THE SIGNATURE OF A TORIC VARIETY

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Abstract. We identify a combinatorial quantity (the alternating sum of the h-vector) defined for any simple polytope as the signature of a toric variety. This quantity was introduced by Charney and Davis in their work, which in particular showed that its non-negativity is closely related to a conjecture of Hopf on the Euler characteristic of a non-positively curved manifold.

We prove positive (or non-negative) lower bounds for this quantity under geometric hypotheses on the polytope, and in particular, resolve a special case of their conjecture. These hypotheses lead to ampleness (or weaker conditions) for certain line bundles on toric divisors, and then the lower bounds follow from calculations using the Hirzebruch Signature Formula.

Moreoever, we show that under these hypotheses on the polytope, the i^{th} L-class of the corresponding toric variety is $(-1)^i$ times an effective class for any i.

1. Introduction

Much attention in combinatorial geometry has centered on the problem of characterizing which non-negative integer sequences $(f_0, f_1, \ldots, f_d)$ can be the *f-vector* $f(P)$ of a d-dimensional convex polytope P, that is, f_i is the number of i-dimensional faces of P; see [3] for a nice survey.

For the class of *simple polytopes*, this problem was completely solved by the combined work of Billera and Lee [4] and of Stanley [33]. A simple d-dimensional polytope is one in which every vertex lies on exactly d edges. McMullen's *g-conjecture* (now the *g-theorem*) gives necessary [33] and sufficient [4] conditions for $(f_0, f_1, \ldots, f_d)$ to be the f-vector of a simple d-dimensional polytope. Stanley's proof of the necessity of these conditions showed that they have a very natural phrasing in terms of the cohomology of the toric variety X_Δ associated to the *(inner) normal fan* Δ of P, and then the Hard Lefschetz Theorem for X_Δ played a crucial role. This construction of X_Δ from Δ requires that P be *rational*, i.e. that its vertices all have rational coordinates with respect to some lattice, which can be achieved by a small perturbation that does not affect $f(P)$. Later, McMullen [29] demonstrated that one can construct a ring $\Pi(P)$, isomorphic (with a doubling of the grading) to the cohomology ring of X_Δ if P is rational, and proved that $\Pi(P)$ formally satisfies the Hard Lefschetz Theorem, using only tools from convex geometry. In particular, he recovered the necessity of the conditions of the g-theorem in this way.

This paper shares a similar spirit with Stanley's proof. We attempt to use further facts about the geometry of X_Δ to deduce information about the f-vector $f(P)$ under certain hypotheses on P. The starting point of our investigation is an

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interpretation of the alternating sum of the h*-vector* which follows from the Hard Lefschetz Theorem. Recall that for a simple polytope P, the h-vector is the sequence $h(P) = (h_0, h_1, \ldots, h_d)$ defined as follows. If we let $f(P,t) := \sum_{i=0}^{d} f_i(P)t^i$, then

$$h(P,t) := \sum_{i=0}^{d} h_i(P)t^i = f(P, t-1).$$

The h-vector has a topological interpretation: h_i is the $2i^{th}$ Betti number for X_Δ, or the dimension of the i^{th}-graded component in McMullen's ring $\Pi(P)$. Part of the conditions of the g-theorem are the Dehn-Sommerville equations $h_i = h_{d-i}$, which reflect Poincaré duality for X_Δ.

Define the alternating sum

$$\begin{aligned}\sigma(P) : &= \sum_{i=0}^{d} (-1)^i h_i(P) \\ &[= h(P,-1) = f(P,-2) = \sum_{i=0}^{d} f_i(P)(-2)^i \ ],\end{aligned}$$

a quantity which is (essentially) equivalent to one arising in a conjecture of Charney and Davis [6], related to a conjecture of Hopf (see Section 5 below). Note that when d is odd, $\sigma(P)$ vanishes by the Dehn-Sommerville equations. When d is even, we have the following result (see Section 2).

Theorem 1.1. *Let P be a simple d-dimensional polytope, with d even. Then $\sigma(P)$ is the signature of the quadratic form $Q(x) = x^2$ defined on the $\frac{d}{2}^{th}$-graded component of McMullen's ring $\Pi(P)$.*

In particular, when P has rational vertices, $\sigma(P)$ is the signature or index $\sigma(X_\Delta)$ of the associated toric variety X_Δ.

