AN OLD, BUT CUTE, CATALAN PROOF

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A few years ago, my colleague Bill Messing [2] suggested to me a cute proof that I'd never seen, and rather liked, of the Catalan enumeration. After consulting the experts on the `domino` list-server about the history, several people (David Callan, Ira Gessel, Greg Kuperberg, Gilles Schaeffer) pointed me to the references by Dorrie [1, pp. 23-24] and by Rémy [3], which both give the same proof. Dorrie credits the proof idea to Rodrigues [4] in 1838.

Theorem 0.1. *Let c_n be the number of nonassociative parenthesizations of a product of $n+1$ symbols $a_1 a_2 \cdots a_{n+1}$. Then*

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Consider the set P_n of nonassociative parenthesizations of a product of *any permutation* $a_{\sigma_1} \cdots a_{\sigma_{n+1}}$ of the same $n+1$ symbols. Clearly the cardinality p_n of P_n satisfies $p_n = (n+1)! c_n$.

Consider the map from $f : P_n \to P_{n-1}$ that erases the symbol a_{n+1} along with the smallest parenthesis pair in which it lies. We claim that every fiber $f^{-1}(\alpha)$ of this map has the same cardinality, namely $2 \cdot (2n-1)$:

- There are $2n-1$ possible factors of α which can get multiplied first with the extra letter a_{n+1} when it is added in. These factors correspond to the $2n-1$ vertices in the rooted binary tree having leaves labelled $a_1, \ldots, a_n$ which is associated with the parenthesization α.
- After choosing this factor in $2n-1$ ways, one pins down the element of the fiber $f^{-1}(\alpha)$ by making one more binary choice of whether a_{n+1} is multiplied before or after that factor.

Hence the cardinality p_n of $|P_n|$ satisfies

$$\begin{aligned} p_n &= 2(2n-1)p_{n-1} \\ p_0 &= 1 \end{aligned} \tag{1}$$

which shows that

$$p_n = 2n(2n-1)\cdots(n+2)(n+1) = \frac{2n!}{n!}$$

and hence

$$c_n = \frac{1}{(n+1)!}p_n = \frac{1}{(n+1)!}\frac{2n!}{n!} = \frac{1}{n+1}\binom{2n}{n}$$

as desired. □

Note that one can reinterpret the crucial recursion (1) as follows. We have shown

$$(n+1)! c_n = 2(2n-1) \cdot n! c_{n-1}$$

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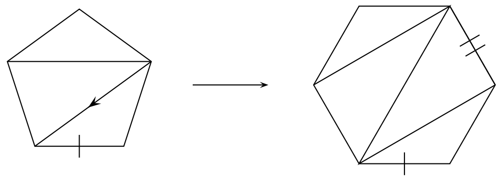
or equivalently, after dividing by $n!$,

$$(n+1)c_n = 2(2n-1) \cdot c_{n-1}. \tag{2}$$

To argue this somewhat more directly (that is, with less labelling, and more symmetry), Greg Kuperberg suggests the following argument, in which c_n is interpreted as the number of triangulations of a convex $(n+2)$-gon, and the two sides of (2) are interpreted as counting two different kinds of markings of triangulations.

First choose a base side of both the $(n+1)$-gon and the $(n+2)$-gon. Mark the $(n+1)$-gon by choosing any edge, either a diagonal or any side including the base side, and orienting it. There are $2(2n-1)$ ways to do this. Mark the $(n+2)$-gon by choosing a side other than the base side. There are $n+1$ ways to do this.

There is a bijection between these two sets of marked triangulations: starting with the marked $(n+1)$-gon triangulation, open the oriented edge to a triangle that points in the direction that the edge points. The new side that you make is the marked side of the $(n+2)$-gon; see the figure below for an example with $n=4$. This proves (2).



References

[1] H. Dörrie, 100 great problems of elementary mathematics. Their history and solution. Dover Publications, Inc., New York, 1982.

[2] W. Messing, personal communication, early 2000's.

[3] J.-L. Rémy, Un procédure itératif de dénombrement d'arbres binaires et son application à leur génération aléatoire. *RAIRO Inform. Theor.* **19** ,1985, 179–195.

[4] Rodrigues, *Journal de Mathématiques* **3** (1838).

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