LOCAL COHOMOLOGY OF STANLEY-REISNER RINGS WITH SUPPORTS IN GENERAL MONOMIAL IDEALS

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1. Introduction

The local cohomology module $H^i_I(R) := \varinjlim \operatorname{Ext}^i_R(R/I^j, R)$ of a noetherian commutative ring R with support in a (non-maximal) ideal I is still a very mysterious object. Recently there have been several instances where more explicit information on local cohomology modules was obtained in special cases. One such case is a combinatorial approach to the local cohomology of affine semigroup rings (e.g., polynomial rings) and at their monomial ideals; see [12, 14, 16, 18, 20, 21]. In this paper, we construct an analogous theory for a Stanley-Reisner ring $k[\Delta]$ using combinatorial topology.

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. Let $\Delta \subseteq 2^{\{1,\dots,n\}}$ be a simplicial complex, and Σ a subcomplex of Δ. Then the Stanley-Reisner ideal I_Δ of Δ is contained in the Stanley-Reisner ideal I_Σ of Σ. We denote the image of I_Σ in $k[\Delta] = S/I_\Delta$ by J. We will study the local cohomology module $H^i_J(k[\Delta])$, making use of its natural $\mathbb{Z}^n$-grading. Hochster gave a famous formula for the $\mathbb{Z}^n$-graded Hilbert function of $H^i_{\mathfrak{m}}(k[\Delta])$, and Terai [18] and Mustaţă [16] gave a similar formula for $H^i_{I_\Sigma}(S)$. Our result, Theorem 3.2, is a generalization of both formulas.

Most results of Mustaţă [16] for $H^i_{I_\Sigma}(S)$ remain true for Gorenstein $k[\Delta]$. In Theorem 4.9, we prove that

$$\operatorname{Ext}^i_{k[\Delta]}(\, k[\Delta]/J,\, k[\Delta]\,) \subseteq \operatorname{Ext}^i_{k[\Delta]}(\, k[\Delta]/J^{[2]},\, k[\Delta]\,) \subseteq \cdots \subseteq H^i_J(k[\Delta]),$$

and these inclusions are well-regulated if we consider the $\mathbb{Z}^n$-grading, where $J^{[j]}$ is the j^{th} "Frobenius power" of J (if $k[\Delta]$ is not Gorenstein, a somewhat weaker result holds). A general commutative ring R and an ideal I usually do not have such a simple property.

There is one big difference between [16] and our case. In [16], it is proved that $\operatorname{Ext}^i_S(S/J, S)$ "determines" $H^i_J(S)$. In our case, $\operatorname{Ext}^i_{k[\Delta]}(\, k[\Delta]/J^{[2]},\, k[\Delta]\,)$ determines $H^i_J(k[\Delta])$, but $\operatorname{Ext}^i_{k[\Delta]}(\, k[\Delta]/J,\, k[\Delta]\,)$ cannot. The reason is that $H^i_J(S)$ is a *straight* module (after suitable degree shifting), but $H^i_J(k[\Delta])$ is only *quasi-straight*; see Section 2.

The paper is structured as follows. In Section 2, we prove some basic properties of $H^i_J(k[\Delta])$. In Section 3, we give a topological formula (Theorem 3.2) for the $\mathbb{Z}^n$-graded Hilbert function of $H^i_J(k[\Delta])$. As an application, we give a topological proof and interpretation of the Lichtenbaum-Hartshorne vanishing theorem for Stanley-Reisner rings (Theorem 3.5). In Section 4, we assume that $k[\Delta]$ is Cohen-Macaulay and study $H^i_J(\omega_{k[\Delta]})$ for the canonical module $\omega_{k[\Delta]}$ of $k[\Delta]$, which is in some sense much easier to treat than $H^i_J(k[\Delta])$. In Section 5, some of the results from Section 4 are used to give a topological formula (Theorem 5.1) for $H^i_J(\omega_{k[\Delta]})$. Of course, when $k[\Delta]$ is Gorenstein, so that $k[\Delta]$ and $\omega_{k[\Delta]}$ are isomorphic up to a shift in grading, this formula is equivalent to that of Section 3. But the equivalence is not trivial, and derived from a variant of Alexander duality (Lemma 6.8). The final section is an appendix, where we collect the tools and the terminology from combinatorial topology that are used in Sections 3 and 5.

We hope that our results will motivate further study on $H^i_J(k[\Delta])$ and provide good examples for the general theory of local cohomology modules.

2. Basic properties

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k. Consider an $\mathbb{N}^n$-grading $S = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S_{\mathbf{a}} = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} k\, x^{\mathbf{a}}$, where $x^{\mathbf{a}} = \prod_{i=1}^n x_i^{a_i}$ is the monomial with exponent vector $\mathbf{a} = (a_1, \dots, a_n)$. We denote the graded maximal ideal $(x_1, \dots, x_n)$ by $\mathfrak{m}$.

Let M be a $\mathbb{Z}^n$-graded S-module, that is, $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}}$ as a k-vector space and $S_{\mathbf{b}}\, M_{\mathbf{a}} \subseteq M_{\mathbf{a}+\mathbf{b}}$ for all $\mathbf{a} \in \mathbb{Z}^n$ and $\mathbf{b} \in \mathbb{N}^n$. Then $M(\mathbf{a})$ denotes the shifted module with $M(\mathbf{a})_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$. We denote the category of $\mathbb{Z}^n$-graded S-modules and their $\mathbb{Z}^n$-graded S-homomorphisms by ${}^*\mathbf{Mod}$. Here, we say $f : M \to N$ is $\mathbb{Z}^n$-graded, if $f(M_{\mathbf{a}}) \subseteq N_{\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{Z}^n$. We denote the category of (usual) S-modules and their S-homomorphisms by $\mathbf{Mod}$.

In this paper, $\mathrm{Hom}_S(M, N)$ always means $\mathrm{Hom}_{\mathbf{Mod}}(M, N)$, even if M and N belong to a subcategory of $\mathbf{Mod}$. If $M, N \in {}^*\mathbf{Mod}$ and M is finitely generated, $\mathrm{Hom}_S(M, N)$ (more generally, $\mathrm{Ext}^i_S(M, N)$) has a natural $\mathbb{Z}^n$-grading with $\mathrm{Hom}_S(M, N)_{\mathbf{a}} = \mathrm{Hom}_{{}^*\mathbf{Mod}}(M, N(\mathbf{a}))$. See [8]. Similarly, if J is a monomial ideal, a local cohomology module $H^i_J(M)$ for M in ${}^*\mathbf{Mod}$ is also $\mathbb{Z}^n$-graded.

For $\mathbf{a} \in \mathbb{Z}^n$, define the following support subsets of $[n] := \{1, 2, \dots, n\}$:

$$\begin{aligned} \mathrm{supp}_+(\mathbf{a}) &:= \{i \mid a_i > 0\} \\ \mathrm{supp}_-(\mathbf{a}) &:= \{i \mid a_i < 0\} \\ \mathrm{supp}(\mathbf{a}) &:= \{i \mid a_i \neq 0\} \end{aligned}$$

We extend this terminology to Laurent monomials $x^{\mathbf{a}}$ in the obvious way; that is, $\mathrm{supp}(x^{\mathbf{a}}) := \mathrm{supp}(\mathbf{a})$, and similarly for $\mathrm{supp}_+, \mathrm{supp}_-$. We use $\mathbf{a} + \{i\}$ to denote $\mathbf{a} + e_i$ for the i-th unit vector $e_i \in \mathbb{Z}^n$. When $\mathbf{a} \in \mathbb{Z}^n$ satisfies $a_i = 0, 1$ for all i, we sometimes identify $\mathbf{a}$ with $F = \mathrm{supp}(\mathbf{a})$. For example, we write M_F for $M_{\mathbf{a}}$.

Definition 2.1 ([19])**.** A $\mathbb{Z}^n$-graded S-module $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}}$ is called *squarefree*, if the following conditions are satisfied.

(a) M is finitely generated and $\mathbb{N}^n$-graded (i.e., $M_{\mathbf{a}} = 0$ if $\mathbf{a} \notin \mathbb{N}^n$).
(b) The multiplication map $M_{\mathbf{a}} \ni y \mapsto x_i y \in M_{\mathbf{a}+\{i\}}$ is bijective for all $i \in [n]$ and all $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ with $a_i \neq 0$.

Examples 2.2. A free module $S(-F)$ is squarefree for all $F \subseteq [n]$. In particular, S itself and the $\mathbb{Z}^n$-graded canonical module $\omega_S = S(-\mathbf{1})$ are squarefree. Here $-\mathbf{1} = (-1, \dots, -1)$.

Let $\Delta \subseteq 2^{[n]}$ be a simplicial complex (i.e., if $F \in \Delta$ and $G \subset F$ then $G \in \Delta$). The *Stanley-Reisner ideal* of Δ is the squarefree monomial ideal $I_\Delta := (x^F \mid F \notin \Delta)$

of S. Any squarefree monomial ideal is I_Δ for some Δ. We say $k[\Delta] = S/I_\Delta$ is the *Stanley-Reisner ring* of Δ. Stanley-Reisner ideals and rings are always squarefree modules. See [19, 20] for further information.

Let $\Sigma \subseteq \Delta \subseteq 2^{[n]}$ be two simplicial complexes. Then we have Stanley-Reisner ideals $I_\Delta \subseteq I_\Sigma$. We always assume that Σ is not the void simplicial complex $\emptyset$, that is, we assume that $I_\Sigma \neq S$, while we allow the case $\Sigma = \{\emptyset\}$ (i.e., $I_\Sigma = \mathfrak{m}$). Throughout this paper, we denote the image of I_Σ in $k[\Delta]$ by J. Sometimes, we also denote I_Σ itself by J. For a $k[\Delta]$-module M, the ideal I_Δ lies in the kernel of the action of S on the local cohomology module $H^i_{I_\Sigma}(M)$ of M regarded as an S-module. Thus $H^i_{I_\Sigma}(M)$ can be regarded as a $k[\Delta]$-module, which is know to be isomorphic to to the local cohomology module $H^i_J(M)$ of M as a $k[\Delta]$-module. Therefore the convention we have chosen is harmless.

Definition 2.3 ([20])**.** A $\mathbb{Z}^n$-graded S-module $M = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} M_\mathbf{a}$ is called *straight*, if the following two conditions are satisfied.

(a) $\dim_k M_\mathbf{a} < \infty$ for all $\mathbf{a} \in \mathbb{Z}^n$.
(b) The multiplication map $M_\mathbf{a} \ni y \mapsto x_i y \in M_{\mathbf{a}+\{i\}}$ is bijective for all $i \in [n]$ and all $\mathbf{a} \in \mathbb{Z}^n$ with $a_i \neq 0$.

Examples 2.4. For a subset $F \subseteq [n]$, let P_F denote the monomial prime ideal $(x_i \mid i \notin F)$ of S. The injective hull $E(S/P_F)$ of S/P_F in the category ${}^*\mathbf{Mod}$ is a straight module. In fact,

$$[E(S/P_F)]_\mathbf{a} = \begin{cases} k & \text{if } \operatorname{supp}_+(\mathbf{a}) \subseteq F, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

and the multiplication map $E(S/P_F)_\mathbf{a} \ni y \mapsto x^\mathbf{b} y \in E(S/P_F)_{\mathbf{a}+\mathbf{b}}$ is always surjective (see [14, Example 1.1]). Note, that $E(S/P_F)$ is not injective in $\mathbf{Mod}$ unless $P_F = \mathfrak{m}$. Nevertheless, (2.1) is still useful. For example, if $E^\bullet$ is an injective resolution of a $\mathbb{Z}^n$-graded module M in ${}^*\mathbf{Mod}$, then $H^i_J(M)$ for a monomial ideal J is isomorphic to $H^i(\Gamma_J(E^\bullet))$, where Γ_J is the endofunctor on ${}^*\mathbf{Mod}$ defined by $\Gamma_J(M) = \{x \in M \mid J^m x = 0 \text{ for } m \gg 0\}$.

In [20], the third author proved that a local cohomology module $H^i_J(\omega_S) = H^i_J(S)(-\mathbf{1})$ is a straight module for any monomial ideal J. But in general, $H^i_J(k[\Delta])$ is not straight even after a degree shift. Thus we have to introduce a new concept.

Definition 2.5. A $\mathbb{Z}^n$-graded S-module $M = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} M_\mathbf{a}$ is called *quasi-straight*, if the following two conditions are satisfied.

(a) $\dim_k M_\mathbf{a} < \infty$ for all $\mathbf{a} \in \mathbb{Z}^n$.
(b) The multiplication map $M_\mathbf{a} \ni y \mapsto x_i y \in M_{\mathbf{a}+\{i\}}$ is bijective for all $i \in [n]$ and all $\mathbf{a} \in \mathbb{Z}^n$ with $a_i \neq 0, -1$.