An important special case of the previously mentioned Charney-Davis conjecture asserts that a certain combinatorial condition on P (namely that of Δ being a *flag* complex; see Section 5) implies $(-1)^{d/2}\sigma(P) \geq 0$. In this paper, we prove this conjecture when P satisfies certain stronger geometric conditions. We also give further conditions which give lower bounds on $(-1)^{d/2}\sigma(P)$. In order to state these results, we give rough definitions of some of these conditions here (see Section 3 for the actual definitions).

Say that the fan Δ is *locally convex* (resp. *locally pointed convex, locally strongly convex*) if every 1-dimensional cone in Δ has the property that the union of all cones of Δ containing it is convex (resp. pointed convex, strongly convex). For example (see Propositions 4.1 and 4.10 below), if each angle in every 2-dimensional face of P is non-acute (resp. obtuse) then Δ will be locally convex (resp. locally strongly convex). It turns out that Δ being locally convex implies that it is flag (Proposition 5.3).

For a simple polytope P with rational vertices, we define an integer $m(P)$ which measures how singular X_Δ is. To be precise, let P in $\mathbb{R}^d$ be rational with respect to some lattice M, and then $m(P)$ is defined to be the least common multiple over all d-dimensional cones σ of the normal fan Δ of the index $[N : N_\sigma]$, where N is the lattice dual to M and N_σ is the sublattice spanned by the lattice vectors on the extremal rays of σ. Note that the condition $m(P) = 1$ is equivalent to the smoothness of the toric variety X_Δ, and such polytopes P are called *Delzant* in the symplectic geometry literature (e.g. [21]).

Now we can state

Theorem 1.2. *Let P be a rational simple d-dimensional polytope with d even, and Δ its normal fan.*

(i) *If Δ is locally convex, then*

$$(-1)^{\frac{d}{2}}\sigma(P) \geq 0.$$

(ii) *If Δ is locally pointed convex, then*

$$(-1)^{\frac{d}{2}}\sigma(P) \geq \frac{f_{d-1}(P)}{3m(P)^{d-1}}.$$

(iii) *If Δ is locally strongly convex, then*

$$(-1)^{\frac{d}{2}}\sigma(P) \geq \text{ coefficient of } x^d \text{ in } \left[\frac{t^d}{m(P)^{d-1}} f(P, t^{-1})\right]_{t\mapsto 1-\frac{x}{\tan(x)}}.$$

We defer a discussion of the relation between Theorem 1.2 (i) and the Charney-Davis conjecture to Section 5. It is amusing to see what Theorem 1.2 says beyond the g-theorem, in the special case where $d = 2$, that is, when P is a (rational) polygon. The g-theorem says exactly that

$$f_1 = f_0 \geq 3,$$

or in other words, every polygon has the same number of edges as vertices, and this number is at least 3. Since

$$(-1)^{\frac{d}{2}}\sigma(P) = f_0(P) - 4,$$

Theorem 1.2 (i) tells us that when Δ is locally convex, we must have $f_0 \geq 4$. In other words, triangles cannot have normal fan Δ which is locally convex, as one can easily check. For $d = 2$, the conditions that Δ is locally pointed convex or locally strongly convex coincide, and Theorem 1.2 (ii),(iii) both assert that under these hypotheses, a (rational) polygon P must have

$$f_0(P) - 4 \;\geq\; \frac{f_1(P)}{3m(P)} \;=\; \frac{f_0(P)}{3m(P)}$$

or after a little algebra,

$$f_0(P) \;\geq\; \frac{12}{3 - \frac{1}{m(P)}}. \tag{1}$$

Since the right-hand side is strictly greater than 4, we conclude that a quadrilateral P cannot have Δ locally pointed convex nor locally strongly convex. This agrees with an easily-checked fact: a quadrilateral P satisfies the weaker condition of having Δ locally convex if and only if P is a rectangle, and rectangles fail to have Δ locally pointed convex. On the other hand, the inequality (1) also implies a not-quite-obvious fact: even though a (rational) pentagon can easily have Δ locally strongly convex, this is impossible if $m(P) = 1$, i.e. there are no Delzant pentagons with this property. It is a fun exercise to show directly that no such pentagon exists, and to construct a Delzant hexagon with this property.

In fact, in the context of algebraic geometry, the proof of Theorem 1.2 gives the following stronger assertion, valid for rational simple polytopes of any dimension d (not necessarily even) about the expansion of the total L-class

$$L(X) = L_0(X) + L_1(X) + \cdots + L_{\frac{d}{2}}(X)$$

where $L_i(X)$ is a cycle in $CH^i(X)_{\mathbb{Q}}$, the Chow ring of X.

