_{i_j}$.
- When one of R or C contains $k+1$, but the other does not, one can check that one either obtains minors which are 0 or of the form $\pm N^{j-1}N_{i_1} \cdots N_{i_j}$.
- When both R, C contain $k+1$, there are three cases, depending upon how R, C intersect. If the symmetric difference $(R - C) \cup (C - R)$ contains more than two elements, then one can check that the minor vanishes. If the symmetric difference has cardinality 2, one can check that the minor will be of the form $\pm N^{j-2}N_{i_1} \cdots N_{i_j}$. If $R = C$, the minor is a determinant of the form

$$\begin{bmatrix} NN_{i_1} & 0 & \cdots & 0 & N_{i_1} \\ 0 & NN_{i_2} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & N_{i_{j-2}} \\ 0 & \cdots & 0 & NN_{i_{j-1}} & N_{i_{j-1}} \\ N_{i_1} & \cdots & N_{i_{j-2}} & N_{i_{j-1}} & k-1 \end{bmatrix}$$

  To evaluate this determinant, add to the last row $-\frac{1}{N}$ times each of the other rows. This does not affect the value of the the determinant, and creates an upper triangular matrix whose determinant is

$$\begin{aligned} (NN_{i_1}) \cdots (NN_{i_{j-1}}) &\left((k-1) - \frac{1}{N}(N_{i_1} + \cdots N_{i_{j-1}}) \right) \\ &= N^{j-2}N_{i_1} \cdots N_{i_{j-1}} \left((k-1)N - (N_{i_1} + \cdots N_{i_{j-1}}) \right) \\ &= N^{j-2}N_{i_1} \cdots N_{i_{j-1}} \left(\sum_{i=1}^{k} N_i - (N_{i_1} + \cdots N_{i_{j-1}}) \right) \\ &= N^{j-2}N_{i_1} \cdots N_{i_{j-1}} \sum_{m \notin \{i_1, \ldots, i_{j-1}\}} N_m \\ &= \sum_{m \notin \{i_1, \ldots, i_{j-1}\}} N^{j-2}N_{i_1} \cdots N_{i_{j-1}} N_m. \end{aligned}$$

  This last sum is divisible by the gcd of the $N^{j-2}N_{i_1} \cdots N_{i_j}$, so the proof of Theorem 1 is complete.

$\square$

Example 4.
As an illustration of Theorem 1, we compute $K(K_{4,4,6})$. In this situation,

$$k = 3,\ N = 4 + 4 + 6 = 14,\ N_1 = 10,\ N_2 = 10,\ N_3 = 8$$

and

$$\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \ \cong\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/40\mathbb{Z}.$$

Thus one has

$$\sigma_1 = 2,\ \sigma_2 = 10\ \sigma_3 = 40,\ g = \gcd(3-1, 8, 10, 10) = 2,\ h = \frac{2 \cdot 10}{2} = 10.$$

Consequently,

$$\begin{aligned} &K(K_{4,4,6}) \\ &\cong (\mathbb{Z}/10\mathbb{Z})^{4-2} \oplus (\mathbb{Z}/10\mathbb{Z})^{4-2} \oplus (\mathbb{Z}/8\mathbb{Z})^{6-2} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/(14 \cdot 40)\mathbb{Z} \\ &\cong (\mathbb{Z}/10\mathbb{Z})^5 \oplus (\mathbb{Z}/8\mathbb{Z})^4 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/560\mathbb{Z}. \end{aligned}$$

We close this section by noting a special case where the form of Theorem 1 simplifies, so that the arithmetic issues disappear.

Corollary 5. *Let $k \geq 2$. For the complete k-partite graph with k blocks of vertices of equal size m, one has*

$$\begin{aligned} K(K_{\underbrace{m, \ldots, m}_{k}}) \cong\ & \mathbb{Z}/(k-1)\mathbb{Z} \ \oplus\ (\mathbb{Z}/(k-1)m\mathbb{Z})^{k(m-2)} \\ & \oplus\ \mathbb{Z}/(k-1)m^2\mathbb{Z} \oplus \left(\mathbb{Z}/k(k-1)m^2\mathbb{Z}\right)^{k-2} \end{aligned}$$

Proof. The bipartite case $k = 2$ follows from Lorenzini's result (2). For $k \geq 3$, apply Theorem 1 after noting that when $n_1 = \cdots = n_k = m$, one has

$$\begin{aligned} N &= km \\ N_i &= (k-1)m \\ \sigma_i &= (k-1)m \\ g &= k-1 \\ h &= (k-1)m^2. \end{aligned}$$

$\square$

4. Cartesian products of complete graphs

The goal of this section is to prove Theorems 2 and 3, both of which will be deduced from Theorem 13. The latter provides a surprisingly simple triangularization of $L(K_{n_1} \times \cdots \times K_{n_k})$ by an integer (unimodular) change-of-basis.