Examples 2.6. Squarefree modules and straight modules are always quasi-straight. The injective module $E(S/P_F)(\mathbf{a})$ is quasi-straight if and only if $a_i = 1, 0$ for all $i \notin F$. If $E(S/P_F)(\mathbf{a})$ is quasi-straight, it can be written as $E(S/P_F)(G)$ for some $G \subseteq [n]$ with $F \cap G = \emptyset$. Note, that $[E(S/P_F)(G)]_\mathbf{a} = k$ if $\operatorname{supp}_+ \mathbf{a} \subseteq F$ and $\operatorname{supp}_- \mathbf{a} \supseteq G$, and $[E(S/P_F)(G)]_\mathbf{a} = 0$ otherwise.

Let M be a quasi-straight module. Then we have $\dim_k M_{\mathbf{a}} = \dim_k M_{\hat{\mathbf{a}}}$ for all $\mathbf{a} \in \mathbb{Z}^n$, where the i^{th} component $\hat{a}_i$ of $\hat{\mathbf{a}} \in \mathbb{Z}^n$ is defined by

$$\hat{a}_i = \begin{cases} 1 & \text{if } a_i \geq 1 \\ 0 & \text{if } a_i = 0 \\ -1 & \text{if } a_i \leq -1. \end{cases}$$

Hence, to know the Hilbert function of a quasi-straight module M, it is enough to know the dimensions $\dim_k M_{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{Z}^n$ with $a_i \in \{-1, 0, 1\}$.

A quasi-straight module M is finitely generated as an S-module if and only if it is $\mathbb{N}^n$-graded. Of course, M is squarefree in this case.

We denote the full subcategory of ${}^*\mathbf{Mod}$ consisting of quasi-straight modules by $\mathbf{qStr}$. One can easily check that this is an abelian subcategory of ${}^*\mathbf{Mod}$ closed under extensions and direct summands.

For further study of $\mathbf{qStr}$, the concept of the incidence algebra of a finite poset is very useful. For the reader's convenience, we recall the basic properties of the incidence algebra here.

Let P be a finite poset. The incidence algebra $A = I(P, k)$ of P over k is the k-vector space with basis $\{e_{x,y} \mid x, y \in P,\ x \leq y\}$ and multiplication defined k-linearly by $e_{x,y}\, e_{z,w} = \delta_{y,z}\, e_{x,w}$. We write e_x for $e_{x,x}$. Then A is a finite dimensional associative k-algebra with $1 = \sum_{x \in P} e_x$. Note, that $e_x\, e_y = \delta_{x,y}\, e_x$.

If M is a right A-module, then $M = \bigoplus_{x \in P} M e_x$ as a k-vector space. We write M_x for $M e_x$. If $f : M \to N$ is an A-linear map of right A-modules, then $f(M_x) \subseteq N_x$. Note, that $M_x e_{x,y} \subseteq M_y$ and $M_x e_{y,z} = 0$ for $y \neq x$.

For each $x \in P$, we can construct an injective object $\bar{E}(x)$ in the category $\mathbf{mod}_A$ of finitely generated right A-modules. Let $\bar{E}(x)$ be a k-vector space with basis $\{\bar{e}_y \mid y \leq x\}$. Then we can regard $\bar{E}(x)$ as a right A-module by

$$\bar{e}_y \cdot e_{z,w} = \begin{cases} \bar{e}_w & \text{if } y = z \text{ and } w \leq x, \\ 0 & \text{otherwise.} \end{cases}$$

Note, that $[\bar{E}(x)]_y = k\bar{e}_y$ if $y \leq x$, and $[\bar{E}(x)]_y = 0$ otherwise.

Proposition 2.7. *The category $\mathbf{mod}_A$ has enough projectives and enough injectives. An indecomposable injective object is isomorphic to $\bar{E}(x)$ for some $x \in P$, and any injective object is a finite direct sum of the copies of $\bar{E}(x)$ for various x.*

Proof. Since A can be regarded as an algebra associated with a *quiver with relation* (c.f. [1]), the assertions follows from [1, Proposition 1.8] and the arguments in [1, pp.61–62]. □

We partially order $\mathbb{Z}^n$ by setting $(a_1, \dots, a_n) \preceq (b_1, \dots, b_n)$ if $a_i \leq b_i$ for all $1 \leq i \leq n$. Denote by $3^{[n]}$ the subposet of $\mathbb{Z}^n$

$$3^{[n]} := \{\mathbf{a} \in \mathbb{Z}^n \mid -1 \leq a_i \leq 1 \text{ for all } i\}.$$

Lemma 2.8. *Let $A := I(3^{[n]}, k)$ be the incidence algebra of the poset $3^{[n]}$ over k. Then there is an equivalence of categories $\mathbf{mod}_A \cong \mathbf{qStr}$.*

Proof. For $N \in \mathbf{mod}_A$, let $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}}$ be a k-vector space with $M_{\mathbf{a}} \cong N_{\hat{\mathbf{a}}}$ for each $\mathbf{a} \in \mathbb{Z}^n$. Then M has an S-module structure such that the multiplication map $M_{\mathbf{a}} \ni y \mapsto x^{\mathbf{b}} y \in M_{\mathbf{a}+\mathbf{b}}$ is induced by $N_{\hat{\mathbf{a}}} \ni y \mapsto e_{\hat{\mathbf{a}},\widehat{(\mathbf{a}+\mathbf{b})}} \cdot y \in N_{\widehat{(\mathbf{a}+\mathbf{b})}}$. By an argument similar to the proof of [21, Theorem 3.2], we can see that M is quasi-straight and the correspondence $\mathbf{mod}_A \ni N \mapsto M \in \mathbf{qStr}$ gives a category equivalence $\mathbf{mod}_A \cong \mathbf{qStr}$. □

Note, that in [21] the author used the term "sheaf on a poset" instead of the equivalent concept of "module over the incidence algebra of the poset."

Proposition 2.9. *The category* $\mathbf{qStr}$ *has enough projectives and enough injectives. An indecomposable injective object in* $\mathbf{qStr}$ *is isomorphic to* $E(S/P_F)(G)$ *for some* $F, G \subseteq [n]$ *with* $F \cap G = \emptyset$.

Proof. Let $A = I(3^{[n]}, k)$ be the incidence algebra of $3^{[n]}$ over k. By the functor $\mathbf{mod}_A \to \mathbf{qStr}$ constructed in Lemma 2.8, an injective object $\bar{E}(\mathbf{a})$, $\mathbf{a} \in 3^{[n]}$, in $\mathbf{mod}_A$ is sent to $E(S/P_F)(G)$, where $F = \mathrm{supp}_+(\mathbf{a})$ and $G = \mathrm{supp}_-(\mathbf{a})$. □

Because of the explicit description of the indecomposable injectives in $\mathbf{qStr}$, we know that an injective object in $\mathbf{qStr}$ is also injective in $^*\mathbf{Mod}$. Hence we deduce the following.

Proposition 2.10. *Let* M *be a quasi-straight module, and* $E^\bullet$ *a minimal injective resolution of* M *in* $^*\mathbf{Mod}$. *Then each* E^i *is quasi-straight.*

Proof. Take a minimal injective resolution of M in $\mathbf{qStr}$. Then this is isomorphic to $E^\bullet$. □

Theorem 2.11. *Let* J *be a monomial ideal of* S. *If* M *is quasi-straight so is the local cohomology module* $H^i_J(M)$ *for all* $i \geq 0$.

Proof. Let $E^\bullet$ be a minimal injective resolution of M in $^*\mathbf{Mod}$. By Proposition 2.10, each E^i, $i \geq 0$, is a direct sum of finitely many copies of quasi-straight modules $E(S/P_F)(G)$ for $F, G \subseteq [n]$ with $F \cap G = \emptyset$. Note, that

$$\Gamma_J(E(S/P_F)(G)) = \begin{cases} E(S/P_F)(G) & \text{if } P_F \supseteq J \text{ (equivalently, } F \in \Sigma), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for each $i \geq 0$, $\Gamma_J(E^i)$ is a direct summand of E^i, and a direct sum of the finitely many copies of $E(S/P_F)(G)$ with $F \in \Sigma$. In particular, $\Gamma_J(E^i)$ is quasi-straight again. Since $\mathbf{qStr}$ is an abelian category, $H^i_J(M) = H^i(\Gamma_J(E^\bullet))$ is quasi-straight. □

Remark 2.12. If M is quasi-straight, so is $\Gamma_J(M) = H^0_J(M)$. Hence $\Gamma_J(-)$ is an endofunctor on $\mathbf{qStr}$, and $H^i_J(-)$ is its right derived functor (on $\mathbf{qStr}$).

Since a Stanley-Reisner ring is square-free, and hence also quasi-straight, we have the following.

Corollary 2.13. *Let* $k[\Delta]$ *be a Stanley-Reisner ring, and* $J \supseteq I_\Delta$ *another monomial ideal of* S. *Then* $H^i_J(k[\Delta])$ *is quasi-straight. In particular,*

$$H^i_J(k[\Delta])_{\mathbf{a}} \cong H^i_J(k[\Delta])_{\hat{\mathbf{a}}}.$$

The following is a generalization of the well-known fact that if $H^i_{\mathfrak{m}}(k[\Delta])$ is finitely generated (or equivalently, if it is of finite length), then $\mathfrak{m}\, H^i_{\mathfrak{m}}(k[\Delta]) = 0$.

Corollary 2.14. *In the situation of Theorem 2.11, if $H^i_J(M)$ is finitely generated as an S-module, then $J\, H^i_J(M) = 0$.*

Proof. Since $H^i_J(M)$ is quasi-straight and finitely generated, it is squarefree. On the other hand, $J^s H^i_J(M) = 0$ for some $s \gg 0$, since $H^i_J(M)$ is finitely generated. Since $H^i_J(M)$ is squarefree, we have $JH^i_J(M) = 0$. □

Moreover, by Theorem 4.5 below, if $H^i_J(k[\Delta])$ is finitely generated as an S-module, $H^i_J(k[\Delta])$ is isomorphic to a submodule of $\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J,\, k[\Delta])$.

Remark 2.15. What we have called a quasi-straight module is essentially the same as a $\mathbf{2}$-*determined module* in Miller [14], where $\mathbf{2} = (2, \ldots, 2) \in \mathbb{N}^n$. More precisely, $M \in {}^*\mathbf{Mod}$ is quasi-straight if and only if $M(-\mathbf{1})$ is $\mathbf{2}$-determined. Miller [14] also defined $\mathbf{a}$-*determined modules* for each $\mathbf{a} \in \mathbb{N}^n$. For readers familiar with [14], it is not hard to check that our methods can be used to show that the category of $\mathbf{a}$-determined modules has enough injectives, and injectives in this category are also injective in ${}^*\mathbf{Mod}$. Thus, for any monomial ideal J, the local cohomology module of $H^i_J(M)$ of an $\mathbf{a}$-determined module M is $\mathbf{a}$-determined again.

In [14], Miller also defined *positively* $\mathbf{a}$-*determined modules* for each $\mathbf{a} \in \mathbb{N}^n$. A positively $\mathbf{a}$-determined module is always finitely generated, and a positively $\mathbf{1}$-determined module is the same thing as what we called a squarefree module. If M is positively $\mathbf{a}$-determined, then the shifted module $M(-\mathbf{1})$ is $(\mathbf{a}+\mathbf{1})$-determined. Thus $H^i_J(M)$ of a positively $\mathbf{a}$-determined module M is $(\mathbf{a}+\mathbf{1})$-determined (after degree shifting by $-\mathbf{1}$).

3. A topological formula for the Hilbert function

In this section we give a (combinatorial) topological formula for the $\mathbb{Z}^n$-graded Hilbert function of the local cohomology module $H^i_J(k[\Delta])$, following Hochster. We use the fact that it may be computed via a Čech complex of localizations of $k[\Delta]$ whose boundary map is similar to the simplicial boundary map.

As before, $\Delta \subseteq 2^{[n]}$ is a simplicial complex, Σ a subcomplex, $I_\Delta \subseteq I_\Sigma$ their associated Stanley-Reisner ideals, with the image of I_Σ in $k[\Delta]$ denoted by J. Let $m_1, m_2, \ldots, m_\mu$ denote the images within $k[\Delta]$ of the square-free monomials corresponding to the faces of $\Delta - \Sigma$ which are minimal with respect to inclusion, so that J is generated by $m_1, m_2, \ldots, m_\mu$. For $f \in k[\Delta]$, let $k[\Delta]_f$ denote the localization of $k[\Delta]$ at the multiplicative set $\{f^n\}_{n \geq 0}$.

The Čech complex $\check{\mathcal{C}}^\bullet$ computing $H^i_J(k[\Delta])$ is the complex

$$0 \to k[\Delta] \to \bigoplus_{1 \leq t_1 \leq \mu} k[\Delta]_{m_{t_1}} \to \bigoplus_{1 \leq t_1 < t_2 \leq \mu} k[\Delta]_{m_{t_1} m_{t_2}} \to \cdots \to k[\Delta]_{m_1 \cdots m_\mu} \to 0$$

where for

$$\check{\mathcal{C}}^i = \bigoplus_{T \subseteq [\mu], |T| = i} k[\Delta]_{\prod_{t \in T} m_t}$$

the coboundary map $\check{\mathcal{C}}^i \to \check{\mathcal{C}}^{i+1}$ has as its (T, T')-entry the following map: if $T \nsubseteq T'$ it is 0, and otherwise if $T' = T \cup \{t'\}$ where t' is the j^{th} smallest element of T', it is $(-1)^j$ times the localization map

$$k[\Delta]_{\prod_{t \in T} m_t} \to \left(k[\Delta]_{\prod_{t \in T} m_t}\right)_{m_{t'}} \cong k[\Delta]_{\prod_{t' \in T'} m_{t'}}.$$

We wish to compute the cohomology of this cochain complex by restricting to each multidegree $\mathbf{a} \in \mathbb{Z}^n$. The next proposition is straightforward (c.f. [4, Lemma 5.3.6]).