Theorem 1.3. *Let $X = X_\Delta$ be a complete toric variety X associated to a simplicial fan Δ. If Δ is locally strongly convex (resp. locally convex), then for each i we have that $(-1)^i L_i(X)$ is effective (resp. either effective or 0).*

For instance, when $i = 1$ this implies that if Δ is locally convex, then

$$\int_X (c_1^2(X) - 2c_2(X)) \cdot H_1 \cdot \ldots \cdot H_{d-2} \leq 0$$

where $\{H_i\}$ are any ample divisor classes. This is reminiscent of the Chern number inequality for the complex spinor bundle of X when this bundle is stable with respect to all polarizations; see e.g. [27].

Notice that if Δ is not locally convex, $(-1)^i L_i(X)$ need not be effective. For example, if Δ is the normal fan of the standard 2-dimensional simplex having vertices at $(0,0), (1,0), (0,1)$, then X is the complex projective plane, and $-L_1(X)$ is represented by the negative of the Poincaré dual of a point.

2. The alternating sum as signature

We wish to prove Theorem 1.1, whose statement we recall here.

Theorem 1.1. *Let P be a simple d-dimensional polytope, with d even. Then $\sigma(P)$ is the signature of the quadratic form $Q(x) = x^2$, defined on the $\frac{d}{2}^{th}$-graded component of McMullen's ring $\Pi(P)$.*

In particular, when P has rational vertices, $\sigma(P)$ is the signature or index $\sigma(X_\Delta)$ of the associated toric variety X_Δ.

Proof. Taking $r = \frac{d}{2}$ in a result of McMullen [29, Theorem 8.6], we find that the quadratic form $(-1)^{\frac{d}{2}} Q(x)$ on the $\frac{d}{2}^{th}$-graded component of $\Pi(P)$ has

$$\sum_{i=0}^{\frac{d}{2}} (-1)^i h_{\frac{d}{2}-i}(P) \text{ positive eigenvalues, and}$$

$$\sum_{i=0}^{\frac{d}{2}-1} (-1)^i h_{\frac{d}{2}-i-1}(P) \text{ negative eigenvalues.}$$

Consequently, the signature $\sigma(Q)$ of Q is

$$\begin{aligned}\sigma(Q) &= (-1)^{\frac{d}{2}} \left[\sum_{i=0}^{\frac{d}{2}} (-1)^i h_{\frac{d}{2}-i}(P) - \sum_{i=0}^{\frac{d}{2}-1} (-1)^i h_{\frac{d}{2}-i-1}(P) \right] \\ &= \sum_{i=0}^{d} (-1)^i h_i(P)\end{aligned}$$

where the second equality uses the *Dehn-Sommerville* equations [29, §4]:

$$h_i(P) = h_{d-i}(P).$$

The second assertion of the theorem follows immediately from McMullen's identification of the ring $\Pi(P)$ with the quotient of the Stanley-Reisner ring of Δ by a certain linear system of parameters [29, §14], which is known to be isomorphic (after a doubling of the grading) with the cohomology of X_Δ [18, §5.2]. $\square$

Remark 2.1.
Starting from any complete rational simplicial fan Δ, one can construct a toric variety X_Δ which will be complete, but not necessarily projective, and satisfies Poincarè duality. The h-vector for Δ can still be defined, and again has an interpretation as the even-dimensional Betti numbers of X_Δ (see [18, §5.2]*). We suspect that the alternating sum of the h-vector is still the signature of this complete toric variety.*

Generalizing in a different direction, to any polytope P which is not necessarily simple, one can associate the normal fan Δ and a projective toric variety X_Δ. Although the (singular) cohomology of X_Δ does not satisfy Poincaré duality, its intersection cohomology (in middle perversity) $IH^{\cdot}(X_\Delta)$ will. There is a combinatorially-defined generalized h-vector which computes these $IH^{\cdot}$ Betti numbers (see [34]*). Moreover, using the Hard Lefschetz Theorem for intersection cohomology and the fact that X_Δ is a finite union of affine subvarieties, the alternating sum of the generalized h-vector equals the signature of the quadratic form on $IH^{\cdot}(X_\Delta)$ defined by the intersection product.*

Remark 2.2.
The special case of the second assertion in Theorem 1.1 is known when X_Δ is smooth (i.e. P is a Delzant polytope); see [31, Theorem 3.12 (3)].

