STANLEY'S SIMPLICIAL POSET CONJECTURE, AFTER M. MASUDA

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Abstract. M. Masuda recently provided the missing piece proving a conjecture of R.P. Stanley on the characterization of f-vectors for Gorenstein* simplicial posets. We propose a slight simplification of Masuda's proof.

Our main result, Theorem 2, was first proved by Masuda [Mas03], completing the missing step in a conjecture of Stanley characterizing the f-vectors of Gorenstein* simplicial posets. This note gives a simplified proof of it, using elementary methods. We begin with some background on simplicial posets; see Stanley [Sta91] for more detail and explanations for assertions not justified here.

A *simplicial poset* P is a finite poset with a minimal element $\hat{0}$ such that every interval $[\hat{0}, p]$ for $p \in P$ is a boolean algebra. We shall work instead with the associated regular cell complex $\Gamma = \Gamma(P)$, whose face poset is P. The (closed) faces of Γ are simplices that meet pairwise in subcomplexes of their boundaries [Sta91]. For simplicity, we identify each face G of Γ (denoted $G \in \Gamma$ in what follows) with the corresponding element of P.

Let $S = \Bbbk[x_G : G \in \Gamma]$ be a polynomial ring over a field $\Bbbk$ in indeterminates indexed by the faces of Γ. The *face ring* of Γ is the quotient $A_\Gamma = S/I_\Gamma$, where $I_\Gamma = \left\langle x_G x_{G'} - x_{G \wedge G'} \sum_F x_F \right\rangle$. Here the summation runs over the minimal faces F among those containing both G and G', and the *meet* $G \wedge G'$ is the largest face in Γ that is contained in both G and G'; its uniqueness (when the sum is nonzero) follows from the fact that G and G' lie in the boolean algebra $[\hat{0}, F]$ for any common upper bound F in the sum.

Write $f_i = f_i(\Gamma)$ for the number of faces of dimension i in Γ, and set $f_{-1} = 1$. Letting $d - 1$ be the dimension of Γ, one has an equivalent encoding of the *f-vector* $(f_{-1}, f_0, f_1, \ldots, f_{d-1})$ via the *h-vector* $(h_0, h_1, \ldots, h_d)$, whose entries are uniquely defined by the equation

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

Stanley [Sta91] completely characterized the possible f-vectors (or h-vectors) when Γ is *Cohen-Macaulay* over $\Bbbk$, and almost characterized the possible f-vectors when Γ satisfies the stronger condition of being *Gorenstein** over $\Bbbk$, that is, when Γ triangulates a $\Bbbk$-homology sphere. When Γ is Cohen-Macaulay, $(h_0, h_1, \ldots, h_d)$ can be interpreted as the Hilbert function of the quotient ring A_Γ/Θ, where Θ is the ideal generated by any linear system of parameters $\theta_1, \ldots, \theta_d$. Consequently $h_i \geq 0$ for all $i = 0, \ldots, d$, and this nonnegativity is sufficient to characterize these h-vectors [Sta91, Theorem 3.10]. The stronger Gorenstein* property further implies that A_Γ/Θ will be a Poincaré duality algebra, and hence $h_i = h_{d-i}$ for all i. This almost characterizes such h-vectors, as shown by the following theorem of Masuda, conjectured by Stanley.

Theorem 1. [Mas03, Corollary 1.2], [Sta91, Remark 5] *Let $(h_0, \ldots, h_d) \in \mathbb{N}^{d+1}$ satisfy $h_i = 1$ and $h_i = h_{d-i}$ for all i. Then there is a Gorenstein* simplicial poset P of rank d with $h_i = h_i(\Gamma(P))$ if and only if either*

- *$h_i > 0$ for all $i = 0, \ldots, d$, or else*
- *$\sum_{i=0}^{d} h_i$ is even.*

Stanley proved half of this theorem, by showing that the above conditions on $(h_0, \ldots, h_d)$ are sufficient to explicitly construct such a Gorenstein* simplicial poset. For the other half, since $f_{d-1} = \sum_{i=0}^{d} h_i$, it only remains to show that the condition $h_i = 0$ for some $i = 1, \ldots, d-1$ forces Γ to have an even number of $(d-1)$-dimensional faces (called *facets*). In fact, Masuda shows (see discussion following [Mas03, Eq. (5.1)]) that the assumption $h_i = 0$ implies the following stronger property.