Proposition 3.1. $\left[k[\Delta]_{\prod_{t \in T} m_t}\right]_{\mathbf{a}}$ *is at most* 1*-dimensional as a* k*-vector space, and is non-vanishing exactly when both*

(i) $\mathrm{supp}_-(\mathbf{a}) \subseteq \mathrm{supp}(\prod_{t \in T} m_t)$, *and*
(ii) $\mathrm{supp}(\prod_{t \in T} m_t) \cup \mathrm{supp}_+(\mathbf{a}) \in \Delta$. $\square$

The non-vanishing condition in the proposition has a nice rephrasing, once we have introduced a little terminology. Regard the map

$$2^{[\mu]} \xrightarrow{s} 2^{[n]}$$
$$T \overset{s}{\mapsto} \mathrm{supp}\left(\prod_{t \in T} m_t\right)$$

as an order-preserving map of Boolean algebras. Then if we regard any simplicial complex on vertex set $[n]$, such as Δ, as an order ideal in $2^{[n]}$, its inverse image $s^{-1}(\Delta)$ is an order ideal in $2^{[\mu]}$, and hence a simplicial complex on $[\mu]$.

We will also use the notions of *stars, deletions,* and *links* of faces in a simplicial complex (see Section 6 (6.1)). Setting $F_+ := \mathrm{supp}_+(\mathbf{a}), F_- := \mathrm{supp}_-(\mathbf{a})$, then Proposition 3.1 may be rephrased as saying that $\left[k[\Delta]_{\prod_{t \in T} m_t}\right]_{\mathbf{a}}$ is 1-dimensional exactly when T is a face of $s^{-1}(\mathrm{star}_\Delta(F_+))$ which does not lie in the subcomplex $s^{-1}(\mathrm{del}_{\mathrm{star}_\Delta(F_+)}(F_-))$. Consequently one can check that there is an isomorphism of cochain complexes of finite-dimensional k-vector spaces, up to shift by -1, between the Čech complex $\check{\mathcal{C}}^\bullet_{\mathbf{a}}$ and the (augmented) simplicial relative cochain complex with coefficients in k for the pair

$$(s^{-1}(\mathrm{star}_\Delta(F_+)), s^{-1}(\mathrm{del}_{\mathrm{star}_\Delta(F_+)}(F_-))).$$

This yields the following theorem.

Theorem 3.2. *Let* $\Sigma \subseteq \Delta$ *be simplicial complexes and let* $\mathbf{a} \in \mathbb{Z}$, $F_+ = \mathrm{supp}_+ \mathbf{a}$ *and* $F_- = \mathrm{supp}_- \mathbf{a}$. *Then*

$$H^i_J(k[\Delta])_{\mathbf{a}} \cong \tilde{H}^{i-1}(s^{-1}(\mathrm{star}_\Delta(F_+)), s^{-1}(\mathrm{del}_{\mathrm{star}_\Delta(F_+)}(F_-)); k) \tag{3.1}$$

where $\tilde{H}^\bullet(-,-;k)$ *denotes simplicial relative reduced cohomology.*

Equivalently,

$$H^i_J(k[\Delta])_{\mathbf{a}} \cong \tilde{H}^{i-1}(\|\mathrm{star}_\Delta(F_+)\| - \|\Sigma\|, \|\mathrm{del}_{\mathrm{star}_\Delta(F_+)}(F_-)\| - \|\Sigma\|; k) \tag{3.2}$$

where $\tilde{H}^\bullet(-,-;k)$ *denotes singular relative reduced cohomology, and* $\|\Delta\|$ *denotes the geometric realization of a simplicial complex* Δ.

Proof. The first equation is immediate from the preceding discussion of the Čech complex. The second is equivalent by Proposition 6.5. $\square$

Readers concerned with effective computation might be slightly disturbed by the use of singular cohomology in Equation (3.2). However, one has also the following equivalent formulation in terms of simplicial cohomology (see Proposition 6.3):

$$H^i_J(k[\Delta])_{\mathbf{a}} \cong \tilde{H}^{i-1}(\mathrm{Sd}(\mathrm{star}_\Delta(F_+) - \Sigma), \mathrm{Sd}(\mathrm{del}_{\mathrm{star}_\Delta(F_+)}(F_-) - \Sigma); k), \tag{3.3}$$

where $\mathrm{Sd}(\Delta - \Sigma)$ means the subcomplex of the barycentric subdivision $\mathrm{Sd}\,\Delta$ induced on the vertices which are barycenters of faces *not* in Σ. Equivalently, it is the order complex of the poset of faces in $\Delta - \Sigma$. (The order complex of the face poset of Δ itself is the cone over $\mathrm{Sd}(\Delta)$ with the vertex $\emptyset$. But, since we assume that $\Sigma \neq \emptyset$, the poset $\Delta - \Sigma$ does not contain $\emptyset$.)

If $\|\,\mathrm{star}_\Delta(F_+)\| - \|\Sigma\| = \emptyset$ and $i = 0$, the right side of Equation (3.2) is a little bit confusing, but its "correct" meaning is given by Equation (3.3). Anyway, it is clear that $H^0_J(k[\Delta]) \neq 0$ if and only if $J = 0$ (i.e., $\Delta = \Sigma$). In this case, $H^0_J(k[\Delta]) = k[\Delta]$.

Corollary 3.3. *With the above notation, if* $[H^i_J(k[\Delta])]_{\mathbf{a}} \neq 0$*, then* $F_+ \in \Sigma$ *and* $F_+ \cup F_- \in \Delta$.

Proof. If $F_+ \notin \Sigma$, then both $\mathrm{Sd}(\mathrm{star}_\Delta(F_+) - \Sigma)$ and $\mathrm{Sd}(\mathrm{del}_{\mathrm{star}_\Delta(F_+)}(F_-) - \Sigma)$ are cones over the barycenter of F_+. Thus the assertion follows from Equation (3.3). If $F_+ \cup F_- \notin \Delta$, then $\mathrm{del}_{\mathrm{star}_\Delta(F_+)}(F_-) = \mathrm{star}_\Delta(F_+)$. Hence the assertion follows from Equation (3.1). $\square$

Hochster's formula on $H^i_{\mathfrak{m}}(k[\Delta])$ ([17, II. Theorem 4.1]) easily follows from Theorem 3.2. Since $\Sigma = \{\emptyset\}$ in this case, we may assume that $\mathbf{a} \in -\mathbb{N}^n$ by Corollary 3.3. Then $\mathrm{star}_\Delta(F_+) = \mathrm{star}_\Delta(\emptyset) = \Delta$, and we have

$$\begin{aligned} H^i_J(k[\Delta])_{\mathbf{a}} &\cong \tilde{H}^{i-1}(\|\Delta\|, \|\,\mathrm{del}_\Delta(F_-)\|; k) \\ &\cong \tilde{H}^{i-1}(\Delta, \mathrm{del}_\Delta(F_-); k) \\ &\cong \tilde{H}^{i-|F|-1}(\mathrm{lk}_\Delta(F_-); k), \end{aligned}$$

where the first isomorphism is Equation (3.2), and the last one is Proposition 6.2 (ii).

The relation between Theorem 3.2 and the results of Mustaţă [16] and Terai [18] on $H^i_{I_\Sigma}(S)$ is also interesting, and will be discussed in Section 5 in a more general setting (see Example 5.3).

We next discuss some consequences of Theorem 3.2. Let $d = \dim \Delta + 1 = \dim k[\Delta]$. The fact that we can express the local cohomology $H^i_J(k[\Delta])$ either in terms of relative simplicial cohomology of complexes on μ vertices (in Equation (3.1)), or complexes of dimension at most $d-1$ (in Equation (3.3)) suggests comparison with vanishing theorems (see [9]) and other results.

For example, the trivial fact that $H^i_J(k[\Delta])$ vanishes for $i > \mu$ (which follows from the computation via Čech complexes) corresponds to the fact that a complex on μ vertices can only have homology in dimensions $\mu - 1$ and below. Somewhat less obvious is the following corollary.

Corollary 3.4. *Assume there are at most* 5 *minimal faces of* Δ *not lying in* Σ*, that is,* $\mu \leq 5$*. Then the dimensions of the graded components* $H^\bullet_J(k[\Delta])_{\mathbf{a}}$ *are independent of the field* k.

Proof. Pairs of simplicial complexes on at most 5 vertices have torsion-free cohomology; one does not obtain torsion until one reaches the minimal triangulation of the real projective plane on 6 vertices. The statement then follows from Equation (3.1) and the universal coefficient theorem in relative cohomology. □

The fact that $H^i_J(k[\Delta])$ vanishes for $i > d = \dim k[\Delta]$ corresponds to the fact that Equation (3.3) expresses each component $H^i_J(k[\Delta])_{\mathbf{a}}$ in terms of relative cohomology for simplicial complexes of dimension at most $d-1$. On the other hand, it is interesting to see what further information the Lichtenbaum-Hartshorne vanishing theorem [10, Theorem 3.1] (LHVT for short) for $k[\Delta]$ tells us about the topology of simplicial complexes.

Theorem 3.5 (LHVT for Stanley-Reisner rings)**.** *Let $\Sigma \subseteq \Delta$ be simplicial complexes, with $d = \dim k[\Delta]$ and $J = I_\Sigma$. Then $H^d_J(k[\Delta]) = 0$ if and only if every $(d-1)$-face of Δ contains a vertex of Σ.*

Although short proofs of the LHVT are known even in the general case, we will give a proof of this special case, to understand its topological meaning.

Proof. For the forward implication, we show the contrapositive. Assume that some $(d-1)$-face F contains no vertex of Σ. Then we claim that the multidegree $\mathbf{a}$ having $F_+ := \mathrm{supp}_+\,\mathbf{a} = \emptyset$, $F_- := \mathrm{supp}_-\,\mathbf{a} = F$ satisfies $H^d_J(k[\Delta])_{\mathbf{a}} \cong k$, and consequently $H^d_J(k[\Delta])$ does not vanish:

$$\begin{aligned}
H^d_J(k[\Delta])_{\mathbf{a}} &\cong \tilde{H}^{d-1}(\|\Delta\| - \|\Sigma\|, \|\operatorname{del}_\Delta(F)\| - \|\Sigma\|; k) \\
&\cong \tilde{H}^{d-1}(\|F\|, \|\partial F\|; k) \\
&\cong \tilde{H}^{d-1}(\|F\|/\|\partial F\|; k) \\
&\cong \tilde{H}^{d-1}(\mathbb{S}^{d-1}; k) \\
&\cong k
\end{aligned}$$

where the first isomorphism is Theorem 3.2, the second isomorphism is excision (we excise away $\|\Delta\| - \|\Sigma\| - \|F\|$, and use the fact that F contains no vertex of Σ), and the third isomorphism is Proposition 6.1.

For the reverse implication, assume every $(d-1)$-face of the $(d-1)$-dimensional complex Δ contains a vertex of the subcomplex Σ. We must show that this implies the combinatorial topological assertion that

$$\tilde{H}^{d-1}(\|\operatorname{star}_\Delta(F_+)\| - \|\Sigma\|, \|\operatorname{del}_{\operatorname{star}_\Delta(F_+)}(F_-)\| - \|\Sigma\|; k) = 0$$

for every pair of disjoint faces F_+, F_- of Δ with $F_+ \amalg F_- \in \Delta$. However, since $\operatorname{star}_\Delta(F_+)$ is again a complex of dimension at most $d-1$, just as Δ was, we can replace $\operatorname{star}_\Delta(F_+)$ by Δ (or equivalently, we may assume $F_+ = \emptyset$). Setting $D := d-1$, we can then rephrase the combinatorial topological assertion we must prove as follows:

> Let Δ be a simplicial complex of dimension at most D, and Σ a subcomplex containing at least one vertex from each D-face of Δ. Then for any face F of Δ,
>
> $$\tilde{H}^D(\|\Delta\| - \|\Sigma\|, \|\operatorname{del}_\Delta(F)\| - \|\Sigma\|; k) = 0.$$

We will actually show that these hypotheses imply that the pair

$$(\|\Delta\| - \|\Sigma\|, \| \operatorname{del}_\Delta(F)\| - \|\Sigma\|)$$

is relatively homotopy equivalent to a pair of subcomplexes of $\operatorname{Sd}\Delta$ of dimension at most $D-1$.