3. Lower bounds derived from the signature theorem

The goal of this section is to explain the various notions used in Theorem 1.2, and to prove this theorem.

We begin with a d-dimensional lattice $M \cong \mathbb{Z}^d$ and its associated real vector space $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. A *polytope* P in $M_\mathbb{R}$ is the convex hull of a finite set of points in $M_\mathbb{R}$. We say that P is *rational* if these points can be chosen to be rational with respect to the lattice M. The *dimension* of P is the dimension of the smallest affine subspace containing it. A *face* of P is the intersection of P with one of its supporting hyperplanes, and a face is always a polytope in its own right. *Vertices* and *edges* of P are 0-dimensional and 1-dimensional faces, respectively. Every vertex of a d-dimensional polytope lies on at least d edges, and P is called *simple* if every vertex lies on exactly d edges.

Let $N = Hom(M, \mathbb{Z})$ be the dual lattice to M and $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ be the dual vector space to $M_\mathbb{R}$, with the natural pairing $M_\mathbb{R} \otimes N_\mathbb{R} \to \mathbb{R}$ denoted by $\langle \cdot, \cdot \rangle$. For a polytope P in $M_\mathbb{R}$, the *normal fan* Δ is the following collection of polyhedral cones in $N_\mathbb{R}$:

$$\Delta = \{\sigma_F : F \text{ a face of } P\},$$

where

$$\sigma_F := \{v \in N_\mathbb{R} : \langle u, v \rangle \leq \langle u', v \rangle \text{ for all } u \in F, u' \in P\}$$

Note that

- the normal fan Δ is a *complete* fan, that is, it exhausts $N_\mathbb{R}$,
- Δ is a *rational* fan, in the sense that its rays all have rational slopes, if P is rational,

- if P is d-dimensional, then every cone σ_F in Δ will be *pointed*, that is, it will contain no proper subspaces of $N_{\mathbb{R}}$,
- P is a simple polytope if and only if Δ is a *simplicial fan*, that is, every cone σ in Δ is simplicial in the sense that its extremal rays are linearly independent.

We next define several affinely invariant conditions on a complete simplicial fan Δ in $N_{\mathbb{R}}$ (and hence on simple polytopes P in $M_{\mathbb{R}}$) that appear in Theorem 1.2. For any collection of polyhedral cones Δ in $N_{\mathbb{R}}$, let $|\Delta|$ denote the support of Δ, that is the union of all of its cones as a point set. Define the *star* and *link* of one of the cones σ in Δ similarly to the analogous notions in simplicial complexes: $\mathrm{star}_\Delta(\sigma)$ is the subfan consisting of those cones τ in Δ such that σ, τ lie in some common cone of Δ, while $\mathrm{link}_\Delta(\sigma)$ is the subfan of $\mathrm{star}_\Delta(\sigma)$ consisting of those cones which intersect σ only at the origin. For a *ray* (i.e. a 1-dimensional cone) ρ of Δ, say that the fan $\mathrm{star}_\Delta(\rho)$ is

- *convex* if its support $|\mathrm{star}_\Delta(\rho)|$ is a convex set in the usual sense,
- *pointed convex* if $|\mathrm{star}_\Delta(\rho)|$ is convex and contains no proper subspace of $N_{\mathbb{R}}$,
- *strongly convex* if furthermore for every cone σ in $\mathrm{link}_\Delta(\rho)$, there exists a linear hyperplane H in $N_{\mathbb{R}}$ which supports $\mathrm{star}_\Delta(\rho)$ and whose intersection with $\mathrm{star}_\Delta(\rho)$ is exactly σ.

Say that Δ is *locally convex* (resp. *locally pointed convex*, *locally strongly convex*) if every ray ρ of Δ has $\mathrm{star}_\Delta(\rho)$ convex (resp. pointed convex, strongly convex). One has the easy implications

$$\text{locally strongly convex} \;\Rightarrow\; \text{locally pointed convex} \;\Rightarrow\; \text{locally convex}\,.$$

We recall here that the affine-lattice invariant $m(P)$ for a rational polytope P was defined (in the introduction) to be the least common multiple of the positive integers $[N : N_\sigma]$ as σ runs over all d-dimensional cones in Δ. Here N_σ is the d-dimensional sublattice of N generated by the lattice vectors on the d extremal rays of σ. In a sense $m(P)$ measures how singular X_Δ is [18, §2.6], with $m(P) = 1$ if and only if X_Δ is smooth, in which case we say that P is *Delzant*.