Throughout this section, I will denote an identity matrix of varying size, whose size can be inferred from the context. Recall also our tacit assumption that $n_i \geq 2$ for each i. We introduce here the equivalence relation for unimodular change-of-bases, and note some of its trivial properties for later use.

Definition 6. *For $A, B \in \mathbb{Z}^{n \times n}$, say that $A \approx B$ if there exists a $P \in GL(n, \mathbb{Z})$ such that $B = PAP^{-1}$.*

Proposition 7. *Note that*

(i) *$A \approx B$ implies $A \sim B$.*
(ii) *$A_\alpha \approx B_\alpha$ for each α implies $\bigoplus_\alpha A_\alpha \approx \bigoplus_\alpha B_\alpha$* $\quad\square$.

The essence of Theorem 13 turns out to be repeated application of a simple fact (Proposition 9 (v) below) that generalizes the row-reductions of $L(K_n)$ used to compute the critical group $K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$ for the complete graph K_n.

Definition 8. *If M and T are square matrices of the same size, $H_n(M,T)$ is the $n \times n$ (block) matrix defined by*

$$H_n(M,T) := \begin{bmatrix} M-T & -T & \cdots & -T \\ -T & M-T & \ddots & \vdots \\ \vdots & \ddots & \ddots & -T \\ -T & \cdots & -T & M-T \end{bmatrix}.$$

Proposition 9. *The construction $H_n(M,T)$ has the following properties.*

(i) $H_n(M,T) + cI = H_n(M + cI, T)$ *for any scalar* c.
(ii) $A \approx B$ *implies* $H_n(A,I) \approx H_n(B,I)$.
(iii) $H_n(\bigoplus_\alpha A_\alpha, I) \approx \bigoplus_\alpha H_n(A_\alpha, I)$.
(iv) $L(K_n \times G) = H_n(L(G), I) + nI$
(v) $H_n(M,T) \approx M^{\oplus(n-2)} \oplus \begin{bmatrix} M & T \\ 0 & M-nT \end{bmatrix}$.

Proof. Assertions (i)-(iv) are completely straightforward from the definitions. For assertion (v), one can check that

$$P \cdot H_n(M,T) \cdot Q = M^{\oplus(n-2)} \oplus \begin{bmatrix} M & T \\ 0 & M-nT \end{bmatrix} \tag{7}$$

where

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -1 & 0 \\ 0 & \cdots & 0 & 1 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ -1 & \cdots & -1 & -1 & -1 & -1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ -1 & \cdots & -1 & -1 & 1-n & -1 \end{bmatrix}$$

interpreting the scalar entries $c = 0, 1, -1, 1-n$ above as scalar matrices cI of the same size as M and T. One can then either check directly that $Q = P^{-1}$, or simply set $M = I$ and $T = 0$ in (7) to reach the same conclusion. ☐

Repeatedly applying parts (iv) and (v) of Proposition 9 to $L(K_{n_1} \times \cdots \times K_{n_k})$ will give rise to upper triangular matrices of a certain form.

Definition 10. *For any integers $m, m_1, \ldots, m_r$, define*

$$M(m; m_1, \ldots, m_r) := \begin{cases} [m] & \text{if } r = 0 \\ \begin{bmatrix} M(m; m_1, \ldots, m_{r-1}) & I \\ 0 & M(m - m_r; m_1, \ldots, m_{r-1}) \end{bmatrix} & \text{otherwise.} \end{cases}$$

Example 11.

$$\begin{aligned}
M(m; -) &= [m]; \\
M(m; m_1) &= \begin{bmatrix} m & 1 \\ 0 & m - m_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & m(m - m_1) \end{bmatrix}; \\
M(m; m_1, m_2) &= \begin{bmatrix} m & 1 & 1 & 0 \\ 0 & m - m_1 & 0 & 1 \\ 0 & 0 & m - m_2 & 1 \\ 0 & 0 & 0 & m - m_1 - m_2 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} m(m - m_1) & 2m - m_1 - m_2 \\ 0 & (m - m_2)(m - m_1 - m_2) \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
a &= \gcd\left(m(m - m_1),\ 2m - m_1 - m_2,\ (m - m_2)(m - m_1 - m_2)\right) \\
b &= \frac{m(m - m_1)(m - m_2)(m - m_1 - m_2)}{a}.
\end{aligned}$$

The following proposition follows easily by induction on r.