To do this, we start with the fact from Proposition 6.3 that the above pair is relatively homotopy equivalent to

$$(\operatorname{Sd}(\Delta - \Sigma), \quad \operatorname{Sd}(\operatorname{del}_\Delta(F) - \Sigma)).$$

The latter is a pair of simplicial complexes of dimension at most D which we will denote by (X, A) for short. We proceed to remove all D-faces of X (or of both X and A) by repeating the following procedure: whenever we find a D-face σ of X (or both X and$A$) that has some $(D-1)$-face τ contained in no other D-face of X (and hence also in no other D-face of A), we remove both σ, τ from X (or from both X and A). Each such removal is either an *elementary collapse* (see [2, (11.1)]) on X which affects no simplices of A, or a simultaneous elementary collapse on X and A. Thus in either case, it does not change the relative homotopy type of the pair (X, A).

To see that this process will remove *all* of the D-faces in X (and A), assume not. That is, assume that at some stage after performing some of these collapses, we are left with a pair (X', A') having some D-faces in X', but every such D-face has all of its $(D-1)$-faces lying in some other D-face, so that no further collapses are possible. Let σ be one of the remaining D-faces in X', and let G be the unique D-face of Δ in which it lies, so that the barycenter b_G of G is one of the vertices of σ. Then $\operatorname{lk}_{X'}(b_G)$ is a subcomplex of the simplicial $(D-1)$-sphere

$$\operatorname{lk}_{\operatorname{Sd}\Delta}(b_G) \cong \partial\,(\operatorname{Sd} G)\,,$$

having these two properties:

(i) It is $(D-1)$-dimensional, by virtue of the existence of σ.
(ii) A $(D-2)$-face τ always lies in exactly two $(D-1)$-faces. (To see this, consider the $(D-1)$-face $\tau \cup b_G$ of X' and use the assumption on X'.)

These two properties force $\operatorname{lk}_{X'}(b_G) = \operatorname{lk}_{\operatorname{Sd}\Delta}(b_G)$. But this contradicts our assumption that G contains at least one vertex of Σ, so that $X' \subseteq X$ is missing at least some of the D-faces of $\operatorname{Sd} G$. $\square$

For the remainder of this section, we regard J as an ideal of $k[\Delta]$, not S. Recall that $\operatorname{grade} J$ is defined to be the length of the longest $k[\Delta]$-sequence contained in J (c.f. [4, §1.2]). It is well-known [4, Proposition 1.2.14] that

$$\operatorname{grade} J \leq \operatorname{ht} J,$$

with equality whenever $k[\Delta]$ is Cohen-Macaulay (c.f. [4, Corollary 2.1.4]). It is also known [9, Theorem 3.8] that

$$\operatorname{grade} J = \min\{i \mid H^i_J(k[\Delta]) \neq 0\}.$$

We are interested in the following problem.

Problem 3.6. Characterize pairs (Δ, Σ) such that $H^i_J(k[\Delta]) = 0$ for all $i \neq \operatorname{grade} J$.

If $\Delta = 2^{\{1,\dots,n\}}$ (i.e., $k[\Delta] = S$), a pair (Δ, Σ) satisfies the condition of Problem 3.6 if and only if $k[\Sigma]$ is Cohen-Macaulay. Similarly, if $\Sigma = \{\emptyset\}$ (i.e., $J = \mathfrak{m}$), the condition of the problem is satisfied if and only if $k[\Delta]$ is Cohen-Macaulay. But, in the general case, the situation is more delicate.

Set $\dim k[\Delta] = d$. If $\Sigma \neq \{\emptyset\}$ (i.e., $J \neq \mathfrak{m}$, equivalently, $d > \operatorname{ht} J \geq \operatorname{grade} J$), all $(d-1)$-faces of Δ must contain a vertex of Σ to satisfy the condition of Problem 3.6, by Theorem 3.5. Assume that $k[\Delta]$ is Cohen-Macaulay and $\dim \Sigma = 0$ (i.e., $\dim(k[\Delta]/J) = 1$). Then $\operatorname{grade} J = d-1$, and hence $H^i_J(k[\Delta]) = 0$ for all $i < d-1$. So, in this case, the condition of Problem 3.6 is satisfied if and only if every facet of Δ contains a vertex of Σ. We will return to this problem in Section 5.

4. Local cohomology modules of the canonical module

The goal of this section is to understand, in the case where $k[\Delta]$ is Cohen-Macaulay, how the local cohomology $H^i_J(\omega_{k[\Delta]})$ is filtered by the modules

$$\operatorname{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[j]}, \omega_{k[\Delta]}) \text{ for } j \geq 1,$$

where $J^{[j]}$ denotes the j^{th} *Frobenius power* of the ideal J (see Definition 4.4). The main result is Theorem 4.9, which shows in particular (Corollary 4.10) that $H^i_J(\omega_{k[\Delta]})$ is determined by $\operatorname{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[2]}, \omega_{k[\Delta]})$.

Denote the category of $\mathbb{Z}^n$-graded $k[\Delta]$-modules by ${}^*\mathbf{Mod}_{k[\Delta]}$, so that ${}^*\mathbf{Mod}_{k[\Delta]}$ is a full subcategory of ${}^*\mathbf{Mod}_S = {}^*\mathbf{Mod}$. We review here some facts about this category which may be found in [8, Chapters 1,2].

If $F \in \Delta$, then $P_F \supseteq I_\Delta$. In this case, we regard P_F as a (prime) ideal of $k[\Delta] = S/I_\Delta$. The category ${}^*\mathbf{Mod}_{k[\Delta]}$ has enough injectives, and an indecomposable injective object in ${}^*\mathbf{Mod}_{k[\Delta]}$ is of the form $E(k[\Delta]/P_F)(\mathbf{a})$ for some $F \in \Delta$ and $\mathbf{a} \in \mathbb{Z}^n$, where

$$E(k[\Delta]/P_F) = \operatorname{Hom}_S(k[\Delta], E(S/P_F)) = \{y \in E(S/P_F) \mid zy = 0 \text{ for all } z \in I_\Delta\ \}$$

is the injective envelope of $k[\Delta]/P_F = S/P_F$ in ${}^*\mathbf{Mod}_{k[\Delta]}$. (Throughout this paper, we use the convention that $E(k[\Delta]/P_F)$ means the injective envelope of $k[\Delta]/P_F$ in ${}^*\mathbf{Mod}_{k[\Delta]}$, not in ${}^*\mathbf{Mod}_S$.) We can easily compute the structure of $E(k[\Delta]/P_F)$.

Lemma 4.1. *We have*

$$[E(k[\Delta]/P_F)]_{\mathbf{a}} = \begin{cases} k & \text{if } \operatorname{supp}_+(\mathbf{a}) \subseteq F \text{ and } \operatorname{supp}_-(\mathbf{a}) \cup F \in \Delta, \\ 0 & \text{otherwise,} \end{cases}$$

and the multiplication map $E(k[\Delta]/P_F)_{\mathbf{a}} \ni y \mapsto x^{\mathbf{b}}y \in E(k[\Delta]/P_F)_{\mathbf{a}+\mathbf{b}}$ is always surjective.

Proof. Since I_Δ is squarefree,

$$E(k[\Delta]/P_F) = \{y \in E(S/P_F) \mid x^G y = 0 \text{ for all } G \notin \Delta\}.$$

For $0 \neq y \in [E(S/P_F)]_{\mathbf{a}}$ and $G \subseteq [n]$, $x^G y \neq 0$ if and only if $\operatorname{supp}_+(\mathbf{a} + G) \subseteq F$. These conditions are also equivalent to $G \subseteq \operatorname{supp}_-(\mathbf{a}) \cup F$. Thus $\operatorname{supp}_-(\mathbf{a}) \cup F$ is the largest subset of $[n]$ whose monomial does not annihilate y. Now the assertion follows from an easy computation. $\square$

Let M be a $\mathbb{Z}^n$-graded S-module. For any multidegree $\mathbf{a} \in \mathbb{Z}^n$, define

$$M_{\succeq \mathbf{a}} := \bigoplus_{\mathbf{b} \succeq \mathbf{a}} M_{\mathbf{b}},$$

a graded submodule of M. When $\mathbf{a} = (-j, \dots, -j)$, we simply denote $M_{\succeq \mathbf{a}}$ by $M_{\succeq -\mathbf{j}}$. In particular, we say $M_{\succeq \mathbf{0}} = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$ is the $\mathbb{N}^n$*-graded part* of M. If M is quasi-straight, then its $\mathbb{N}^n$-graded part is squarefree.

Lemma 4.2. *If M is quasi-straight, it is a *essential extension (c.f.* [4, §3.6]*) of any of its submodules $M_{\succeq -\mathbf{j}}$ for $j \geq 1$, that is, for any homogeneous element $0 \neq y \in M$, there is some $z \in S$ such that $0 \neq zy \in M_{\succeq -\mathbf{j}}$.*

Proof. For a homogeneous element $0 \neq y \in M_{\mathbf{a}}$, set $T = \{i \mid a_i < -1\}$, and $z := \prod_{i \in T} x_i^{-1-a_i} \in S$. Since M is quasi-straight, we have $0 \neq zy \in M_{\succeq -\mathbf{1}} \subseteq M_{\succeq -\mathbf{j}}$. □

Lemma 4.3. *Let $j \geq 1$ be an integer. If M is a quasi-straight module, then $N := M_{\succeq -\mathbf{j}}$ satisfies the following conditions*

(a) $\dim_k N_{\mathbf{a}} < \infty$ *for all* $\mathbf{a} \in \mathbb{Z}^n$.
(b) *The multiplication map $N_{\mathbf{a}} \ni y \mapsto x_i y \in N_{\mathbf{a}+\{i\}}$ is bijective for all $i \in [n]$ and all $\mathbf{a} \succeq -\mathbf{j}$ with $a_i \neq 0, -1$.*

Conversely, if a $\mathbb{Z}^n$-graded S-module N with $N = N_{\succeq -\mathbf{j}}$ satisfies conditions (a) and (b), then there is a quasi-straight module M with $M_{\succeq -\mathbf{j}} \cong N$. Such an M is unique up to isomorphism.

Proof. Straightforward. □

Definition 4.4. As in the previous sections, J is the image of a monomial ideal I_Σ ($\supseteq I_\Delta$) of S in $k[\Delta]$. We denote the ideal $(\,(x^G)^i \mid G \notin \Sigma\,)$ of $k[\Delta]$ by $J^{[i]}$. More precisely, $J^{[i]}$ is the natural image $(\,(x^G)^i \mid G \notin \Sigma\,) + I_\Delta$ in $k[\Delta]$. We call $J^{[i]}$ the i^{th} *Frobenius power* of J. Note, that $J^{[1]} = J$.

Theorem 4.5. *Suppose that the $\mathbb{Z}^n$-graded $k[\Delta]$-module M is quasi-straight as an S-module. For all $i, j \geq 0$ and $-\mathbf{j} = (-j, \dots, -j)$, we have*

$$[\, H_J^i(M)\,]_{\succeq -\mathbf{j}} = [\, \mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[j+1]}, M)\,]_{\succeq -\mathbf{j}}.$$

In particular,

$$[\, H_J^i(k[\Delta])\,]_{\succeq -\mathbf{j}} = [\, \mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[j+1]}, k[\Delta])\,]_{\succeq -\mathbf{j}}.$$

To prove the theorem, we need the following lemma.

Lemma 4.6. *Let $\mathbf{a} \in \mathbb{Z}^n$ with $\mathbf{a} \succeq -\mathbf{j} = (-j, \dots, -j)$. For a monomial prime ideal $P = P_F$ with $F \in \Delta$,*

$$[\, \mathrm{Hom}_{k[\Delta]}(k[\Delta]/J^{[j+1]}, E(k[\Delta]/P))\,]_{\succeq \mathbf{a}} = \begin{cases} [\, E(k[\Delta]/P)\,]_{\succeq \mathbf{a}} & \text{if } J \subseteq P, \\ 0 & \text{otherwise.} \end{cases}$$

In other words,

$$[\, \mathrm{Hom}_{k[\Delta]}(k[\Delta]/J^{[j+1]}, E(k[\Delta]/P))\,]_{\succeq \mathbf{a}} = [\, \Gamma_J(E(k[\Delta]/P))\,]_{\succeq \mathbf{a}}.$$

Proof. For all $0 \neq y \in E(k[\Delta]/P)$, we have $\mathrm{ann}(y) \subseteq P$. If $J \not\subseteq P$, then $J^{[j+1]} \not\subseteq P$. Thus $\mathrm{Hom}_{k[\Delta]}(k[\Delta]/J^{[j+1]}, E(k[\Delta]/P)) = 0$ in this case. On the other hand, for all $y \in [E(k[\Delta]/P)]_{\mathbf{a}}$, we have $\mathrm{ann}(y) \supseteq P^{[j+1]}$ by Lemma 4.1. If $J \subseteq P$, we have $J^{[j+1]} \subseteq P^{[j+1]}$. Hence $J^{[j+1]} y = 0$ and $y \in [\,\mathrm{Hom}_{k[\Delta]}(k[\Delta]/J^{[j+1]}, E(k[\Delta]/P))\,]_{\succeq \mathbf{a}}$. □

Lemma 4.7. *Let M be a $\mathbb{Z}^n$-graded $k[\Delta]$-module whose $\mathbb{N}^n$-graded part is squarefree as an S-module, and $E^\bullet$ a minimal injective resolution of M in ${}^*\mathbf{Mod}_{k[\Delta]}$. Then each E^i, $i \geq 0$, is a direct sum of the copies of $E(k[\Delta]/P_F)(\mathbf{a})$ for some $F \in \Delta$ and $\mathbf{a} \in \mathbb{N}^n$.*

Proof. Let M' be the $\mathbb{N}^n$-graded part of M. Since M' is squarefree, the injective envelope $E_S(M')$ of M' in ${}^*\mathbf{Mod}_S$ is a direct sum of the copies of unshifted $E(S/P_F)$ by an argument in §3 of [21]. Since the injective envelope $E_{k[\Delta]}(M')$ of M' in ${}^*\mathbf{Mod}_{k[\Delta]}$ is $\mathrm{Hom}_S(k[\Delta], E_S(M'))$, it is a direct sum of unshifted $E(k[\Delta]/P_F)$. By the injectivity of $E_{k[\Delta]}(M')$, we have a $\mathbb{Z}^n$-graded map $f : M \to E_{k[\Delta]}(M')$ extending $M' \hookrightarrow E_{k[\Delta]}(M')$. By definition, the $\mathbb{N}^n$-graded part of $M'' := \ker(f)$ is 0. Hence the $\mathbb{N}^n$-graded part of the injective envelope $E_{k[\Delta]}(M'')$ is 0, and $E_{k[\Delta]}(M'')$ is a direct sum of the copies of $E(S/P_F)(\mathbf{a})$ for $\mathbf{a} \in \mathbb{N}^n \setminus \{0\}$. There is a $\mathbb{Z}^n$-graded map $M \to E_{k[\Delta]}(M'')$, which extends $M'' \hookrightarrow E_{k[\Delta]}(M'')$. Since $E^0 = E_{k[\Delta]}(M)$ is a direct summand of $E_{k[\Delta]}(M') \oplus E_{k[\Delta]}(M'')$, it is a direct sum of the copies of $E(S/P_F)(\mathbf{a})$ for $\mathbf{a} \in \mathbb{N}^n$.

