NOTE ON A THEOREM OF ENG

VICTOR REINER

In [2, Theorem 1], O. Eng proved the following theorem. Let (W, S) be a finite Coxeter system, J a subset of S, W^J the distinguished left coset representatives for the parabolic subgroup W_J in W, and w_0 the longest element of W, which is always an involution.

Theorem 1.

$$\left[\sum_{w \in W^J} q^{l(w)}\right]_{q=-1} = \#\{w \in W^J : w_0 w W_J = w W_J\}.$$

Eng proved this by appeal to the classification of finite irreducible Coxeter systems, and asked for a case-free proof. We give such a proof via geometry in the case where (W, S) is a *Weyl group*.

Associated to a Weyl group (W, S) is a complex semisimple algebraic group G with a choice of a Borel subgroup B. The choice of $J \subseteq S$ gives rise to a parabolic subgroup P containing B, with the following well-known property (see e.g. [4]): the smooth complex projective variety G/P has a Schubert cell decomposition indexed by W^J (or the cosets W/W_J) which gives rise to the following expression for its Poincare polynomial:

$$\sum_i \beta_{2i}(G/P)\ q^i = \sum_{w \in W^J} q^{l(w)}$$

where β_k denotes the k^{th} Betti number. Since G/P is smooth and projective, it is a Kähler manifold, and consequently its *signature* or *index* $I(G/P)$ has the following expression (see [3, p. 125], [5, p. 208]) in terms of its Hodge numbers $h^{i,j}(G/P)$:

$$\sigma(G/P) = \sum_{i,j} (-1)^i h^{i,j}(G/P).$$

Since the homology of G/P is additively generated by the fundamental classes of the Schubert subvarieties, $h^{i,j}(G/P) = 0$ for $i \neq j$. Using this and the equation $\beta_k(G/P) = \sum_{i+j=k} h^{i,j}(G/P)$, we conclude that

$$I(G/P) = \sum_{i,j} (-1)^i h^{i,j}(G/P) = \sum_i (-1)^i \beta_{2i}(G/P) = \left[\sum_{w \in W^J} q^{l(w)}\right]_{q=-1}$$

On the other hand, Connolly and Nagano [1, p.38] calculated this index $I(G/P)$, using the fact that the Schubert class corresponding to the coset wW_J is dual under the intersection pairing to the Schubert class corresponding to $w_0 w W_J$. Hence the intersection form when written with respect to the Schubert cycle basis looks like the permutation matrix for the involution w_0 permuting the cosets $\{wW_J\}_{w \in W^J}$. Consequently, its signature $I(G/P)$ is the number of cosets fixed by this involution, which is the right-hand side of Eng's Theorem.

References

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School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: reiner@math.umn.edu