Theorem 2. [Mas03] *If Γ is Gorenstein* and $h_i(\Gamma) = 0$ for some i strictly between 0 and d, then for every subset $V = \{v_1, \ldots, v_d\}$ of vertices, the number of facets in Γ having vertex set V is even.*

Proof. Since the quotient A_Γ/Θ is a Poincaré duality algebra, one has $h_d = 1$, that is, its degree d piece (its *socle*) is a 1-dimensional vector space over $\Bbbk$. Using the relations in I_Γ, one can see that the product $x_{v_1} \cdots x_{v_d}$ in A_Γ is congruent to the sum $\sum_F x_F$ as F ranges over all facets having V as their vertex set. If $h_i = 0$, then as observed in [Mas03], the image of $x_{v_1} \cdots x_{v_d}$ modulo Θ will be zero, because it has factors (such as $x_{v_1} \cdots x_{v_i}$) lying in the vanishing i^{th}-graded component $(A_\Gamma/\Theta)_i$. Hence the image of $\sum_F x_F$ modulo Θ must be zero in $(A_\Gamma/\Theta)_i$. Since this sum takes place in the socle of A_Γ/Θ, it is therefore enough to prove the following general claims:

(i) the images modulo Θ of the variables x_F for facets F containing the vertices $v_1, \ldots, v_d$ are all the same up to $\pm$ sign in the 1-dimensional socle, and
(ii) these images $\bar{x}_F$ are all nonzero.

Indeed, since the sum $\sum_F x_F$ modulo Θ is zero in the socle, (i) and (ii) would imply that there must be an even number of terms in the sum.

Claims (i) and (ii) follow from Proposition 5 and Corollary 7, below, since Γ being a $\Bbbk$-homology sphere implies that it is also a pseudomanifold. □

The rest of this note proves results implying Claims (i) and (ii). We do not assume that Γ is Gorenstein* anywhere in what follows, unless explicitly stated otherwise.

For a monomial $m = x_{G_1} \cdots x_{G_r}$ in A_Γ, define m to be *standard* if $G_1 \subseteq \cdots \subseteq G_r$ is a (weak) chain in Γ. The following is a slight strengthening of Stanley's observation [Sta91, Lemma 3.9] that A_Γ is integral over its subalgebra generated by $(A_\Gamma)_1$.

Lemma 3. *The variables x_G for $G \in \Gamma$ generate A_Γ as a* module *over the subalgebra of A_Γ generated by $(A_\Gamma)_1$. In fact, A_P is spanned $\Bbbk$-linearly by monomials mx_G in which m is a monomial in the variables x_v for the vertices v of G.*

Proof. Every element of A_Γ is a sum of standard monomials by [Sta91, Lemma 3.4]. Let $x_{G_1} \cdots x_{G_r}$ be a standard monomial, so $G_1 \subseteq \cdots \subseteq G_r$. For each face $G \in \Gamma$, denote by x^G the product of all variables x_v for vertices v of G. Now, for every index $i < r$, use the defining relations of A_Γ to replace x_{G_i} with $x^{G_i} - \sum x_{G'}$, the sum being over all minimal faces $G' \neq G_i$ containing the vertices of G_i. Observe that $x_{G'}x_{G_r} = 0$ in A_Γ for the faces G' in the sum, because no face contains both G_r and a face other than G_i with the same vertices as G_i. Hence $x_{G_1} \cdots x_{G_r}$ is equal to x_{G_r} times a monomial in the variables x_v for vertices v of G. □

Abusing notation slightly, let Θ denote a linear system of parameters $\theta_1, \ldots, \theta_d$ for A_Γ, and $\Bbbk[\Theta]$ the polynomial subalgebra of A_Γ that they generate. After choosing an ordering on the vertices of Γ, one can express Θ as a $d \times n$ matrix whose rows are $\theta_1, \ldots, \theta_d$. As observed by Masuda [Mas03, Lemma 3.1], given any facet F of Γ, one can compose the finite extension $\Bbbk[\Theta] \hookrightarrow A_\Gamma$ with the surjection $A_\Gamma \twoheadrightarrow A_{[\hat{0},F]}$ that sends all variables x_G for $G \not\subset F$ to zero. Because the composite $\Bbbk[\Theta] \to A_{[\hat{0},F]}$ must also be finite, and since $A_{[\hat{0},F]}$ is a polynomial ring on the variables $\{x_v\}_{v \in F}$, the $d \times d$ submatrix Θ_F of Θ with columns indexed by vertices in F has nonzero determinant $\det(\Theta_F)$. For any element $y \in A_\Gamma$, denote by $\bar{y}$ the image of y in the quotient ring A_Γ/Θ.

Lemma 4. *(cf.* [Ful93, §5.2 Lemma, p. 107]*) Suppose that Γ is* pure*, meaning that its facets all have dimension $d-1$. Let m be a monomial in the variables x_v for vertices v in a facet F. Then, for any face G of F, the image $\bar{m}\bar{x}_G$ of mx_G in A_Γ/Θ equals a sum of terms $\bar{m}'\bar{x}_G$ in which each monomial m' is a product of variables x_v for distinct vertices v outside of G.*

Proof. Invertibility of Θ_F implies that for any vertex v of F, the $\Bbbk$-span of $\theta_1, \ldots, \theta_d$ contains a linear form $\theta' = x_v + \sum_{w \notin F} c_w x_w$ for some constants $c_w \in \Bbbk$.