We can now recall the statements of Theorems 1.2 and 1.3.

Theorem 1.2. *Let P be a simple d-dimensional polytope in $M_{\mathbb{R}}$, which is rational with respect to M, and Δ its normal fan in $N_{\mathbb{R}}$. Assume d is even.*

(i) *If Δ is locally convex, then*
$$(-1)^{\frac{d}{2}}\sigma(P) \geq 0.$$

(ii) *If Δ is locally pointed convex, then*
$$(-1)^{\frac{d}{2}}\sigma(P) \geq \frac{f_{d-1}(P)}{3m(P)^{d-1}}.$$

(iii) *If Δ is locally strongly convex, then*
$$(-1)^{\frac{d}{2}}\sigma(P) \geq \textit{ coefficient of } x^d \textit{ in } \left[\frac{t^d}{m(P)^{d-1}}\, f(P, t^{-1})\right]_{t \mapsto 1 - \frac{x}{\tan(x)}}.$$

Theorem 1.3. *Let $X = X_\Delta$ be a complete toric variety X associated to a simplicial fan Δ, and let the expansion of the total L-class be*

$$L(X) = L_0(X) + L_1(X) + \cdots + L_{\frac{d}{2}}(X)$$

where $L_i(X)$ is a cycle in $CH^i(X)_{\mathbb{Q}}$, the Chow ring of X.

If Δ is locally strongly convex (resp. locally convex), then for each i we have that $(-1)^i L_i(X)$ is effective (resp. either effective or 0).

The remainder of this section is devoted to the proof of these theorems. We begin by recalling some toric geometry. As a general reference for toric varieties, we rely on Fulton [18], although many of the facts we will use can also be found in Oda's book [31] or Danilov's survey article [9].

Let X denote the toric variety X_Δ. Simpleness of P implies that X is an orbifold [18, §2.2]. Recall that irreducible toric divisors[1] on X correspond in a one-to-one fashion with the codimension 1 faces of P, or to 1-dimensional rays in the normal fan Δ. Number these toric divisors on X as $D_1, ..., D_m$. Intersection theory for these D_i's is studied in Chapter 5 of [18]. Every D_i is a toric variety in its own right with at worst orbifold singularities. Moreover $D = \bigcup_{i=1}^m D_i$ is a simple normal crossing divisor on X [18, §4.3].

Next we want to express the signature of X in terms of these D_i's. When X is a smooth variety, a consequence of the hard Lefschetz Theorem is that its signature $\sigma(X)$ can be expressed in terms of the Hodge numbers of X as follows [23, Theorem 15.8.2]:

$$\sigma(X) = \Sigma_{p,q=0}^d (-1)^q h^{p,q}(X).$$

By the Dolbeault Theorem, $h^{p,q}(X) = \dim H^q(X, \Omega_X^p)$, and hence the signature can be expressed in terms of twisted holomorphic Euler characteristics

$$\sigma(X) = \sum_{p=0}^d \chi(X, \Omega_X^p)$$

where $\chi(X,E) := \sum_{q=0}^d (-1)^q \dim H^q(X,E)$. Using the Riemann-Roch formula, we can write

$$\chi(X, \Omega_X^p) = \int_X ch(\Omega_X^p) Td_X,$$

where ch is the Chern character and Td_X is the Todd class of X. Therefore

$$\sigma(X) = \int_X \sum_{p=0}^d ch(\Omega_X^p) Td_X.$$

When X is smooth, $\sum_{p=0}^d ch(\Omega_X^p) Td_X$ equals the Hirzebruch L-class $L(X)$ of X (see page 16 in [23, Theorem 15.8.2] for example) and we recover the Hirzebruch Signature Formula

$$\sigma(X) = \int_X L(X). \tag{2}$$

If X is a projective variety with at worst orbifold singularities, the hard Lefschetz, Dolbeault, and Riemann-Roch Theorems continue to hold, and we can take the sum $\sum_{p=0}^d ch(\Omega_X^p) Td_X$ as a definition of $L(X)$. Since we can express $L(X)$ in terms of Chern roots of the orbi-bundle Ω_X^1 and Chern classes for orbi-bundles satisfy

[1] Actually these are $\mathbb{Q}$-Cartier divisors on the orbifold X.