The next term E^1 of $E^\bullet$ is the injective envelope of $E_{k[\Delta]}(M)/M$. Since the $\mathbb{N}^n$-graded parts of M and $E_{k[\Delta]}(M)$ are squarefree, that of $E_{k[\Delta]}(M)/M$ is also. Hence we can apply the above argument to $E_{k[\Delta]}(M)/M$. Similarly, we can prove the assertion for E^i, $i \geq 2$, by induction on i. □

Proof of Theorem 4.5. Let $E^\bullet$ be a minimal injective resolution of M in ${}^*\mathbf{Mod}_{k[\Delta]}$. Then $E^\bullet$ consists of $E(k[\Delta]/P_F)(\mathbf{a})$ for $\mathbf{a} \in \mathbb{N}^n$ by Lemma 4.7. For a $\mathbb{Z}^n$-graded module N, we have $N(\mathbf{a})_{\succeq -\mathbf{j}} = N_{\succeq \mathbf{a}-\mathbf{j}}(\mathbf{a})$. Hence

$$[\,\mathrm{Hom}_{k[\Delta]}(k[\Delta]/J^{[j+1]}, E^\bullet)\,]_{\succeq -\mathbf{j}} = [\,\Gamma_J(E^\bullet)\,]_{\succeq -\mathbf{j}}$$

by Lemma 4.6. The i^{th} cohomology of the complex of the left side is isomorphic to $[\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[j+1]}, M)]_{\succeq -\mathbf{j}}$ and that of the right side is $[H^i_J(M)]_{\succeq -\mathbf{j}}$. □

Remark 4.8. (1) In the situation of Theorem 4.5, $[\,\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[j+1]}, M)\,]_{\succeq -\mathbf{j}}$ "determines" $H^i_J(M)$ for each $j \geq 1$, since $H^i_J(M)$ is quasi-straight. And $H^i_J(M)$ is a *essential extension of $[\,\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[j+1]}, M)\,]_{\succeq -\mathbf{j}}$ for all $j \geq 1$. Of course, $[\,\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[2]}, M)\,]_{\succeq -\mathbf{1}}$ is the smallest among these modules. In Example 4.13, we will see that $[\,\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J, M)\,]_{\succeq \mathbf{0}}$ is not enough to recover $H^i_J(M)$.

(2) In general, we have

$$[\,\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[j+1]}, k[\Delta])\,]_{\succeq -\mathbf{j}} \neq \mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[j+1]}, k[\Delta]).$$

For example, assume $k[\Delta]$ is not Gorenstein and $J = \mathfrak{m}$. Then $\mathrm{Ext}^i_{k[\Delta]}(k, k[\Delta]) \neq 0$ for all $i \geq d := \dim k[\Delta]$. But $H^i_{\mathfrak{m}}(k[\Delta]) = 0$ for all $i > d$. Hence the $\mathbb{N}^n$-graded part of $\mathrm{Ext}^i_{k[\Delta]}(k, k[\Delta])$ is 0 for $i > d$.

We say $\omega_{k[\Delta]} := \mathrm{Ext}_S^{n-d}(k[\Delta], \omega_S)$ with $d = \dim k[\Delta]$ is the *canonical module* of $k[\Delta]$. The following is a generalization of [16, Theorem 1.1].

Theorem 4.9. *If $k[\Delta]$ is Cohen-Macaulay,*

$$[\, H_J^i(\omega_{k[\Delta]})\,]_{\succeq -\mathbf{j}} \cong \mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[j+1]}, \omega_{k[\Delta]})$$

for all $j \geq 0$.

Proof. For convenience, we say a $\mathbb{Z}^n$-graded S-module M is $\mathbf{j}$-*graded*, if $M_{\succeq -\mathbf{j}} = M$. By Theorem 4.5, it suffices to prove that $\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[j+1]}, \omega_{k[\Delta]})$ is $\mathbf{j}$-graded. Set $d = \dim k[\Delta]$. It is well-known that

$$\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[j+1]}, \omega_{k[\Delta]}) \cong \mathrm{Ext}_S^{n-d+i}(k[\Delta]/J^{[j+1]}, \omega_S)$$

as $\mathbb{Z}^n$-graded S-modules. In fact, both of them are isomorphic to the graded Matlis dual of $H_\mathfrak{m}^{d-i}(k[\Delta]/J^{[j+1]})$. Hence it suffices to show that $\mathrm{Ext}^i_S(k[\Delta]/J^{[j+1]}, \omega_S)$ is $\mathbf{j}$-graded. Let $F_\bullet$ be a $\mathbb{Z}^n$-graded minimal free resolution of $k[\Delta]/J^{[j+1]}$ over S. By an argument using Taylor's resolution, we see that $F_\bullet$ is a direct sum of free modules $S(-\mathbf{a})$ with $\mathbf{a} \preceq \mathbf{j} + \mathbf{1}$. Set $G^\bullet := \mathrm{Hom}_S(F_\bullet, \omega_S)$. Since $\omega_S = S(-\mathbf{1})$, $G^\bullet$ consists of free modules $S(-\mathbf{b})$ with $\mathbf{b} \succeq -\mathbf{j}$. In particular, $G^\bullet$ is a complex of $\mathbf{j}$-graded modules. Thus $H^i(G^\bullet) = \mathrm{Ext}^i_S(k[\Delta]/J^{[j+1]}, \omega_S)$ is $\mathbf{j}$-graded. $\square$

The following is clear from Theorem 4.9.

Corollary 4.10. *If $k[\Delta]$ is Cohen-Macaulay, we can construct $H_J^i(\omega_{k[\Delta]})$ from its submodule $\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[2]}, \omega_{k[\Delta]})$ by Lemma 4.3. In particular, if $k[\Delta]$ is Gorenstein, $H_J^i(k[\Delta])$ is determined by a submodule $\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[2]}, k[\Delta])$.*

Remark 4.11. Assume that $k[\Delta]$ is Cohen-Macaulay. From Theorem 4.9, we see that $\mathrm{depth}(k[\Delta]/J^{[2]}) = \mathrm{depth}(k[\Delta]/J^{[i]})$ and $\mathrm{Ass}(k[\Delta]/J^{[2]}) = \mathrm{Ass}(k[\Delta]/J^{[i]})$ for all $i \geq 2$. The former statement also follows from an argument using lcm-lattices (see [7]). In fact, for all $i \geq 2$, the lcm-lattice of $I_\Sigma^{[i]} + I_\Delta$ is isomorphic to that of $I_\Sigma^{[2]} + I_\Delta$.

For a squarefree monomial ideal $I_\Delta \subset S$, Lyubeznik [13] proved a beautiful formula on the cohomological dimension of S at I_Δ, which states that

$$\mathrm{cd}(S, I_\Delta) := \max\{i \mid H^i_{I_\Delta}(S) \neq 0\} = \dim S - \mathrm{depth}(S/I_\Delta). \tag{4.1}$$

We can extend this result to a monomial ideal J of a Gorenstein Stanley-Reisner ring $k[\Delta]$, using the second Frobenius power $J^{[2]}$.

Corollary 4.12. *Let the notation be as above. Assume that $k[\Delta]$ is Gorenstein. We have*

$$\mathrm{cd}(k[\Delta], J) = \dim k[\Delta] - \mathrm{depth}(k[\Delta]/J^{[2]}).$$

Proof. By Corollary 4.10, $H^i_J(k[\Delta]) \neq 0$ if and only if $\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[2]}, k[\Delta]) \neq 0$. Since $k[\Delta]$ is Gorenstein, $\mathrm{Ext}^i_{k[\Delta]}(k[\Delta]/J^{[2]}, k[\Delta])$ is isomorphic to the Matlis dual of $H_\mathfrak{m}^{d-i}(k[\Delta]/J^{[2]})$ up to degree shifting, where $d = \dim k[\Delta]$. Hence $H^i_J(k[\Delta]) \neq 0$ if and only if $H_\mathfrak{m}^{d-i}(k[\Delta]/J^{[2]}) \neq 0$. Since $\max\{i \mid H^i_\mathfrak{m}(k[\Delta]/J^{[2]})\} = \mathrm{depth}(k[\Delta]/J^{[2]})$, we are done. $\square$

If $\Delta = 2^{[n]}$ (i.e., $k[\Delta] = S$), the above corollary and Equation (4.1) imply that $\operatorname{depth}(S/I_\Delta) = \operatorname{depth}(S/I_\Delta^{[i]})$ for all $i \geq 1$. This also follows from the fact that the lcm-lattice of $I_\Delta^{[i]}$ is isomorphic to that of I_Δ for all $i \geq 1$.

Example 4.13. Set $S = k[x, y, z, w]$, $I_\Delta = (xz, yw)$, and $J = (z, w)$. That is, Δ consists of four edges of a quadrilateral and Σ consists of three edges of Δ. Then $k[\Delta]$ is Gorenstein, and $\dim k[\Delta] - \operatorname{depth}(k[\Delta]/J) = 0$. But an easy calculation (e.g. via Theorem 3.2) shows that $[H_J^1(k[\Delta])]_{(1,0,0,-1)} \neq 0$, although $H_J^i(k[\Delta]) = 0$ for all $i \geq 2$. Thus $\operatorname{cd}(k[\Delta], J) = 1$, so the direct analog of (4.1) does not hold for $J \subset k[\Delta]$. On the other hand, $\operatorname{depth}(k[\Delta]/J^{[2]}) = 1$, and in fact, $J^{[2]} \subset k[\Delta]$ has an embedded prime (x, w, z) (this also means that $(x, w, z) \in \operatorname{Ass}(H_J^1(k[\Delta]))$ by Corollary 4.14 below). Hence $\dim k[\Delta] - \operatorname{depth}(k[\Delta]/J^{[2]}) = 1$, in agreement with Corollary 4.12.

To state the next result, we introduce the following notion. For a $k[\Delta]$-module M and an integer i, set

$$\operatorname{Ass}^i(M) = \{P \in \operatorname{Ass}(M) \mid \dim(k[\Delta]/P) = \dim k[\Delta] - i\}.$$

Corollary 4.14. *Assume that $k[\Delta]$ is Cohen-Macaulay and $\dim k[\Delta] = d$. If $P \in \operatorname{Ass}(H_J^i(\omega_{k[\Delta]}))$, then $\dim(k[\Delta]/P) \leq d - i$ and*

$$\operatorname{Ass}^i(H_J^i(\omega_{k[\Delta]})) = \operatorname{Ass}^i(k[\Delta]/J^{[2]}) \supseteq \{P_F \mid F \text{ is a facet of } \Sigma \text{ with } |F| = d - i\ \}.$$

To prove this result, we need the following lemma.