Proposition 12. *For any scalar c, one has*

$$M(m + c; m_1, \ldots, m_r) = M(m; m_1, \ldots, m_r) + cI. \quad \square$$

We can now triangularize $L(K_{n_1} \times \cdots \times K_{n_k})$. For this purpose, given any subset $S = \{s_1 < \cdots < s_r\} \subseteq [k]$, define

$$M(N; n_S) := M(N; n_{s_1}, \ldots, n_{s_r}).$$

Theorem 13.

$$L(K_{n_1} \times \cdots \times K_{n_k}) \approx \bigoplus_{S \subseteq [k]} M(N; n_S)^{\oplus \prod_{i \in [k] - S} (n_i - 2)}.$$

Consequently,

$$\mathbb{Z} \oplus K(K_{n_1} \times \cdots \times K_{n_k}) \cong \bigoplus_{S \subseteq [k]} \operatorname{coker} M(N; n_S)^{\oplus \prod_{i \in [k] - S} (n_i - 2)}.$$

Proof. We use induction on k, with base case $k = 1$ easily checked to follow directly from Proposition 9 (iv) and (v).

In the inductive step, one has

$$\begin{aligned}
L(K_{n_1} \times \cdots \times K_{n_k}) &= L((K_{n_1} \times \cdots \times K_{n_{k-1}}) \times K_{n_k}) \\
&= H_{n_k}(L(K_{n_1} \times \cdots \times K_{n_{k-1}}) + n_k I, I)
\end{aligned}$$

by Proposition 9(iv). By induction on k, one has

$$L(K_{n_1} \times \cdots \times K_{n_{k-1}}) \approx \bigoplus_{S \subseteq [k-1]} M(N - n_k; n_S)^{\oplus \prod_{i \in [k-1]-S} (n_i - 2)}$$

and hence by Proposition 12

$$L(K_{n_1} \times \cdots \times K_{n_{k-1}}) + n_k I \approx \bigoplus_{S \subseteq [k-1]} M(N; n_S)^{\oplus \prod_{i \in [k-1]-S} (n_i - 2)}.$$

Therefore by Proposition 9(ii) and (iii),

$$\begin{aligned} L(K_{n_1} \times \cdots \times K_{n_k}) &\approx H_{n_k}\left(\bigoplus_{S \subseteq [k-1]} M(N; n_S)^{\oplus \prod_{i \in [k-1]-S} (n_i - 2)}, I \right) \\ &\approx \bigoplus_{S \subseteq [k-1]} H_{n_k}(M(N; n_S), I)^{\oplus \prod_{i \in [k-1]-S} (n_i - 2)}. \end{aligned}$$

After noting that Proposition 9(v) along with the definition of $M(N; n_S)$ shows

$$H_{n_k}(M(N; n_S), I) \approx M(N; n_S)^{\oplus (n_k - 2)} \oplus M(N; n_{S \cup \{k\}}),$$

one can then apply Proposition 7(ii) to give

$$\begin{aligned} L(K_{n_1} \times \cdots \times K_{n_k}) &\approx \bigoplus_{S \subseteq [k-1]} \Big(M(N; n_S)^{\oplus (n_k - 2) \prod_{i \in [k-1]-S} (n_i - 2)} \\ &\qquad \oplus M(N; n_{S \cup \{k\}})^{\oplus \prod_{i \in [k-1]-S} (n_i - 2)} \Big) \\ &= \bigoplus_{S \subseteq [k]} M(N; n_S)^{\oplus \prod_{i \in [k]-S} (n_i - 2)}. \end{aligned}$$

□

Theorems 2 and 3 require a tiny bit of further row-reduction on the matrices $M(N; n_S)$, provided by the following proposition.

Proposition 14. *If $r \geq 1$,*

$$M(m; m_1, \ldots, m_r) \sim I_{2^{r-1}} \oplus M(m; m_1, \ldots, m_{r-1}) M(m - m_r; m_1, \ldots, m_{r-1}).$$

Proof. Note that for any integer square matrices A, B of the same size, one has

$$\begin{bmatrix} A & I \\ 0 & B \end{bmatrix} \sim \begin{bmatrix} A & I \\ -AB & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & I \\ -AB & 0 \end{bmatrix} \sim I \oplus AB.$$

Apply this with

$$\begin{aligned} A &= M(m; m_1, \ldots, m_{r-1}) \\ B &= M(m - m_r; m_1, \ldots, m_{r-1}). \end{aligned}$$

□

Proof of Theorem 3.

