NOTE ON THE PFAFFIAN MATRIX-TREE THEOREM

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Kirchoff's celebrated Matrix-Tree Theorem gives a determinant counting spanning trees in a graph. It has at least three different well-known proofs: one via the Binet-Cauchy Theorem (see e.g. [5, §2.2]), one via a deletion-contraction induction (see e.g. [2, §13.2]), and one due to Chaiken [1] via a sign-reversing involution. Recently Masbaum and Vaintrob [3] proved a beautiful analogue of the Matrix-Tree Theorem giving a Pfaffian that enumerates spanning trees in a 3-uniform hypergraph. Their proof used the analogue of deletion-contraction induction. The purpose of this note is to give a simple proof via a sign-reversing involution, analogous to [1]. We find this proof illuminating, because it explains the cancellations involved directly.

We recall the main result of [3], along with some background and notation. Let $[n] := \{1, 2, \ldots, n\}$ be the vertex set for the complete 3-uniform hypergraph $K_n^{(3)}$, which has edge set $\binom{[n]}{3}$. Say that a subhypergraph $T = ([n], E(T))$ where $E(T) \subset \binom{[n]}{3}$ is a *spanning tree* if the associated bipartite graph $B(T)$ on vertex set $E(T) \sqcup [n]$, having an edge from $\{i, j, k\}$ in $E(T)$ to each of the vertices i, j, k in $[n]$, is a spanning tree for this bipartite graph. An example is depicted in Figure 1. It is easy to see by "leaf induction" that $K_n^{(3)}$ will have no spanning trees unless n is odd.

Let $\{y_{ijk}\}_{1 \leq i,j,k \leq n}$ be a set of variables which are *skew-symmetric* in the ordered indices (i, j, k), that is, $y_{\sigma(i)\sigma(j)\sigma(k)} = \mathrm{sign}(\sigma) y_{ijk}$ for all permutations $\sigma \in \mathfrak{S}_3$,

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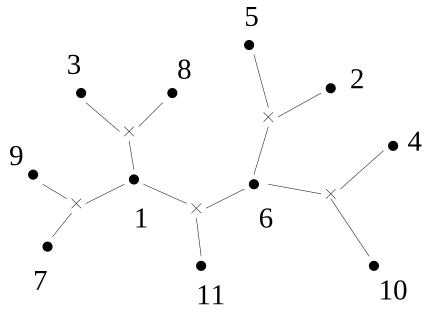


FIGURE 1. The spanning tree T in $K_{11}^{(3)}$ with edge set $E(T) = \{\{1, 3, 8\}, \{1, 7, 9\}, \{1, 6, 11\}, \{2, 5, 6\}, \{4, 6, 10\}\}$, depicted via an embedding of the associated bipartite graph $B(T)$.

and in particular $y_{iik} = 0$. To each spanning tree T associate a signed monomial y_T in these variables as follows. Embed the bipartite graph $B(T)$ in the plane, and walk *clockwise* around its perimeter, recording the vertices from $[n]$ in the order that they are first encountered to give a permutation $\sigma = \sigma_1 \dots \sigma_n$. Then $y_T := \mathrm{sign}(\pi) \prod_{\{i,j,k\} \in E(T)} y_{ijk}$, where we assume that for each i, j, k in $E(T)$, the ordering (i, j, k) used in y_{ijk} is clockwise around the vertex of $B(T)$ corresponding to $\{i, j, k\}$ in the planar embedding. An example is shown in Figure 1, where if one starts walking clockwise from the vertex 1, one produces $\pi = 1\ 7\ 9\ 3\ 8\ 6\ 5\ 2\ 4\ 10\ 11 = (2758)(394)$, having $\mathrm{sign}(\pi) = -1$, and yielding $y_T = -y_{1,3,8}\, y_{1,7,9}\, y_{1,6,11}\, y_{2,6,5}\, y_{4,10,6}$. It is not hard to see that the definition of y_T is independent of the various choices involved (the choice of planar embedding of $B(T)$, the starting point for walking around, the clockwise orderings around vertices).