Lemma 4.15 (c.f. [6, Theorem 1.1])**.** *Let M be a finitely generated $k[\Delta]$-module. If $P \in \operatorname{Ext}_R^i(M, \omega_{k[\Delta]})$, then $\dim(k[\Delta]/P) \leq d - i$. And we have $\operatorname{Ass}^i(M) = \operatorname{Ass}^i(\operatorname{Ext}_{k[\Delta]}^i(M, \omega_{k[\Delta]}))$ for all $i \geq 0$.*

Proof. The proof of [6, Theorem 1.1] also works here. □

Proof of Corollary 4.14. Since $\operatorname{Ext}_{k[\Delta]}^i(k[\Delta]/J^{[2]}, \omega_{k[\Delta]}) \subseteq H_J^i(\omega_{k[\Delta]})$ is a *essential extension by Theorem 4.9, we have $\operatorname{Ass}(H_J^i(\omega_{k[\Delta]})) = \operatorname{Ass}(\operatorname{Ext}_{k[\Delta]}^i(k[\Delta]/J^{[2]}, \omega_{k[\Delta]}))$. Hence the first statement and the equality in the displayed equality follow from Lemma 4.15. The last inclusion holds, since $\operatorname{Ass}(k[\Delta]/J^{[2]}) \supseteq \operatorname{Ass}(k[\Delta]/J) = \operatorname{Ass}(k[\Sigma]) = \{P_F \mid F \text{ is a facet of } \Sigma\}$. □

Remark 4.16. Even $H_J^i(S)$ can have an embedded prime, see [16, Example 1]. And the inclusion in Corollary 4.14 can be strict, as Example 4.13 shows.

5. A topological formula for the Hilbert function in the Gorenstein case

In Section 3, we gave a combinatorial formula for the $\mathbb{Z}^n$-graded Hilbert function of $H_J^i(k[\Delta])$ for general Δ. If Δ is Cohen-Macaulay, we can give a formula (Theorem 5.1) for the Hilbert function of $H_J^i(\omega_{k[\Delta]})$ for the canonical module $\omega_{k[\Delta]}$ of $k[\Delta]$. Hence, if Δ is Gorenstein, we get a new formula (Corollary 5.2) on $H_J^i(k[\Delta])$.

Theorem 5.1. *Assume that $k[\Delta]$ is Cohen-Macaulay and has dimension d, and $\Sigma \subseteq \Delta$ a subcomplex of Δ. As before, let J denote the image of I_Σ in $k[\Delta] = S/I_\Delta$.*

Then for any multidegree $\mathbf{a}$ *in* $\mathbb{Z}^n$, *setting* $F_+ := \operatorname{supp}_+ \mathbf{a}, F_- := \operatorname{supp}_- \mathbf{a}$, *one has*

$$\begin{aligned} H^i_J(\omega_{k[\Delta]})_{\mathbf{a}} &\cong \tilde{H}_{d-i-1}(\operatorname{star}_\Delta(F_-) \cap \Sigma, \operatorname{del}_{\operatorname{star}_\Delta(F_-)\cap\Sigma}(F_+); k) \\ &\cong \tilde{H}_{d-i-|F_+|-1}(\operatorname{lk}_{\operatorname{star}_\Delta(F_-)\cap\Sigma}(F_+); k). \end{aligned}$$

Proof. The second expression is equivalent to the first by Proposition 6.2, so we need only prove the first.

A minimal injective resolution of ω_S in ${}^*\mathbf{Mod}$ is given by

$$D^\bullet : 0 \to D^0 \to D^1 \to \cdots \to D^n \to 0,$$

$$D^i = \bigoplus_{\substack{F\subseteq[n]\\ |F|=n-i}} E(S/P_F),$$

and the differential is composed of $\pm f : E(S/P_F) \to E(S/P_{F\setminus\{j\}})$ for $j \in F$, where $f : E(S/P_F) \to E(S/P_{F\setminus\{j\}})$ is induced by the natural surjection $S/P_F \to S/P_{F\setminus\{j\}}$ (see [4, §5.7]). Then $D^\bullet_{k[\Delta]} := \operatorname{Hom}_S(k[\Delta], D^\bullet)[n-d]$ gives a minimal injective resolution of $\omega_{k[\Delta]}$ in ${}^*\mathbf{Mod}_{k[\Delta]}$. We see that $D^\bullet_{k[\Delta]}$ is of the form

$$D^\bullet_{k[\Delta]} : 0 \to D^0_{k[\Delta]} \to D^1_{k[\Delta]} \to \cdots \to D^d_{k[\Delta]} \to 0,$$

$$D^i_{k[\Delta]} = \bigoplus_{\substack{F\in\Delta\\ |F|=d-i}} E(k[\Delta]/P_F),$$

and the differential is induced by that of $D^\bullet$. Note, that the signs in this differential are given in the same way as the augmented chain complex $\tilde{C}_\bullet(\Delta)$ of Δ. Since J is a monomial ideal, we have $H^i_J(\omega_{k[\Delta]}) = H^i(\Gamma_J(D^\bullet_{k[\Delta]}))$. Note that

$$\Gamma_J(D^i_{k[\Delta]}) = \bigoplus_{\substack{F\in\Sigma\\ |F|=d-i}} E(k[\Delta]/P_F).$$

By Lemma 4.1, $[E(k[\Delta]/P_F)]_{\mathbf{a}} = k$ if $F \in \operatorname{star}_\Delta(F_-) - \operatorname{del}_{\operatorname{star}_\Delta(F_-)}(F_+)$, and $[E(k[\Delta]/P_F)]_{\mathbf{a}} = 0$ otherwise. Hence, $[\Gamma_J(D^\bullet_{k[\Delta]})]_{\mathbf{a}}$ is isomorphic to

$$\tilde{C}_{d-\bullet-1}(\operatorname{star}_\Delta(F_-) \cap \Sigma, \operatorname{del}_{\operatorname{star}_\Delta(F_-)\cap\Sigma}(F_+); k)$$

as a chain complex of k-vector spaces. We are done. □

If $\mathbf{a} \in \mathbb{N}^n$ (i.e., $F_- = \emptyset$), the above theorem simplifies because $\operatorname{star}_\Delta(F_-)\cap\Sigma = \Sigma$. Consequently, we obtain for $\mathbf{a} \in \mathbb{N}^n$ with $F := \operatorname{supp} \mathbf{a}$,

$$H^i_J(\omega_{k[\Delta]})_{\mathbf{a}} \cong \tilde{H}_{d-i-|F|-1}(\operatorname{lk}_\Sigma(F); k). \tag{5.1}$$

If $\mathbf{a} \in \mathbb{N}^n$, we have

$$H^i_J(\omega_{k[\Delta]})_{\mathbf{a}} \cong \operatorname{Ext}^i_{k[\Delta]}(k[\Delta]/J, \omega_{k[\Delta]})_{\mathbf{a}} \cong H^{d-i}_{\mathfrak{m}}(k[\Delta]/J)_{-\mathbf{a}}$$

by Theorem 4.9 and local duality [4, Theorem 3.5.8]. Thus (5.1) also follows from a well-known formula of Hochster ([17, Theorem 4.1]).

Let Δ be a Gorenstein complex (over k). Set $\Delta' := \operatorname{core}(\Delta)$, a k-homology sphere. Let V be the vertex set of Δ', so that if $W := [n] - V$, then $\Delta = 2^W * \Delta'$ and $\omega_{k[\Delta]} = k[\Delta](-W)$, see [17, 4].

Corollary 5.2. *Suppose that $k[\Delta]$ is Gorenstein with $\dim k[\Delta] = d$, and Σ a subcomplex of Δ. Keeping the same notation as above, and defining for $\mathbf{a} \in \mathbb{Z}^n$,*

$$F := (F_+ \cup W) - F_- = F_+ \cup (W - F_-)$$

one has

$$\begin{aligned} H^i_J(k[\Delta])_{\mathbf{a}} &\cong \tilde{H}_{d-i-1}(\mathrm{star}_\Delta(F_-) \cap \Sigma, \mathrm{del}_{\mathrm{star}_\Delta(F_-)\cap\Sigma}(F); k) \\ &\cong \tilde{H}_{d-i-|F|-1}(\mathrm{lk}_{\mathrm{star}_\Delta(F_-)\cap\Sigma}(F); k) \quad \square \end{aligned} \tag{5.2}$$

Example 5.3. Consider the case $k[\Delta] = S$ (that is, Δ is the full simplex $2^{[n]}$). In this case, $\mathrm{star}_\Delta(F_-) \cap \Sigma$ is always Σ. And we have

$$H^i_J(S)_{\mathbf{a}} \cong \tilde{H}_{n-i-|F|-1}(\mathrm{lk}_\Sigma(F); k)$$

for all $\mathbf{a} \in \mathbb{Z}^n$, where $F = \{i \mid a_i \geq 0\}$. This re-proves a formula of Terai [18].

Now, we consider Problem 3.6 again.

Theorem 5.4. *Suppose that Δ is a Cohen-Macaulay complex of dimension $d-1$, and Σ is an $(s-1)$-dimensional subcomplex of Δ. Then $H^i_J(\omega_{k[\Delta]}) = 0$ for all $i \neq d-s$ if and only if for every face F of Δ the subcomplex $\mathrm{star}_\Delta(F) \cap \Sigma$ is a Cohen-Macaulay complex of dimension $s-1$.*

Proof. The "if" part is immediate from Theorem 5.1. We will prove the "only if" part. Set $\Sigma'(F) := \mathrm{star}_\Delta(F) \cap \Sigma$ for $F \in \Delta$. It suffices to show that

$$\tilde{H}_{i-|G|-1}(\mathrm{lk}_{\Sigma'(F)}(G)) = 0 \quad \text{for all } i \neq s \text{ and all } G \subseteq [n]. \tag{5.3}$$

We can easily check that $\mathrm{lk}_{\Sigma'(F\setminus G)}(G) = \mathrm{lk}_{\Sigma'(F)}(G)$. So we may assume that $F \cap G = \emptyset$. Then there is some $\mathbf{a} \in \mathbb{Z}^n$ with $\mathrm{supp}_+(\mathbf{a}) = G$ and $\mathrm{supp}_-(\mathbf{a}) = F$. So (5.3) follows from Theorem 5.1. $\square$

The next corollary is immediate from Theorem 5.4.

Corollary 5.5. *Suppose that $k[\Delta]$ is Gorenstein. Then $H^i_J(k[\Delta]) = 0$ for all $i \neq \mathrm{grade}\, J$ (i.e., the condition of Problem 3.6 is satisfied) if and only if for every face $F \in \Delta$ the subcomplex $\mathrm{star}_\Delta(F) \cap \Sigma$ is Cohen-Macaulay and has the same dimension as Σ.*

Remark 5.6. (1) Assume that a pair (Δ, Σ) satisfies the condition of Theorem 5.4. Then Σ itself is Cohen-Macaulay since $\mathrm{star}_\Delta(\emptyset) \cap \Sigma = \Sigma$. And, for any facet F of Δ, $\{G \in \Sigma \mid G \subseteq F\} = \mathrm{star}_\Delta(F) \cap \Sigma$ is a Cohen-Macaulay complex of dimension $s-1$. In particular, any facet of Δ contains a facet of Σ.

(2) Here we will give a ring-theoretic meaning of $\mathrm{star}_\Delta(F) \cap \Sigma$. Let $k[\Delta]_{x^F}$ be the localization of $k[\Delta]$ at the multiplicatively closed set generated by the monomial x^F. Then the kernel of the composition $S \twoheadrightarrow k[\Delta] \to k[\Delta]_{x^F}$ is $I_{\mathrm{star}_\Delta(F)}$. Thus $I_{\mathrm{star}_\Delta(F)\cap\Sigma} = I_{\mathrm{star}_\Delta(F)} + I_\Sigma$.

Example 5.7. The combinatorial condition of Theorem 5.4 is somewhat complicated to check. But the following examples satisfy the condition.

(1) Let Σ be the s-skeleton of a Cohen-Macaulay complex Δ. In this case, $\mathrm{star}_\Delta(F) \cap \Sigma$ is the s-skeleton of the Cohen-Macaulay complex $\mathrm{star}_\Delta(F) = 2^F * \mathrm{lk}_\Delta(F)$. Hence it is Cohen-Macaulay.

(2) Let Δ be a $(d-1)$-dimensional *balanced* Cohen-Macaulay complex, $T \subset [d]$ a subset with $|T| = s$, and $\Sigma = \Delta_T$ the *rank-selected* subcomplex (c.f. [17, III, §4]). Then the pair (Δ, Σ) satisfies the condition of Theorem 5.4. In fact, for any $F \in \Delta$, $\mathrm{star}_\Delta(F)$ is a balanced Cohen-Macaulay complex again, and $\mathrm{star}_\Delta(F) \cap \Sigma$ coincides with the rank-selected subcomplex $(\mathrm{star}_\Delta(F))_T$, hence it is a Cohen-Macaulay complex of dimension s by [17, III, Thorem 4.5].

We can explain this example ring-theoretically. As in [17, III, §4], let $\kappa : [n] \to [d]$ be the coloring map of Δ. For $j \in [d]$, define

$$\theta_j = \sum_{\kappa(i)=j} x_i \in k[\Delta].$$

By [17, Proposition 4.3], $\theta_1, \dots, \theta_d$ is a system of parameters of $k[\Delta]$. If $\kappa(i) = j$, then $x_i\theta_j = x_i^2$ in $k[\Delta]$. Set $J' := (\, \theta_j \mid j \notin T\,)$. Since $J = (x_i \mid \kappa(i) \notin T)$, we have $J \supset J' \supset J^{[2]}$ and hence $J = \sqrt{J'}$. Therefore, $J \subset k[\Delta]$ is a *set theoretic complete intersection* of codimension $d-s$, and $H^i_J(k[\Delta]) = H^i_{J'}(k[\Delta]) = 0$ for all $i > d-s$ (this is true even if $k[\Delta]$ is not Cohen-Macaulay). On the other hand, since $k[\Delta]$ is Cohen-Macaulay, $H^i_J(k[\Delta]) = 0$ for all $i < d-s = \mathrm{grade}(J)$. By the same reason, we also have $H^i_J(\omega_{k[\Delta]}) = 0$ for $i \neq d-s$. So Remark 5.6 (1) gives another proof of [17, III, Theorem 4.5], which states that Σ is Cohen-Macaulay in this case.