From Proposition 14, the Smith normal form of $M(N; n_S) = M(N; n_{s_1}, \ldots, n_{s_r})$ contains at least 2^{r-1} ones, which is half its size. Hence the total number of ones in the Smith normal form of $L(K_{n_1} \times \cdots \times K_{n_k})$ is at least half the combined size of the matrices $M(N; n_S)$, where $\emptyset \neq S \subseteq [k]$. Since the total size of

$L\left(K_{n_1} \times \cdots \times K_{n_k}\right)$ is $\prod_{i=1}^{k} n_i$, and corresponding to $S = \emptyset$ there are $\prod_{i=1}^{k}(n_i - 2)$ one-by-one matrices of the form $M(N; -) = [N]$, this number of ones is at least

$$\frac{1}{2}\left(\prod_{i=1}^{k} n_i - \prod_{i=1}^{k}(n_i - 2)\right).$$

Since $L\left(K_{n_1} \times \cdots \times K_{n_k}\right)$ also has a 0 in its Smith normal form, the number of invariant factors is at most

$$\prod_{i=1}^{k} n_i - \frac{1}{2}\left(\prod_{i=1}^{k} n_i - \prod_{i=1}^{k}(n_i - 2)\right) - 1 = \frac{1}{2}\left(\prod_{i=1}^{k} n_i + \prod_{i=1}^{k}(n_i - 2)\right) - 1. \quad \square$$

As was mentioned in the Introduction, H. Bai [1, Theorem 1.1] showed that in the case $n_1 = \cdots = n_k = 2$, the number of invariant factors is exactly $2^{k-1} - 1$, so that the bound in Theorem 3 is tight in this case. Unfortunately, it is not tight in general, as shown by the following example.

Example 15.
Proposition 14 implies that

$$\begin{aligned}
&M(N; n_1, n_2, n_3) \\
&\sim I_{2^{r-1}} \oplus \\
&\begin{bmatrix}
N(N-n_3) & 2N - n_1 - n_3 & 2N - n_2 - n_3 & 2 \\
0 & (N-n_1)(N-n_1-n_3) & 0 & \alpha \\
0 & 0 & (N-n_2)(N-n_2-n_3) & \beta \\
0 & 0 & 0 & \gamma
\end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= 2N - 2n_1 - n_2 - n_3 \\
\beta &= 2N - n_1 - 2n_2 - n_3 \\
\gamma &= (N - n_1 - n_2)(N - n_1 - n_2 - n_3).
\end{aligned}$$

The presence of the entry 2 in the upper right corner of this last matrix has surprisingly subtle consequences for the structure of the critical group $L(K_{n_1} \times K_{n_2} \times K_{n_3})$. In particular, if at least one other entry in the above matrix is odd, then the gcd of the entries will be 1, so $K(K_{n_1} \times K_{n_2} \times K_{n_3})$ will have fewer invariant factors than the upper bound given in Theorem 3. For example, one can compute by brute force that

$$K(K_3 \times K_2 \times K_2) \cong \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/35\mathbb{Z} \oplus \mathbb{Z}/420\mathbb{Z}$$

which has 4 invariant factors, one fewer than the predicted upper bound:

$$\frac{1}{2}\left(\prod_{i=1}^{k} n_i + \prod_{i=1}^{k}(n_i - 2)\right) - 1 = \frac{1}{2}(3 \cdot 2 \cdot 2 + 1 \cdot 0 \cdot 0) - 1 = 5.$$

Also, this entry of 2 can cause the critical group structure to depart from naive guesses based on the eigenvalues of the Laplacian. For example, (1) predicts the tree number

$$\kappa(K_4 \times K_4 \times K_4) = \frac{4^3 \cdot 4^3 \cdot 4^3 \cdot 8^9 \cdot 8^9 \cdot 8^9 \cdot 12^{27}}{4 \cdot 4 \cdot 4},$$

and so one might naively hope that the critical group $K(K_4 \times K_4 \times K_4)$ is a direct sum of cyclic groups all of the form $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/12\mathbb{Z}$. This is contradicted by the presence of the 2 in the upper right of the matrix above, which via Theorem 13, shows that $\mathbb{Z}/2\mathbb{Z}$ will occur as a summand of the critical group.

