FAKE DEGREES FOR REFLECTION ACTIONS ON ROOTS

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ABSTRACT. A finite irreducible real reflection group of rank ℓ and Coxeter number h has root system of cardinality $h{\cdot}\ell$. It is shown that the *fake degree* for the permutation action on its roots is divisible by $[h]_q = 1+q+q^2+\cdots+q^{h-1}$, and that in simply-laced types it equals $[h]_q \cdot \sum_{i=1}^{\ell} q^{d_i^*}$ where $d_i^* = e_i - 1$ are the *codegrees* and e_i are the *exponents*.

1. INTRODUCTION

Consider a complex reflection group $W \subset GL(V)$ with $V = \mathbb{C}^\ell$, acting by linear substitutions on the polynomial algebra $S = \mathrm{Sym}(V^*) \cong \mathbb{C}[x_1, \ldots, x_n]$. Both Shephard and Todd [9] and Chevalley [2] proved that the invariant subalgebra is again a polynomial algebra $S^W = \mathbb{C}[f_1, \ldots, f_\ell]$ for some homogeneous polynomials f_i, and that the *coinvariant algebra* S/I where $I = (f_1, \ldots, f_\ell)$ carries a graded version of the regular representation. Thus for any finite-dimensional $\mathbb{C}W$-module U, the intertwiner space $\mathrm{Hom}_W(U, S/I) \cong (U^* \otimes S/I)^W$ is a graded $\mathbb{C}$-vector space, whose *q-dimension* or *Hilbert series* has been called its *fake degree*

$$f^U(q) = \mathrm{Hilb}(\ \mathrm{Hom}_W(U, S/I)\ ,\ q).$$

Since $f^U(1) = \dim_{\mathbb{C}} \mathrm{Hom}(U, \mathbb{C}W) = \dim_{\mathbb{C}} U$, one may regard $f^U(q)$ as a q-analogue of the degree $\dim_{\mathbb{C}} U$. For example, the fake degree $f^{V^*}(q)$ of the *dual reflection representation* V^* is determined by the *degrees* $d_1 \le \cdots \le d_\ell$ of the invariants $f_1, \ldots, f_\ell$ via[1] $f^{V^*}(q) = \sum_{i=1}^{\ell} q^{d_i - 1}$. One also defines the *codegrees* $d_1^* \le \cdots \le d_\ell^*$ via the fake degree polynomial $f^V(q) = \sum_{i=1}^{\ell} q^{d_i^* + 1}$ of the representation V itself.

We focus here on the case where W acts on $V = \mathbb{C}^\ell$ as the complexification of an irreducible *real* reflection group, so that one has $V \cong V^*$ and $f^V(q) = f^{V^*}(q)$. In this setting, one defines the *exponents* $(e_1, \ldots, e_\ell)$ by $e_i = d_i - 1 = d_i^* + 1$, and the *Coxeter number* $h = d_\ell$. Choose a *root system* Φ, containing one opposite pair $\{\pm\alpha\}$ of normals to each reflecting hyperplane, stable under the W-action. Given any W-stable subset Φ' of Φ, we will consider the fake degree polynomial $f^{\Phi'}(q) := f^U(q)$ for the W-permutation action $U = \mathbb{C}\Phi'$. Recall [1, Chap. VI, §1, no. 2], [3, S 3.18] that the cardinality of Φ has formula $|\Phi| = h \cdot \ell$.

Theorem 1. *Let W be an irreducible finite real reflection group, with root system Φ, and Coxeter number h. Then for any W-stable subset of Φ' of Φ,*

(i) *$f^{\Phi'}(q)$ is divisible by $[h]_q = \frac{1-q^n}{1-q}$, and*

(ii) *when W is simply-laced, $f^{\Phi}(q) = [h]_q \cdot (q^{d_1^*} + \cdots + q^{d_\ell^*})$.*

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[1]This follows as a consequence of Solomon's result [10] that the W-invariant *differential forms* with polynomial coefficients $S \otimes \wedge^k V$ form a free S^W-module with basis elements $df_{i_1} \wedge \cdots \wedge df_{i_k}$.