Naturally, since Theorem 3.2 and Corollary 5.2 both express the same local cohomology in topological terms when $k[\Delta]$ is Gorenstein, there must be some topological explanation for their equivalence.

For example, consider the case where $\Delta = 2^{[n]}$ and $k[\Delta] = S$, as in Example 5.3. Terai's formula (and our Corollary 5.2) states that

$$[H^i_{I_\Sigma}(S)]_{(-1,\dots,-1)} \cong \tilde{H}_{n-i-1}(\Sigma; k),$$

while Theorem 3.2 gives

$$[H^i_{I_\Sigma}(S)]_{(-1,\dots,-1)} \cong \tilde{H}^{i-1}(\|\Delta\| - \|\Sigma\|, \|\partial\Delta\| - \|\Sigma\|; k),$$

where for $\Delta = 2^{[n]}$ we have $\partial\Delta = 2^{[n]} - [n]$. For simplicity, assume that $I_\Sigma \neq 0$ (i.e., $\Sigma \neq 2^{[n]}$). Then $\|\Delta\| - \|\Sigma\|$ is contractible (since it is star-shaped with respect to the barycenter of the maximum face $[n]$ of Δ), and hence $\tilde{H}^{i-1}(\|\Delta\| - \|\Sigma\|; k) = 0$ for all i. So the long exact sequence for cohomology shows that

$$\tilde{H}^{i-1}(\|\Delta\| - \|\Sigma\|, \|\partial\Delta\| - \|\Sigma\|; k) \cong \tilde{H}^{i-2}(\|\partial\Delta\| - \|\Sigma\|; k).$$

Thus Alexander duality on the $(n-2)$-sphere $\|\partial\Delta\|$ connects Theorem 3.2 and Terai's formula.

In the general case, the situation is much more complicated. But, as we now explain, the relation between Theorem 3.2 and Corollary 5.2 is based on the "disguised" version of Alexander duality (Lemma 6.8), giving further confirmation for the observation (see e.g. [14]) that in a combinatorial setting, Matlis duality corresponds to Alexander duality.

Recall that Corollary 5.2 says

$$\begin{aligned} H^i_J(k[\Delta])_{\mathbf{a}} &\cong \tilde{H}_{d-i-|F|-1}(\mathrm{lk}_{\mathrm{star}_\Delta(F_-)\cap\Sigma}(F); k) \\ &\cong \tilde{H}_{d-i-|F|-1}(\mathrm{lk}_{\mathrm{star}_\Delta(F_+\cup F_-)\cap\Sigma}(F); k), \end{aligned}$$

where the second isomorphism is due to the easily checked equality

$$\mathrm{lk}_{\mathrm{star}_\Delta(A)\cap\Sigma}(B) = \mathrm{lk}_{\mathrm{star}_\Delta(C\cup A)\cap\Sigma}(B) \text{ if } C \subseteq B.$$

On the other hand, Theorem 3.2 says

$$\begin{aligned} H^i_J(k[\Delta])_{\mathbf{a}} &\cong \tilde{H}^{i-1}(s^{-1}(\mathrm{star}_\Delta(F_+)), s^{-1}(\mathrm{del}_{\mathrm{star}_\Delta(F_+)}(F_-)); k) \\ &\cong \tilde{H}^{i-1}(s^{-1}(\mathrm{star}_\Delta(F_+ \cup F_-)), s^{-1}(\mathrm{del}_{\mathrm{star}_\Delta(F_+\cup F_-)}(F_-); k), \end{aligned}$$

where the first isomorphism is our original statement, and the second isomorphism is Proposition 6.6.

So we need to show that when Δ is Gorenstein, and with the above notation,

$$\begin{aligned} &\tilde{H}_{d-i-|F|-1}(\mathrm{lk}_{\mathrm{star}_\Delta(F_+\cup F_-)\cap\Sigma}(F); k) \cong \\ &\tilde{H}^{i-1}(s^{-1}(\mathrm{star}_\Delta(F_+ \cup F_-)), s^{-1}(\mathrm{del}_{\mathrm{star}_\Delta(F_+\cup F_-)}(F_-)); k). \end{aligned}$$

Note, that all complexes in this last equation are subcomplexes of

$$\mathrm{star}_\Delta(F_+ \cup F_-) = 2^F * 2^{F_-} * \mathrm{lk}_{\Delta'}((F_+ \cup F_-) - W).$$

Note also, that the last complex in the above join is a link of a face in the homology sphere $\Delta' = \mathrm{core}(\Delta)$, hence a homology sphere itself. Using Equation 6.3, we can replace

$$\begin{aligned} \mathrm{star}_\Delta(F_+ \cup F_-) &\text{ with } \Delta \\ \mathrm{star}_\Delta(F_+ \cup F_-) \cap \Sigma &\text{ with } \Sigma \\ F_- &\text{ with } G \end{aligned}$$

(since we may assume that $F_+ \cup F_- \in \Delta$ by Corollary 3.3, we can keep the assumption $\Sigma \neq \emptyset$) and then we need only prove the following combinatorial topological statement:

> If $\Delta = 2^F * 2^G * \mathbb{S}$ for some homology sphere $\mathbb{S}$ and Σ is a subcomplex of Δ then
>
> $$\tilde{H}_{d-i-|F|-1}(\mathrm{lk}_\Sigma(F); k) \cong \tilde{H}^{i-1}(s^{-1}(\Delta), s^{-1}(\mathrm{del}_\Delta(G)); k).$$
>
> where $d - 1 = \dim \Delta$.

To prove this, first note that if F is not a face of Σ, then the left-hand-side vanishes. The right-hand side will also vanish, since then some monomial m which divides x^F will lie in $I_\Sigma \setminus I_\Delta$ and give rise to a relative cone point in $(s^{-1}(\Delta), s^{-1}(\mathrm{del}_\Delta(G)))$.

So we may assume F is a face of Σ, and then Proposition 6.7 implies

$$\tilde{H}^{i-1}(s_\Sigma^{-1}(\Delta), s_\Sigma^{-1}(\mathrm{del}_\Delta(G)); k) \cong \tilde{H}^{i-1}(s^{-1}_{\mathrm{lk}_\Sigma F}(2^G * \mathbb{S}), s^{-1}_{\mathrm{lk}_\Sigma F}(\mathrm{del}_{2^G*\mathbb{S}}(G)); k).$$

Now we can further replace

$$\begin{aligned} 2^G * \mathbb{S} &\text{ with } \Delta \\ \mathrm{lk}_\Sigma F &\text{ with } \Sigma \end{aligned}$$

and then what we must prove is as follows:

> If $\Delta = 2^G * \mathbb{S}$ for some homology sphere $\mathbb{S}$, and Σ is a subcomplex of Δ then
>
> $$\tilde{H}_{d-i-1}(\Sigma; k) \cong \tilde{H}^{i-1}(s^{-1}(\Delta), s^{-1}(\mathrm{del}_\Delta(G)); k)$$
>
> where $d - 1 = \dim \Delta$.

Starting with the right-hand side we have

$$\begin{aligned}\tilde{H}^{i-1}(s^{-1}(\Delta), s^{-1}(\mathrm{del}_\Delta(G)); k) &= \tilde{H}^{i-1}(s^{-1}(\Delta), s^{-1}(\partial\Delta); k)\\ &\cong \tilde{H}^{i-1}(\|\Delta\| - \|\Sigma\|, \|\partial\Delta\| - \|\Sigma\|; k)\\ &\cong \tilde{H}_{d-i-1}(\Sigma; k)\end{aligned}$$

where the second-to-last isomorphism uses Proposition 6.5, and the last isomorphism is the disguised Alexander duality lemma, Lemma 6.8.

6. Appendix: tools from combinatorial topology

In this section we collect various definitions and facts from combinatorial topology needed in the paper. Good references for much of this material are [15], [3, Section 4.7], [2].

Let Δ be a *simplicial complex* on vertex set V, that is a collection of subsets F (called *faces*) of V which is closed under inclusion. The *dimension* $\dim F$ of a face F is one less than its cardinality $|F|$, and the dimension $\dim\Delta := \max_{F\in\Delta}\dim F$. A maximal face under inclusion is called a *facet*. The topological space which is the *geometric realization* of Δ will be denoted $\|\Delta\|$. We write $X \simeq Y$ to mean that the topological spaces X, Y are homotopy equivalent, and $X \cong Y$ to mean that they are homeomorphic. When using these symbols with simplicial complexes, we mean that the corresponding geometric realizations are homotopy equivalent or homeomorphic.

Given a face F of Δ, we can define three subcomplexes called the *star, deletion,* and *link* of F within Δ as follows:

$$\begin{aligned}\mathrm{star}_\Delta(F) &:= \{G \in \Delta : G \cup F \in \Delta\}\\ \mathrm{del}_\Delta(F) &:= \{G \in \Delta : F \not\subseteq G\}\\ \mathrm{lk}_\Delta(F) &:= \{G \in \Delta : G \cup F \in \Delta, G \cap F = \emptyset\}\end{aligned} \tag{6.1}$$

Note, that $\mathrm{lk}_\Delta(F)$ is a simplicial complex on the vertex set $V - F$. Given two simplicial complexes Δ_1, Δ_2 on disjoint vertex sets V_1, V_2, their *simplicial join* $\Delta_1 * \Delta_2$ is the simplicial complex on vertex set $V_1 \amalg V_2$ having a face $F_1 \amalg F_2$ for every F_i in Δ_i, $i = 1, 2$. One can also define the corresponding topological join $X * Y$ of two spaces X, Y (see [15, Section 62]) so that

$$\|\Delta_1 * \Delta_2\| \cong \|\Delta_1\| * \|\Delta_2\|$$

With these definitions, one can check that

$$\begin{aligned}\mathrm{star}_\Delta(F) \cup \mathrm{del}_\Delta(F) &= \Delta\\ \mathrm{star}_\Delta(F) \cap \mathrm{del}_\Delta(F) &= \partial F * \mathrm{lk}_\Delta(F)\\ \mathrm{star}_\Delta(F) &= 2^F * \mathrm{lk}_\Delta(F)\end{aligned} \tag{6.2}$$

where ∂F denotes the boundary complex of F, i.e., all proper subsets of F.

The *suspension* $\mathrm{Susp}\,\Delta$ of a simplicial complex (or a topological space) Δ is the join $\mathbb{S}^0 * \Delta$, where $\mathbb{S}^0$ is a complex (space) having two disconnected vertices. More generally, when Δ_1 triangulates a d-sphere, one has $\Delta_1 * \Delta_2 \cong \mathrm{Susp}^{d+1}\Delta_2$.

For a subspace Y of a topological space X, we denote X/Y the *quotient space* that shrinks Y to a point. The following fact is well-known

Proposition 6.1. *For any subcomplex Δ' of a simplicial complex Δ, one has*

$$\tilde{H}^\bullet(\Delta, \Delta'; k) \cong \tilde{H}^\bullet(\|\Delta\|, \|\Delta'\|; k) \cong \tilde{H}^\bullet(\|\Delta\|/\|\Delta'\|; k)$$

where the cohomology in the leftmost expression is simplicial, and in the two on the right is singular. □

The following proposition is a straightforward and well-known consequence of Equations (6.2).

Proposition 6.2. *For any face F in a simplicial complex Δ, we have*

(i)
$$\|\Delta\|/\|\operatorname{del}_\Delta(F)\| \simeq \operatorname{Susp}^{|F|} \|\operatorname{lk}_\Delta(F)\|.$$

(ii)
$$\tilde{H}^i(\Delta, \operatorname{del}_\Delta(F); k) \cong \tilde{H}_{i-|F|}(\operatorname{lk}_\Delta F; k). \quad \square$$

Let Sd denote the barycentric subdivision operator on simplicial complexes [15, Section 15]. Given a subcomplex Σ of a simplicial complex Δ, let $\operatorname{Sd}(\Delta - \Sigma)$ denote the subcomplex of $\operatorname{Sd}\Delta$ induced on the vertices which are not barycenters of faces in Σ.

Proposition 6.3 (c.f. [3, Lemma 4.7.27])**.** *Let Δ be a simplicial complex, and Δ', Σ two subcomplexes. Then the pair of spaces $(\|\Delta\| - \|\Sigma\|, \|\Delta'\| - \|\Sigma\|)$ is relatively homotopy equivalent to the (geometric realizations) of the pair*

$$(\operatorname{Sd}(\Delta - \Sigma), \quad \operatorname{Sd}(\Delta' - \Sigma)). \quad \square$$

We will sometimes wish to refer to the topology of a finite poset P, by which we mean the geometric realization of its *order complex*, the simplicial complex on vertex set P having the totally ordered subsets of P as faces.