Recall the definition of the Pfaffian: any $N \times N$ skew-symmetric matrix $B = (b_{ij})_{1 \le i,j \le N}$ has $\det(B) = 0$ if N is odd, and when N is even, $\det(B) = \mathrm{Pf}(B)^2$ where $\mathrm{Pf}(B) \in \mathbb{Z}[b_{ij}]$ is the *Pfaffian* polynomial, uniquely defined up to $\pm$ sign. It can be defined by the following formula:

$$\mathrm{Pf}(B) = \sum_{\substack{\text{perfect matchings} \\ m \text{ of } [N]}} (-1)^{\mathrm{cross}(m)} \prod_{\{i<j\} \in E(m)} b_{ij} \tag{1}$$

where $\mathrm{cross}(m)$ is the *crossing number* of m given by

$$\mathrm{cross}(m) := \#\{i < j < k < l : \{i, k\}, \{j, l\} \in E(m)\}.$$

We will need the following easily verified fact about crossing numbers (see e.g. [4, Lemma 2.1]).

Lemma 1. *Given a perfect matching m of $[n]$ in which the vertices $i, i+1$ are not matched, if $s_i m$ denotes the perfect matching obtained from m by swapping their labels, then $\mathrm{cross}(s_i m)$ and $\mathrm{cross}(m)$ differ by 1.* □

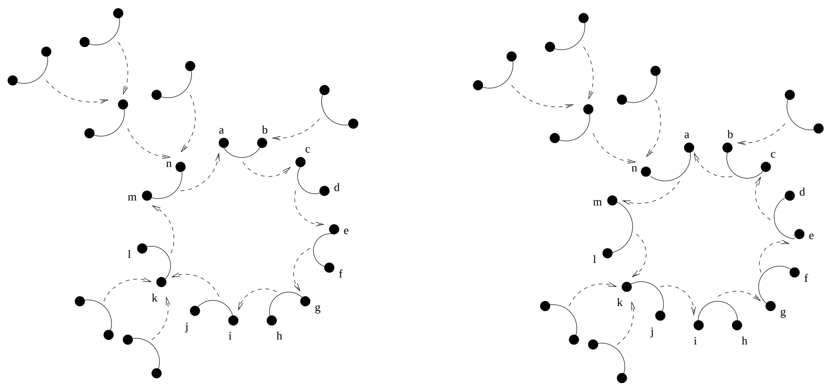
Define an $n \times n$ skew-symmetric matrix $A = (a_{ij})_{1 \le i,j \le n}$ by $a_{ij} = \sum_{k=1}^{n} y_{ijk}$, and let $A^{(i)}$ denote the skew-symmetric matrix obtained from A by removing both the i^{th} row and column. The Pfaffian Matrix-Tree Theorem of Masbaum and Vaintrob can then be stated as follows.

Theorem 2. [3, Theorem 5.1] *Let A be the matrix defined above. For odd n, and $1 \le i \le n$,*

$$\mathrm{Pf}(A^{(i)}) = (-1)^{i-1} \sum_{\substack{\textit{spanning trees} \\ T \textit{ in } K_n^{(3)}}} y_T.$$

Proof. For convenience we may assume that $i = n$, since simultaneously swapping rows i, j and columns i, j multiplies the Pfaffian by -1 (see e.g. [4, Lemma 2.3(b)]). We expand the Pfaffian into a signed sum of monomials via (1):

$$\begin{aligned}
\mathrm{Pf}(A^{(n)}) &= \sum_{\substack{\text{perfect matchings} \\ m \text{ of } [n-1]}} (-1)^{\mathrm{cross}(m)} \prod_{\{i<j\} \in E(m)} \left(\sum_{k=1}^{n} y_{ijk} \right) \\
&= \sum_{\substack{(m,f):\ m \text{ a perfect matching of } [n-1], \\ f: E(m) \to [n]}} (-1)^{\mathrm{cross}(m)} \prod_{\{i<j\} \in E(m)} y_{i,j,f(\{i,j\})}
\end{aligned}$$



(a) (b)

FIGURE 2. Local structure of two bad pairs (m, f) and $\iota(m, f)$ having $y_{(m,f)} = -y_{\iota(m,f)}$ which will cancel in the left-hand side of (2). Matching edges $E(m)$ are depicted as solid arcs, and the functions $f : E(m) \to [n]$ are depicted by dashed arrows.