We note that neither of these subtleties arises in $M(N; n_1, n_2)$; see Example 11. Consequently, if one desires, one can use Theorem 13 along with the calculations in Example 11 to write down an explicit expression for $K(K_{n_1} \times K_{n_2})$ as a direct sum of cyclic groups.

For the proof of Theorem 2, we require one further simple lemma about p-primary components $\mathrm{Syl}_p(G)$ of abelian groups G.

Lemma 16. *(cf.* [1, Proposition 3.1]*) Let G be an abelian group, and let α, β be two endomorphisms $G \to G$ satisfying $\beta - \alpha = m \cdot 1_G$ for some integer m.*

Then for any prime p that does not divide m, one has

$$\mathrm{Syl}_p(\operatorname{coker} \alpha\beta) \cong \mathrm{Syl}_p(\operatorname{coker} \alpha \oplus \operatorname{coker} \beta).$$

Proof. Consider the maps

$$\begin{array}{ccc} \operatorname{coker} \alpha\beta & \stackrel{\phi}{\to} & \operatorname{coker} \alpha \oplus \operatorname{coker} \beta \\ g + \operatorname{im} \alpha\beta & \mapsto & (g + \operatorname{im} \alpha, g + \operatorname{im} \beta) \end{array}$$

$$\begin{array}{ccc} \operatorname{coker} \alpha \oplus \operatorname{coker} \beta & \stackrel{\psi}{\to} & \operatorname{coker} \alpha\beta \\ (g_1 + \operatorname{im} \alpha, g_2 + \operatorname{im} \beta) & \mapsto & \beta g_1 - \alpha g_2 + \operatorname{im} \alpha\beta \end{array}$$

One can check that ϕ, ψ are well-defined, using the fact that $\beta - \alpha = m \cdot 1_G$ implies β, α commute, and hence

$$\operatorname{im} \alpha\beta = \operatorname{im} \beta\alpha \subseteq \operatorname{im} \alpha, \operatorname{im} \beta.$$

A straightforward calculation shows that the composite maps $\phi\psi, \psi\phi$ both coincide with scalar multiplications by m, and hence induce inverse isomorphisms on p-primary components. $\square$

Proof of Theorem 2.

Note that Propositions 12 and 14 imply that Lemma 16 can be applied with

$$\begin{aligned} \alpha &= M(N; n_{s_1}, \ldots, n_{s_{r-1}}) \\ \beta &= M(N - n_{s_r}; n_{s_1}, \ldots, n_{s_{r-1}}) \\ m &= n_r \end{aligned}$$

to show that for any prime p not dividing n_r, one has

$$\begin{aligned} &\mathrm{Syl}_p \mathrm{coker} M(N; n_{s_1}, \ldots, n_{s_r}) \cong \\ &\quad \mathrm{Syl}_p \left(\mathrm{coker} M(N; n_{s_1}, \ldots, n_{s_{r-1}}) \oplus \mathrm{coker} M(N - n_{s_r}; n_{s_1}, \ldots, n_{s_{r-1}}) \right) \end{aligned} \tag{8}$$

Note that $\mathrm{coker} M(m; -) \cong \mathbb{Z}/m\mathbb{Z}$, except in the case where $m = 0$ so that $\mathrm{coker} M(0; -) = \mathrm{coker}(0) = \mathbb{Z}$. Hence one can iterate (8) to conclude that if p divides none of $n_{s_1}, \ldots, n_{s_r}$, one has

$$\mathrm{Syl}_p(\mathrm{coker} M(N; n_S)) \cong \mathrm{Syl}_p \left(\bigoplus_{T \subseteq S, T \neq [k]} \mathbb{Z}/(N - N_T)\mathbb{Z} \right),$$

where $N_T := \sum_{i \in T} n_i$. Applying this to Theorem 13 gives

$$\begin{aligned}
\mathrm{Syl}_p(K(K_{n_1} \times \cdots \times K_{n_k})) &\cong \mathrm{Syl}_p\left(\bigoplus_{S \subseteq [k]} \mathrm{coker} M(N; n_S)^{\oplus \prod_{i \notin S}(n_i - 2)} \right) \\
&\cong \bigoplus_{S \subseteq [k]} \mathrm{Syl}_p\left(\bigoplus_{T \subseteq S, T \neq [k]} \mathbb{Z}/(N - N_T)\mathbb{Z}^{\oplus \prod_{i \notin S}(n_i - 2)} \right) \\
&\cong \bigoplus_{\emptyset \neq T' \subseteq [k]} \mathrm{Syl}_p\left(\bigoplus_{S' \subseteq T', T \neq [k]} \mathbb{Z}/N_{T'}\mathbb{Z}^{\oplus \prod_{i \in S'}(n_i - 2)} \right) \\
&\cong \bigoplus_{\emptyset \neq T' \subseteq [k]} \mathrm{Syl}_p\,(\mathbb{Z}/N_{T'}\mathbb{Z})^{\oplus \prod_{i \in T'}(n_i - 1)} \qquad \square
\end{aligned}$$