Lemma 6.4. *(Relative Quillen Fiber Lemma) Let $f : P \to Q$ be an order-preserving map of posets, and $Q' \subseteq Q$ be an induced subposet. If the posets $f^{-1}(Q_{\leq q})$ and $f^{-1}(Q'_{\leq q'})$ are contractible for every $q \in Q, q' \in Q'$, then f induces a relative homotopy equivalence of the pairs $(P, f^{-1}(Q'))$ and (Q, Q').*

Proof. By the usual Quillen Fiber Lemma [2, (10.11)] [3, Lemma 4.7.29], f induces homotopy equivalences between P and Q and between $f^{-1}(Q')$ and Q'. But this is sufficient for f to induce a relative homotopy equivalence of the pairs by [5, Lemma 3]. □

We recall some terminology which was used in Section 3. Let Σ be a subcomplex of a simplicial complex Δ on vertex set $[n]$, with $I_\Sigma = (m_1, \ldots, m_\mu) + I_\Delta$, i.e., $m_1, \ldots, m_\mu$ are monomials corresponding to the minimal faces of $\Delta - \Sigma$. Define the order-preserving map

$$2^{[\mu]} \xrightarrow{s} 2^{[n]}$$
$$F \overset{s}{\mapsto} \operatorname{supp} \prod_{i \in F} m_i.$$

Note, that since s is order-preserving, $s^{-1}(\Delta)$ is a simplicial complex on vertex set $[\mu]$, as is $s^{-1}(\Delta')$ for any other subcomplex Δ' of Δ. When we wish to emphasize the dependence of this map s on the pair (Δ, Σ), we will write $s_{\Delta,\Sigma}$. However, note that if Δ' is a subcomplex of Δ, then those monomials in $\{m_i\}_{i=1}^{\mu}$ which correspond to faces of Δ' generate the image of $I_{\Sigma\cap\Delta'}$ in $k[\Delta']$, and as a consequence,

$$s^{-1}_{\Delta,\Sigma}(\Delta') = s^{-1}_{\Delta',\Sigma\cap\Delta'}(\Delta'). \tag{6.3}$$

Proposition 6.5. *If Δ' is a subcomplex of Δ, then $(s^{-1}_{\Sigma}(\Delta), s^{-1}_{\Sigma}(\Delta'))$ is relatively homotopy equivalent to the pair of spaces $(\|\Delta\| - \|\Sigma\|, \|\Delta'\| - \|\Sigma\|)$. In particular, $s^{-1}_{\Sigma}(\Delta)$ is homotopy equivalent to $\|\Delta\| - \|\Sigma\|$.*

Proof. The first assertion specializes to the second for $\Delta' = \emptyset$.

To prove the first assertion, note that by Proposition 6.3 it is equivalent to show $(s^{-1}_{\Sigma}(\Delta), s^{-1}_{\Sigma}(\Delta'))$ is relatively homotopy equivalent to the pair $(\mathrm{Sd}(\Delta-\Sigma), \mathrm{Sd}(\Delta'-\Sigma))$. By the definition of barycentric subdivision, the latter is a pair of order complexes for the posets of faces in $(\Delta - \Sigma, \Delta' - \Sigma)$. Thus if we momentarily regard $\Delta - \Sigma, \Delta' - \Sigma$ as posets, the result would follow from Lemma 6.4 if we can show

- $s^{-1}_{\Delta,\Sigma}(\Delta) = s^{-1}_{\Delta,\Sigma}(\Delta - \Sigma)$ (and similarly for Δ'), and
- for each face F in Δ but not in Σ, the subposet $s^{-1}_{\Delta,\Sigma}((\Delta - \Sigma)_{\subseteq F})$ is contractible (and similarly for Δ').

To see the first assertion, note that if $G \subseteq [\mu]$ lies in $s^{-1}_{\Delta,\Sigma}(\Delta)$ then this means $\mathrm{supp}(\prod_{j\in G} m_j)$ is a face F of Δ which does not lie in Σ (since its support contains the support of at least one m_j in I_Σ). For the second assertion, note that $s^{-1}_{\Delta,\Sigma}((\Delta - \Sigma)_{\subseteq F})$ contains a unique maximum element, namely $\{j \in [\mu] \mid \mathrm{supp}\, m_j \subseteq F\}$. $\square$

The next proposition follows immediately using Equations (6.2) and excision.

Proposition 6.6.

$$\tilde{H}^i(s^{-1}(\Delta), s^{-1}(\mathrm{del}_\Delta(F)); k) \cong \tilde{H}^i(s^{-1}(\mathrm{star}_\Delta F), s^{-1}(\mathrm{del}_{\mathrm{star}_\Delta(F)}(F); k). \quad \square$$

The next proposition is used in Section 5 in showing the equivalence of Corollary 5.2 and Theorem 3.2 when $k[\Delta]$ is Gorenstein.

Proposition 6.7. *Let Φ be a simplicial complex, $\Phi' \subseteq \Phi$ a subcomplex. Let Δ, Δ' be simplicial complexes which are joins*

$$\Delta := 2^F * \Phi$$
$$\Delta' := 2^F * \Phi'$$

in which F is a simplex on a vertex set disjoint from that of Φ, Φ'. Assume Σ is a subcomplex of Δ containing the face F. Then the pair $(s^{-1}_{\Delta,\Sigma}(\Delta), s^{-1}_{\Delta,\Sigma}(\Delta'))$ is relatively homotopy equivalent to the pair $(s^{-1}_{\Phi,\mathrm{lk}_\Sigma(F)}(\Phi), s^{-1}_{\Phi,\mathrm{lk}_\Sigma(F)}(\Phi'))$.

Equivalently, via Proposition 6.5, the pair $(\|\Delta\| - \|\Sigma\|, \|\Delta'\| - \|\Sigma\|)$ is relatively homotopy equivalent to the pair $(\|\Phi\| - \|\mathrm{lk}_\Sigma(F)\|, \|\Phi'\| - \|\mathrm{lk}_\Sigma(F)\|)$.

Proof. By Proposition 6.3, it suffices to show $(\mathrm{Sd}(\Delta - \Sigma), \mathrm{Sd}(\Delta' - \Sigma))$ is relatively homotopy equivalent to $(\mathrm{Sd}(\Phi - \mathrm{lk}_\Sigma(F)), \mathrm{Sd}(\Phi' - \mathrm{lk}_\Sigma(F)))$.

Regarding these as pairs of order complexes for the pairs of posets $(\Delta - \Sigma, \Delta' - \Sigma)$ and $(\Phi - \mathrm{lk}_\Sigma(F), \Phi - \mathrm{lk}_\Sigma(F))$, we can try to apply Lemma 6.4. To this end, define an order-preserving poset map

$$\Delta - \Sigma \xrightarrow{f} \Delta$$
$$G \xmapsto{f} G \cup F.$$

Every face in the image of f clearly contains F, so one can define a map on the image in the reverse direction that sends $H \mapsto H - F$. It is easy to check that this gives an isomorphism between the image poset $f(\Delta - \Sigma)$ (resp. $f(\Delta' - \Sigma)$) and the poset $\Phi - \mathrm{lk}_\Sigma(F)$ (resp. $\Phi' - \mathrm{lk}_\Sigma(F)$). Thus it only remains to show that f induces a relative homotopy equivalence, i.e., the subposet $f^{-1}(\Delta_{\subseteq H})$ of $\Delta - \Sigma$ is contractible for every H in the image of f. But $f^{-1}(\Delta_{\subseteq H})$ has H as its unique maximum element H. □

Our last result is a disguised form of Alexander duality, which turns out to be the crux of the equivalence of Corollary 5.2 and Theorem 3.2 when $k[\Delta]$ is Gorenstein.

Lemma 6.8. *Let $\mathbb{S}$ be a simplicial homology sphere (over some coefficient ring), and $\Delta = 2^F * \mathbb{S}$ for some simplex F (possibly empty) on a vertex set disjoint from $\mathbb{S}$. Let $d = \dim \Delta$, and Σ a subcomplex of Δ.*

Then

$$\tilde{H}_i(\Sigma; k) \cong \tilde{H}^j(\|\Delta\| - \|\Sigma\|, \|\partial\Delta\| - \|\Sigma\|; k)$$

for $i + j = d - 1$, where $\partial\Delta = \partial F * \mathbb{S}$ (and the coefficient ring has been suppressed for the sake of brevity).

Proof. First deal with the easy case where F is empty. Then $\Delta = \mathbb{S}$ is a homology d-sphere, and $\partial\Delta = \emptyset$. Consequently

$$\begin{aligned}\tilde{H}^j(\|\Delta\| - \|\Sigma\|, \|\partial\Delta\| - \|\Sigma\|; k) &\cong \tilde{H}^j(\|\mathbb{S}\| - \|\Sigma\|, \emptyset; k)\\ &\cong \tilde{H}^j(\|\mathbb{S}\| - \|\Sigma\|; k)\\ &\cong \tilde{H}_i(\Sigma; k) \text{ for } i + j = d - 1,\end{aligned}$$

where the last isomorphism is Alexander duality within $\mathbb{S}$ [15, Section 71].

If $F \neq \emptyset$, then ∂F is a sphere, and hence $\partial\Delta = \partial F * \mathbb{S}$ is another homology sphere. The key idea is to enlarge Δ in a controlled way, so as to obtain yet another homology sphere $\hat{\mathbb{S}}$, use excision, and then rely on Alexander duality within $\hat{\mathbb{S}}$. Define this new homology sphere $\hat{\mathbb{S}}$ by adding a cone vertex v over the subcomplex $\partial\Delta$:

$$\hat{\mathbb{S}} := \Delta \cup (v * \partial\Delta).$$

Proposition 6.9. *$\hat{\mathbb{S}}$ is a homology d-sphere.*

Assuming this proposition for the moment, we can complete the proof of Lemma 6.8. We have the following isomorphisms, which are justified below:

$$\begin{aligned}\tilde{H}^j(\|\Delta\| - \|\Sigma\|, \|\partial\Delta\| - \|\Sigma\|; k) &\cong \tilde{H}^j(\|\hat{\mathbb{S}}\| - \|\Sigma\|, \|v * \partial\Delta\| - \|\Sigma\|; k) \\ &\cong \tilde{H}^j(\|\hat{\mathbb{S}}\| - \|\Sigma\|; k) \\ &\cong \tilde{H}_i(\Sigma; k) \text{ for } i + j = d - 1.\end{aligned}$$

The first isomorphism above comes from excision: we excise away

$$\big(\|v * \partial\Delta\| - \|\partial\Delta\|\big) - \|\Sigma\|$$

from the pair on the right-hand side.

The second isomorphism above is due to the fact that $\|v * \partial\Delta\| - \|\Sigma\|$ is a contractible subspace of $\|\hat{\mathbb{S}}\| - \|\Sigma\|$: it is star-shaped with respect to v, since Σ can only intersect the cone $v * \partial\Delta$ in a subset of its base $\partial\Delta$. Consequently the homology of $\|v * \partial\Delta\| - \|\Sigma\|$ vanishes, and then the long exact sequence for the pair $(\|\hat{\mathbb{S}}\| - \|\Sigma\|, \|v * \partial\Delta\| - \|\Sigma\|)$ gives the isomorphism.

The last isomorphism above is Alexander duality within $\hat{\mathbb{S}}$, using Proposition 6.9. $\square$

Proof of Proposition 6.9. We must show for every face G in $\hat{\mathbb{S}}$, that $\mathrm{lk}_{\hat{\mathbb{S}}}(G)$ has the homology of a sphere of the same dimension as $\mathrm{lk}_{\hat{\mathbb{S}}}(G)$.

We first deal with the case where $G = \emptyset$. That is, we must show $\hat{\mathbb{S}}$ itself has the homology of a d-sphere. This follows from Mayer-Vietoris applied to

$$\begin{aligned}\hat{\mathbb{S}} &= \Delta \cup (v * \partial\Delta) \\ \partial\Delta &= \Delta \cap (v * \partial\Delta)\end{aligned}$$

and the fact that $\partial\Delta = \partial F * \mathbb{S}$ is a homology sphere.

We now deal with two other cases for G, depending on whether it contains v.
Case 1: $v \notin G$. If we write $G = F' \amalg G'$ where $F' \subseteq F$ and G' is a face of $\mathbb{S}$, then one easily checks that

$$\mathrm{lk}_{\hat{\mathbb{S}}}\, G = \Delta' \cup (v * \partial\Delta')$$

where Δ' is the join $2^{F-F'} * \mathrm{lk}_{\mathbb{S}}\, G'$. Since Δ' is again a simplex joined with a homology sphere (like Δ was), this link has the same form as $\hat{\mathbb{S}}$, and hence has the correct homology by the $G = \emptyset$ case that we just did.
Case 2: $v \in G$. Then one easily checks that

$$\mathrm{lk}_{\hat{\mathbb{S}}}\, G = \mathrm{lk}_{v*\partial\Delta}\, G = \mathrm{lk}_{\partial\Delta}(G - v).$$

Hence we are done in this case because $\partial\Delta$ is a homology $(d-1)$-sphere. $\square$

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