Given a pair (m, f) as above, define $y_{(m,f)} = (-1)^{\mathrm{cross}(m)} \prod_{\{i<j\} \in E(m)} y_{i,j,f(\{i,j\})}$. Then we must show

$$\sum_{(m,f) \text{ as above}} y_{(m,f)} = \sum_{\substack{\text{spanning trees} \\ T \text{ in } K_n^{(3)}}} y_T. \tag{2}$$

The method of sign-reversing involution is to

- first classify the pairs (m, f) as *good* or *bad* in some fashion,
- produce a fixed-point free involution $\iota : (m, f) \mapsto \iota(m, f)$ on the set of all bad pairs, with the property that $y_{(m,f)} = -y_{\iota(m,f)}$, so that they cancel in pairs on the left-hand side of (2),
- show that there is a bijection $(m, f) \mapsto T(m, f)$ from the set of good pairs to the set of spanning trees in $K_n^{(3)}$, with the property that $y_{(m,f)} = y_{T(m,f)}$.

To this end, for any pair (m, f) we wish to associate a new function $\tilde{f}$ mapping $E(m) \cup \{n\}$ into itself:

$$\tilde{f}(n) = n$$

$$\tilde{f}(\{i, j\}) = \begin{cases} \{k, l\} & \text{if } f(\{i, j\}) \in \{k, l\} \\ n & \text{if } f(\{i, j\}) = n. \end{cases}$$

We will say that (m, f) is *bad* if there exists some edge $\{i, j\}$ which is a *recurrent element* for $\tilde{f}$, that is, there exists some $M > 0$ having $\tilde{f}^M(\{i, j\}) = \{i, j\}$. Say (m, f) is *good* otherwise, that is, if n is the *only* recurrent element for $\tilde{f}$.

When (m, f) is bad, there must exist at least one directed cycle as depicted in Figure 2(a), which for definiteness we will assume contains the smallest labelled vertex of any such cycle. Define the involution without fixed points $\iota : (m, f) \mapsto \iota(m, f)$ on the set of bad pairs by replacing this cycle with the one depicted in Figure 2(b) to obtain $\iota(m, f)$. Note that $y_{(m,f)} = \pm y_{\iota(m,f)}$; it is our immediate goal to show that one always has $y_{(m,f)} = -y_{\iota(m,f)}$. This is easy to check directly in the very special case where the vertex labels going around the cycle in Figures 2(a) or (b) are $1, 2, 3, \ldots$ in clockwise order.

For the general case, one can reduce to this special case by performing a sequence of adjacent transpositions $s_i = (i\ i+1)$ to the labels of the vertices in both (m, f) and $\iota(m, f)$, and then one must check that $\frac{y_{(m,f)}}{y_{\iota(m,f)}} = \frac{y_{(s_i m, s_i f)}}{y_{\iota(s_i m, s_i f)}}$. This is checked in two cases, depending on whether $i, i+1$ are matched in m or not, using Lemma 1 in the latter case. The details are straightforward, and left to the reader.

When (m, f) is good, the fact that n is the only recurrent element for $\tilde{f}$ implies that the digraph associated to $\tilde{f}$ is a tree (in the usual graph-theoretic sense) directed toward n. It is then easy to see that the 3-uniform hypergraph $T(m, f)$ having

$$E(T(m,f)) := \{\{i, j, f(\{i,j\})\} : \{i,j\} \in E(m)\}$$

is a spanning tree in $K_n^{(3)}$, and $y_{(m,f)} = \pm y_{T(m,f)}$. Conversely, for each spanning tree T in $K_n^{(3)}$, there is a *unique* good pair (m, f) such that $T(m, f) = T$: for every edge in $E(T)$ containing n, say $\{i, j, n\}$, the pair $\{i, j\}$ must form a matching edge in $E(m)$, with $f(\{i, j\}) = n$, and one reconstructs the rest of (m, f) similarly, by searching through T away from the root vertex n.

It remains to show that $y_{(m,f)} = +y_{T(m,f)}$. This is easy to check in the special case where the T has a planar embedding such that the permutation π obtained from reading clockwise around the perimeter is the identity permutation. For the general case, one can again reduce to this special case by performing a sequence of adjacent transpositions $s_i = (i\ i+1)$ to the labels of the vertices in both (m, f) and $T(m, f)$. One must check that $\frac{y_{(m,f)}}{y_{T(m,f)}} = \frac{y_{(s_i m, s_i f)}}{y_{T(s_i m, s_i f)}}$. As before, this is a straightforward verification involving two cases, depending on whether $i, i+1$ are matched in m or not, using Lemma 1 in the latter case. $\square$

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