2. Proof of assertion (*i*)

In the proof, one may assume without loss of generality that W acts transitively on the subset Φ' of Φ. The desired divisibility will then be deduced from Lemma 2 below, applied to a *Coxeter element* of W. The statement of the lemma involves Springer's notion [11] of a *regular element* c in W, with a *regular eigenvalue* ζ, meaning $c(v) = \zeta v$ for an eigenvector v lying on none of the reflecting hyperplanes for W. Then c and ζ have the same multiplicative order n. Denote by C the cyclic subgroup $\langle c \rangle$ generated by c.

Lemma 2. [8, Thm. 8.2] *Let W be a complex reflection group acting transitively on a finite set X, and c in W a regular element of order n, with a regular eigenvalue ζ. Then for all m, the fake degree $f^X(q) := f^U(q)$ for the W-permutation action $U = \mathbb{C}X$ satisfies*

$$f^X(\zeta^m) \quad = \quad \#\{x \in X : c^m(x) = x\}.$$

In particular, $f^X(q)$ is divisible by $[n]_q$ if and only if C acts freely on X.

Proof. For the sake of completeness, we recall the proof from [8]. Springer [11] extended the work of Shephard-Todd and Chevalley by proving one has an isomorphism $W \times C$-representations

$$S/I \cong \mathbb{C}W \tag{2.1}$$

where W acts as before, and where C acts on $\mathbb{C}W$ via *right-translation*, and on S/I via scalar substitutions $c(x_i) = \zeta^{-1} \cdot x_i$. Equivalently, c scales the d^{th} homogeneous component $(S/I)_d$ by the scalar ζ^{-d}.

Now identify the transitive W-permutation representation $\mathbb{C}X$ with a coset action $\mathbb{C}[W/W']$ for some subgroup W' of W. Then one has an isomorphism $\mathrm{Hom}_W(\mathbb{C}[W/W'], S/I) \cong (S/I)^{W'}$, and one can reformulate the fake degree:

$$f^X(q) = \mathrm{Hilb}((S/I)^{W'}, q). \tag{2.2}$$

Taking W'-fixed spaces in (2.1) give an isomorphism of C-representations

$$(S/I)^{W'} \cong (\mathbb{C}W)^{W'} \cong \mathbb{C}X \tag{2.3}$$

and the result now follows by comparing the trace of c^m on the two ends of (2.3). □

To finish the proof of assertion (i), one applies Lemma 2 to a finite *real* reflection group W, with *Coxeter generators* $S = \{s_1, \ldots, s_\ell\}$, and with $c = s_1 s_2 \cdots s_\ell$ a *Coxeter element*. It is known that all Coxeter elements lie in a single W-conjugacy class, that they have multiplicative order $h = d_\ell$, and that they are regular elements having $\zeta = e^{\frac{2\pi i}{h}}$ as a regular eigenvalue; see [3, §3.16, 3.17]. Furthermore, it is known [1, Chap. V, §1, no. 11] that the cyclic group C generated by a Coxeter element c acts freely on the roots Φ. Assertion (i) now follows from Lemma 2.

3. Proof of assertion (*ii*)

We first recall a bit more of the root geometry for finite real reflection groups, in order to further reformulate the fake degree $f^{\Phi'}(q)$; see e.g. [3, Chapters 1, 5].

Assume W is the complexification of a real reflection group acting on $V_{\mathbb{R}} \cong \mathbb{R}^\ell$, that preserves a positive definite inner product $(-,-)$ on $V_{\mathbb{R}}$. The reflecting hyperplanes dissect $V_{\mathbb{R}}$ into open simplicial cones called *chambers*, which are permuted simply-transitively by W. Choosing one such chamber C to be the *dominant chamber*, every W-orbit contains exactly one point in its closure $\bar{C}$. The root system decomposes as $\Phi = \Phi_+ \sqcup -\Phi_+$, where the positive roots Φ_+ are those having positive inner product with the points of C. This also distinguishes the subset of *simple roots* $\{\alpha_1, \ldots, \alpha_\ell\}$ inside Φ_+, whose nonnegative linear combinations contain Φ_+, and whose corresponding *simple reflections* $S = \{s_1, \ldots, s_\ell\}$ gives rise to a Coxeter presentation (W, S) for W. The above discussion implies that every W-orbit of roots contains a unique *dominant* representative α_0 lying in $\bar{C}$, whose isotropy subgroup W_{α_0} is a *standard parabolic subgroup* generated by some[2] subset S.