We first find a sum as in the lemma in which each m' is only squarefree (but may involve vertices of G), by induction on the sum of all exponents ≥ 2 on variables x_v in m. Suppose x_v^2 divides m, and write $m = x_v \ell$. Use the linear form θ' to write

$$\bar{m}\bar{x}_G = -\sum_{w \notin F} c_w \bar{x}_w \bar{\ell} \bar{x}_G.$$

If $\bar{x}_w \bar{\ell} \bar{x}_G \neq 0$, then some facet F' containing G also contains all vertices appearing in $x_w \ell$. By induction, $\bar{x}_w \bar{\ell} \bar{x}_G$ can be rewritten as desired.

Now assume that m is squarefree, and use a similar argument, this time by induction on the number of variables x_v dividing m for vertices $v \in G$. The fact that each w in θ' is not in F ensures that we re-create the squarefree hypothesis at each stage. □

Proposition 5. *If Γ is pure, the images in A_Γ/Θ of the variables x_G for faces $G \in \Gamma$ span $\Bbbk$-linearly. In particular, if Γ is Gorenstein*, then $\bar{x}_F \neq 0$ in A_Γ/Θ for some facet F.*

Proof. By Lemmas 3 and 4, every element of A_Γ can be expressed mod Θ as a sum of monomials of the form $m'x_{G'}$, where m' is a product of variables x_v for distinct vertices $v \notin G'$. But in A_Γ, such a monomial $m'x_{G'}$ equals the sum of the variables x_G as G runs over all faces minimal with respect to the property that they contain both G' and all vertices v for which x_v divides m'. □

Say that two facets F, F' in a pure $(d-1)$-dimensional complex *share a thin ridge* if their intersection is a $(d-2)$-face contained in no other $(d-1)$-faces.

Proposition 6. *When Γ is pure, any two facets F, F' sharing a thin ridge will have $\bar{x}_F = \pm\frac{\det(\Theta_{F'})}{\det(\Theta_F)}\bar{x}_{F'}$ in A_Γ/Θ.*

Proof. For convenience of notation, denote $\frac{\det(\Theta_{F'})}{\det(\Theta_F)}$ by δ. Let $G = F \wedge F'$, and let $v_F, v_{F'}$ be their corresponding vertices not in G. There are two possibilities for these vertices: either $v_F = v_{F'}$ or else $v_F \neq v_{F'}$.

When $v_F = v_{F'} = v$, we get $x_v x_G = x_F + x_{F'}$. The argument in Lemma 4 shows that, modulo Θ, we can write $\bar{x}_v$ as a linear combination of degree 1 elements $\bar{x}_v$ for vertices v not in F. Since G lies only in the facets F and F', this implies that $\bar{x}_F + \bar{x}_{F'} = \bar{x}_v\bar{x}_G = 0$. Hence in this case, $\bar{x}_F = -\bar{x}_{F'}$ (which equals $-\delta\bar{x}_{F'}$, since $\Theta_{F'} = \Theta_F$).

When $v_F \neq v_{F'}$, we get $x_{v_F}x_G = x_F$, and $x_{v_{F'}}x_G = x_{F'}$. After multiplying the matrix Θ on the left by Θ_F^{-1}, one of the rows gives (by Cramer's rule) a linear form $x_{v_F} \mp \delta x_{v_{F'}} + \sum c_v x_v$ in the ideal Θ, where the sum is over all vertices $v \neq v_{F'}$ that do not lie in F, and the c_v are constants in $\Bbbk$. Hence

$$\bar{x}_F \;=\; \bar{x}_{v_F}\bar{x}_G \;=\; \Big(\pm\delta x_{v_{F'}} - \sum c_v x_v\Big)\bar{x}_G \;=\; \pm\delta\bar{x}_{v_{F'}}\bar{x}_G \;=\; \pm\delta\bar{x}_{F'},$$

where the third equality holds because $x_v x_G = 0$ whenever v does not lie in $F \cup F'$. $\square$

Corollary 7. *Assume Γ is pure and F, F' are facets with the same set of vertices that are connected by a sequence $F = F_1, \ldots, F_r = F'$ in which F_j and F_{j-1} share a thin ridge for $2 \leq j \leq r$. Then $\bar{x}_F = \pm\bar{x}_{F'}$.*

Proof. Use Proposition 6: the product of ratios of determinants telescopes to ± 1. $\square$

Example 8. Let Γ be obtained by slitting a hollow tetrahedron along a single edge, and attaching a "pita pocket" of two triangles sewn together along two of their common edges. Suppose that the two pita triangles have vertices 123, while the slit tetrahedron has vertices 2345. Take a path from from the top pita triangle to the bottom pita triangle by traversing the facets with vertices 234, then 345, then 235, and finally back to the bottom pita triangle. Writing $[ijk] = \det(\Theta_F)$ when F has vertex set ijk (these are *Plücker coordinates*), the sequence of ratios of determinants is $[234]/[123]$, then $[345]/[234]$, then $-[235]/[345]$, and finally $[123]/[235]$. Note that the product of all these is -1, which is also the sign obtained by flipping from the top pita triangle to the bottom one along one of the two codimension 1 faces they share.

References

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