where the second-to-last isomorphism comes from interchanging the order of summations, along with a change of summation index so that S', T' are the complements within $[k]$ of the sets S, T, respectively. The last isomorphism uses the identity

$$\sum_{S' \subseteq T'} \prod_{i \in S'} (n_i - 2) = \prod_{i \in T'} (1 + (n_i - 2)) = \prod_{i \in T'} (n_i - 1).$$

Corollary 17. *(H. Bai,* [1, Theorem 1.2]*) For any odd prime p, the 1-skeleton of the k-cube $Q_k = \underbrace{K_2 \times \cdots \times K_2}_{k \text{ times}}$ has*

$$\mathrm{Syl}_p(K(Q_k)) \cong \mathrm{Syl}_p\left(\bigoplus_{\ell=1}^{k} \mathbb{Z}/\ell\mathbb{Z} \right)^{\oplus \binom{k}{\ell}}.$$

Proof. Let $n_1 = \cdots = n_k = 2$ and apply Theorem 2, giving

$$\begin{aligned}
\mathrm{Syl}_p(K(Q_k)) &\cong \mathrm{Syl}_p\left(\bigoplus_{\emptyset \neq S \subseteq [k]} \mathbb{Z}/N_S\mathbb{Z} \right)^{\oplus \prod_{i \in S}(2-1)} \\
&\cong \mathrm{Syl}_p\left(\bigoplus_{\emptyset \neq S \subseteq [k]} \mathbb{Z}/2|S|\mathbb{Z} \right) \\
&\cong \mathrm{Syl}_p\left(\bigoplus_{\ell=1}^{k} \mathbb{Z}/2\ell\mathbb{Z} \right)^{\oplus \binom{k}{\ell}} \\
&\cong \mathrm{Syl}_p\left(\bigoplus_{\ell=1}^{k} \mathbb{Z}/\ell\mathbb{Z} \right)^{\oplus \binom{k}{\ell}}.
\end{aligned}$$

$\square$

References

[1] H. Bai, On the critical group of the n-cube, preprint, 2002.
[2] N.L. Biggs, Algebraic graph theory, second edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1993.
[3] N.L. Biggs, Algebraic potential theory on graphs. *Bull. London Math. Soc.* **29** (1997), 641–682.

[4] N.L. Biggs, Chip-firing and the critical group of a graph. *J. Algebraic Combin.* **9** (1999), 25–45.
[5] R. Cori and D. Rossin, On the sandpile group of dual graphs. *European J. Combin.* **21** (2000), 447–459.
[6] D. Dhar, Self-organized critical state of sandpile automaton models. *Phys. Rev. Lett.* **64** (1990), 1613–1616.
[7] D. Dhar, P. Ruelle, S. Sen, and D.-N. Verma, Algebraic aspects of abelian sandpile models. *J. Phys. A* **28** (1995), 805–831.
[8] D. Dummit and R.M. Foote, Abstract algebra. Prentice Hall, Inc., Englewood Cliffs, NJ, 1991.
[9] C. Godsil, G. Royle, Algebraic graph theory. *Graduate Texts in Mathematics* **207**. Springer-Verlag, New York, 2001.
[10] R.P. Lewis, The number of spanning trees of a complete multipartite graph. *Discrete Math.* **197/198** (1999), 537–541.
[11] D.J. Lorenzini, A finite group attached to the Laplacian of a graph. *Discrete Math.* **91** (1991), 277–282.
[12] D. Wagner, The critical group of a directed graph, ArXiv preprint `math.CO/0010241`, 2003.

School of Mathematics, University of Minnesota, Minneapolis MN 55455
E-mail address: `jaco0457@tc.umn.edu`

Department of Mathematics, Harvey Mudd College, Claremont, CA 91711
E-mail address: `aniedermaier@hmc.edu`

School of Mathematics, University of Minnesota, Minneapolis MN 55455
E-mail address: `reiner@math.umn.edu`