Proposition 3. *Let W be a finite real reflection group W with root system Φ and positive roots Φ_+. Let Φ' be a W-orbit of roots, with unique dominant representative α_0. Then the fake degree for the W-permutation action on Φ' can be expressed as*

$$f^{\Phi'}(q) = \sum_{\alpha \in \Phi'} q^{d(\alpha_0, \alpha)}$$

where $d(\alpha_0, \alpha)$ is the Coxeter group length $\ell_S(w)$ of the minimum length representative w for the coset $wW_{\alpha_0} = \{u \in W : u(\alpha_0) = \alpha\}$.

Proof. Note that S is a free S^W-module, because $S^W = \mathbb{C}[f_1, \ldots, f_\ell]$ is a polynomial ring. One obtains S^W-module splittings for the ring inclusions $S^{W_{\alpha_0}} \subset S$ and $S^W \subset S^{W_{\alpha_0}}$ by averaging over W_{α_0} and over coset representatives for W/W_{α_0}, respectively. Hence $S^{W_{\alpha_0}}$ is also a free S^W-module, with

$$f^{\Phi'}(q) = \mathrm{Hilb}((S/I)^{W_{\alpha_0}}, q) = \frac{\mathrm{Hilb}(S^{W_{\alpha_0}}, q)}{\mathrm{Hilb}(S^W, q)}.$$

For any standard parabolic subgroup W' of W, such as $W' = W_{\alpha_0}$ or $W' = W$ itself, one has [3, §3.15] that $\mathrm{Hilb}(S^{W'}, q)^{-1} = (1-q)^\ell \sum_{w \in W'} q^{\ell_S(w)}$. Therefore

$$f^{\Phi'}(q) = \frac{\sum_{w \in W} q^{\ell_S(w)}}{\sum_{w \in W_{\alpha_0}} q^{\ell_S(w)}} = \sum_{w} q^{\ell_S(w)} \tag{3.1}$$

where in this last sum, w runs over the minimum-length coset representatives for the cosets wW_{α_0} in W/W_{α_0}. □

The crux of the proof of assertion (ii) will be the following lemma[3]. It relates, for simply-laced root systems with highest root α_0, the quantity $d(\alpha_0, \alpha)$ to the *root height* of α, which we recall here; see [1, Chap. VI, §8], [3, §3.20], [12, §3] for further discussion. When W is a crystallographic root system Φ, with simple roots $\{\alpha_1, \ldots, \alpha_\ell\}$, for every root α in Φ the unique expression $\alpha = \sum_{i=1}^{\ell} c_i \alpha_i$ has *integer* coefficients c_i, and one defines the *height* $\mathrm{ht}(\alpha) = \sum_{i=1}^{\ell} c_i$. There is a unique *highest root* α_0, achieving the maximum height $\mathrm{ht}(\alpha_0) = h - 1$, and this highest root α_0 is always dominant.

Lemma 4. *Let W be a simply-laced root Weyl group with root system Φ, positive roots Φ_+, and highest root α_0. Then any root α in Φ has*

$$d(\alpha_0, \alpha) = \begin{cases} \mathrm{ht}(\alpha_0) - \mathrm{ht}(\alpha) & \text{if } \alpha \in \Phi_+, \\ \mathrm{ht}(\alpha_0) - \mathrm{ht}(\alpha) - 1 & \text{if } \alpha \in -\Phi_+. \end{cases}$$

[2] Although we will not need this information here, the table at the beginning of Section 4 lists the type for these standard parabolic subgroups W_{α_0}. When W is crystallographic and α_0 is the highest root, W_{α_0} is generated by the simple reflections of W not adjacent to the extra node s_0 in the extended Dynkin diagram for the affine Weyl group $\widetilde{W}$.

[3] This lemma is similar in spirit to results of Stembridge [12, §2,3] on a quantity that he calls the *depth* $d(\alpha)$ of the root α, closely related to the quantity $d(\alpha_0, \alpha)$ defined here.

Proof. Rescale all roots α so that $(\alpha, \alpha) = 2$, and consequently (α, β) lies in $\{0, \pm 1, \pm 2\}$ for all pairs of roots α, β. For any simple root, the formula

$$s_i(\beta) = \beta - (\beta, \alpha_i)\alpha_i$$

shows that applying the simple reflection s_i to a root $\beta \neq \pm\alpha_i$ has the following effect on its height:

$$\mathrm{ht}(s_i\beta) = \begin{cases} \mathrm{ht}(\beta) & \text{if } (\beta, \alpha_i) = 0 \\ \mathrm{ht}(\beta) + 1 & \text{if } (\beta, \alpha_i) = -1 \\ \mathrm{ht}(\beta) - 1 & \text{if } (\beta, \alpha_i) = +1. \end{cases}$$

When $\beta = \pm\alpha_i$, one has $\mathrm{ht}(\beta) = \pm 1$, and $\mathrm{ht}(s_i(\beta)) = -\mathrm{ht}(\beta) = \mp 1$.

Consequently, when starting with the highest root α_0, and applying a sequence of simple reflections s_i, the height can drop by at most one at each stage, except when one crosses from a simple root to its negative. This implies that the expression on the right side in the lemma (call it $b(\alpha)$) gives a lower bound on the length $\ell_S(w)$ for any w sending α_0 to α. Thus $d(\alpha_0, \alpha) \geq b(\alpha)$.

To show $d(\alpha_0, \alpha) \leq b(\alpha)$, induct on $b(\alpha)$. In the base case $b(\alpha) = 0$, so $\alpha = \alpha_0$ and $d(\alpha_0, \alpha) = 0$ also. In the inductive step, $b(\alpha) \neq 0$ implies $\alpha \neq \alpha_0$, so (as we are in the simply-laced case) α is not dominant, and there exists some simple root α_i with $(\alpha, \alpha_i) < 0$. It suffices to show that $b(s_i\alpha) = b(\alpha) - 1$.

If $(\alpha, \alpha_i) = -1$ then $\mathrm{ht}(s_i\alpha) = \mathrm{ht}(\alpha) + 1$, and either both $\alpha, s_i(\alpha)$ lie in Φ_+ or both lie in $-\Phi_+$, so $b(s_i\alpha) = b(\alpha) - 1$.

If $(\alpha, \alpha_i) = -2$ then $\alpha = -\alpha_i$, so that $s_i\alpha = +\alpha_i$, and again $b(s_i\alpha) = b(\alpha) - 1$. □

The proof of assertion (ii) requires one more well-known fact [3, §3.20], relating the distribution of root heights to the exponents $e_i = d_i^* + 1$:

$$\sum_{\alpha \in \Phi_+} q^{\mathrm{ht}(\alpha)} = \sum_{i=1}^{\ell} (q^1 + q^2 + \cdots + q^{e_i}). \tag{3.2}$$

For W simply-laced, there is only one orbit Φ, whose dominant root α_0 is the highest root, with $\mathrm{ht}(\alpha_0) = h - 1$. Combining Proposition 3, Lemma 4, (3.2) gives

$$\begin{aligned} f^{\Phi}(q) &= \sum_{\alpha \in \Phi_+} q^{h-1-\mathrm{ht}(\alpha)} + \sum_{\alpha \in -\Phi_+} q^{h-2-\mathrm{ht}(\alpha)} \\ &= \sum_{i=1}^{\ell} (q^{h-e_i-1} + q^{h-e_i} + \cdots + q^{h-2}) + (q^{h-1} + q^h + \cdots + q^{h+e_i-2}) \\ &= (1-q)^{-1} \sum_{i=1}^{\ell} (q^{h-e_i-1} - q^{h+e_i-1}) \\ &= (1-q)^{-1} \sum_{i=1}^{\ell} (q^{e_i-1} - q^{h+e_i-1}) \end{aligned}$$

where the last equality used the fact [3, §3.16] that $h - e_i = e_{\ell+1-i}$. Therefore

$$f^{\Phi}(q) = \frac{1-q^h}{1-q} \cdot \sum_{i=1}^{\ell} q^{e_i-1} = [h]_q \cdot \sum_{i=1}^{\ell} q^{d_i^*}$$

as desired.

4. Remarks and questions

4.1. **Further divisibilities.** The table below records the polynomial $f^{\Phi'}(q)/[h]_q$ for all root orbits Φ' in real reflection groups. The last column tabulates the additional data $\gcd([h]_q, \sum_{i=1}^{\ell} q^{d_i^*})$, relevant for Proposition 5 below.

W	h	$\Phi' = W.\alpha_0$	W_{α_0} type	$f^{\Phi'}(q)/[h]_q$	$\gcd([h]_q, \sum_i q^{d_i^*})$
A_{n-1}	n	Φ	A_{n-3}	$[n-1]_q$	1
B_n	$2n$	$\{\pm e_i \pm e_j\}$ $\{\pm e_i\}$	$A_1 \times B_{n-2}$ B_{n-1}	$[n-1]_{q^2}$ 1	$[n]_{q^2}$
D_n	$2(n-1)$	Φ	$A_1 \times D_{n-2}$	$\frac{[n-2]_{q^2}[n]_q}{[2]_q}$	1
E_6	12	Φ	A_5	$[2]_{q^4}[3]_{q^3}$	1
E_7	18	Φ	D_6	$\frac{[2]_{q^6}}{[2]_{q^2}}[7]_{q^2}$	1
E_8	30	Φ	E_7	$[2]_{q^{10}}[4]_{q^6}$	1
F_4	12	either orbit	B_3	$[2]_{q^4}$	$[2]_{q^6}$
H_3	10	Φ	$A_1 \times A_1$	$[3]_{q^2}$	1
H_4	30	Φ	H_3	$[2]_{q^6}[2]_{q^{10}}$	1
$I_2(m)$ m even	m	either orbit	A_1	1	1 if $\frac{m}{2}$ odd $[2]_{q^2}$ if $\frac{m}{2}$ even
$I_2(m)$ m odd	m	Φ	$-$	$[2]_q$	1

The table exhibits case-by-case two facts for which we lack uniform proofs.

Proposition 5. *For finite real W with one root orbit,* $\gcd([h]_q, \sum_{i=1}^{\ell} q^{d_i^*}) = 1$.

Using (2.1), Proposition 5 is equivalent to the assertion that, when W has only one orbit of roots, every power c^m of a Coxeter element c acts on V with nonzero trace.

Proposition 6. *For finite real W which are* **at most doubly-laced***, meaning that its Coxeter presentation relations $(s_i s_j)^{m_{ij}} = e$ all have $m_{ij} \leq 4$, every W-stable root subset Φ' has fake degree $f^{\Phi'}(q)$ divisible by $\sum_{i=1}^{\ell} q^{d_i^*}$.*

4.2. **Original motivation.** Theorem 1 was originally observed case-by-case while computing the fake degree of a certain *irreducible* representation of simply-laced W, arising naturally in [7, Chapter 3]. One can decompose the W-permutation representation $\mathbb{C}[\Phi']$ of any real reflection group W on a root orbit Φ' into two direct summands, namely its *symmetric* and *antisymmetric* components $\mathbb{C}[\Phi']^+, \mathbb{C}[\Phi']^-$ with respect to the W-equivariant involution that simultaneously swaps each $+\alpha, -\alpha$. A straightforward calculation then shows the following.

Proposition 7. *Let W be a finite real reflection group W and Φ' an orbit of its roots. Then any one of the three fake degrees for $\mathbb{C}[\Phi'], \mathbb{C}[\Phi']^+, \mathbb{C}[\Phi']^-$ determines the others via the relations $f^{\Phi'}(q) = f^{\Phi',+}(q) + f^{\Phi',-}(q)$ and $f^{\Phi',-}(q) = q \cdot f^{\Phi',+}(q)$.*

It was further shown in [7, Chapter 3] that, for irreducible real reflection groups W, and any root orbit Φ', the antisymmetric component $\mathbb{C}[\Phi']^-$ has W-irreducible decomposition which is *multiplicity-free.* In the simply-laced case, it has only two irreducible constituents: $\mathbb{C}[\Phi]^- = V \oplus U$ where V is the reflection representation V of degree ℓ, and U is another W-irreducible, of degree $|\Phi^+| - \ell = \frac{h-2}{2} \cdot \ell$. Using

Proposition 7, one can check that Theorem 1(ii) is equivalent to the assertion that this W-irreducible U has fake degree $f^U(q) = q^2 \cdot \frac{[h-2]_q}{[2]_q} \cdot \sum_{i=1}^{\ell} q^{e_i}$.

4.3. **M-V cycles.** Lemma 4 has a geometric interpretation. It is well-known that for a standard parabolic subgroup W' of a Weyl group W associated to simple complex algebraic group G and Borel subgroup B, one can identify the invariant subalgebra $(S/I)^{W'}$ with the cohomology $H^*(G/P)$ of G/P where $P = \langle B, W' \rangle$. The Schubert cell decomposition of G/P lets one express its Poincaré polynomial in terms of lengths of minimal coset representatives for W/W'. The expression (3.1) then arises in this way when $W' = W_{\alpha_0}$ for a dominant root α_0.

When α_0 happens to be the highest root of a simply-laced root system, the cone over the variety G/P also arises as a Schubert variety in the affine Grassmannian. The cell decomposition of G/P as above can be used to give a decomposition of this cone into *Mirković-Vilonen cycles* introduced in [5]. In this picture, the dimension formula for the Mirković-Vilonen cycles is equivalent to Lemma 4; see Mirković and Vilonen [5, Theorem 3.2] with $\lambda = \alpha_0$, and also Ngô and Polo [6, Lemme 7.4].

4.4. **A-D-E quivers?** For simply-laced W, the W-action permuting the roots can be modeled by *reflection functors* acting on the the bounded derived category of quiver representations, with a Coxeter element c corresponding to the *Auslander-Reiten translation.* Here the W-equivariant map from an object to its dimension vector factors through the quotient category that mods out by the square of the shift map; see the discussion of the *periodic Auslander-Reiten quiver* by Kirillov and Thind [4]. Does Theorem 1(ii) reflect something lurking in this quiver picture?

References

[1] N. Bourbaki, Lie groups and Lie algebras, Chapters 4–6. *Elements of Mathematics*, Springer-Verlag, Berlin, 2002.

[2] C. Chevalley, Invariants of finite groups generated by reflections. *Amer. J. Math.* **77** (1955), 778–782.

[3] J.E. Humphreys, Reflection groups and Coxeter groups. *Cambridge Studies in Advanced Mathematics* **29**. Cambridge University Press, Cambridge, 1990.

[4] A.A. Kirillov and J. Thind, Coxeter elements and periodic Auslander-Reiten quiver. *J. Algebra* **323** (2010), 1241–1265.

[5] I. Mirkovic and K. Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)* **166** (2007), 95–143.

[6] B.C. Ngô and P. Polo, Résolutions de Demazure affines et formule de Casselman-Shalika géométrique. *J. Algebraic Geom.* **10** (2001), no. 3, 515–547

[7] V. Reiner, F. Saliola and V. Welker, Spectra of symmetrized shuffling operators, `arXiv:1102.2460`.

[8] V. Reiner, D. Stanton and D. White, The cyclic sieving phenomenon. *J. Combin. Theory Ser. A* **108** (2004), 17–50.

[9] G.C. Shephard and J.A. Todd, Finite unitary reflection groups. *Canadian J. Math.* **6**, (1954). 274–304.

[10] L. Solomon, Invariants of finite reflection groups. *Nagoya Math. J,* **22** (1963), 57–64.

[11] T.A. Springer, Regular elements of finite reflection groups. *Invent. Math.* **25** (1974) 159–198.

[12] J.R. Stembridge, Quasi-minuscule quotients and reduced words for reflections. *J. Algebraic Combin.* **13** (2001) 275–293.

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