the same functorial properties as for the usual Chern classes, the same holds true for $L(X)$. For example we will use the splitting principle in the proof of the next lemma, where we write $L(X)$ in terms of toric data.[2]

Lemma 3.1.

$$\begin{aligned}(-1)^{\frac{d}{2}}\sigma(X) &= (-1)^{\frac{d}{2}}\int_X L(X)\\ &= \sum_{p=1}^{d/2}\ \sum_{\substack{n_1+\ldots+n_p=d/2\\ n_i>0;\, i_1<\ldots<i_p}} b_{n_1}\cdots b_{n_p}\ (-1)^p\, D_{i_1}^{2n_1}\cdot\ldots\cdot D_{i_p}^{2n_p}\end{aligned}$$

where $D_{j_1}\cdot\ldots\cdot D_{j_d}$ denotes the intersection number for the d divisors $D_{j_1},\ldots,D_{j_d}$ on X, and b_n are the coefficients in the expansion

$$\frac{\sqrt{x}}{\tanh\sqrt{x}} = 1-\textstyle\sum_{n=1}^{\infty}(-1)^n\, b_n x^n.$$

That is, $b_n = \frac{2^{2n}B_n}{(2n)!}$ where B_n is the n^{th} Bernoulli number.

Proof. Recall [30] that the L-class is a multiplicative characteristic class corresponding to the power series $\frac{\sqrt{x}}{\tanh\sqrt{x}}$, as we are about to explain. In our situation, we need to compute the L-class of a holomorphic orbi-bundle E, namely the tangent orbi-bundle of X. For the purposes of this computation, we can treat E like a genuine vector bundle (see e.g. [8, Appendix A]). We will assume that E can be stably split into a direct sum of line bundles, that is

$$E\oplus O_X^{\oplus(m-d)}\cong\bigoplus_{i=1}^{m}L_i.$$

Then $c(E)=\prod_{i=1}^{m}(1+x_i)$ where x_i is the first Chern class $c_1(L_i)$, i.e. the x_i's are *stable Chern roots* of E. We then have

$$c(E\otimes_{\mathbb{R}}\mathbb{C}) = c(E\oplus\overline{E}) = \prod_{i=1}^{m}(1+x_i)\prod_{j=1}^{m}(1-x_j) = \prod_{i=1}^{m}(1-x_i^2).$$

The L-class is then computed by the formula

$$L(E)=\prod_{i=1}^{m}(1-\sum_{n\geq 1}(-1)^n b_n x_i^{2n})$$

where b_n is the positive number defined in the Lemma.

For example, in terms of Pontrjagin classes of X we have

$$\begin{aligned}L_1(X) &= \frac{1}{3}p_1\\ L_2(X) &= \frac{1}{45}\left(7p_2-p_1^2\right)\\ L_3(X) &= \frac{1}{945}\left(62p_3-13p_2p_1+2p_1^3\right).\end{aligned}$$

[2]For a general orbifold X, not necessary an algebraic variety, Kawasaki [26] expressed the signature of X in terms of integral of certain curvature forms, thus generalizing the Hirzebruch signature formula in a different way.

To use the Hirzebruch Signature Formula (2), we need to express $L(X)$ in terms of toric data, specifically intersection numbers of the toric divisors $D_1, ..., D_m$ on X discussed above. To relate these divisors with characteristic classes of X, we consider the exact sequence of sheaves:

$$0 \to \Omega_X^1 \to \Omega_X^1(\log D) \xrightarrow{\text{Res}} \bigoplus_{i=1}^m O_{D_i} \to 0$$

where Ω_X^1 (resp. $\Omega_X^1(\log D)$) is the sheaf of differentials on X (resp. differentials on X with logarithmic poles along D). Notice that $\Omega_X^1(\log D)$ is a trivial sheaf of rank d [18, §4.3]. On the other hand, there is an exact sequence of sheaves

$$0 \to O_X(-D_i) \to O_X \to O_{D_i} \to 0$$

for each toric divisor D_i. From the functorial properties of Chern classes, we have

$$c\left(\Omega_X^1\right) = \Pi_{i=1}^m c\left(O_X(-D_i)\right) = \Pi_{i=1}^m \left(1 - D_i\right).$$

Here we have identified a divisor D_i with the Poincaré dual of the first Chern class of its associated line bundle. Since Ω_X^1 is the sheaf of sections of the cotangent bundle on X, we can write the total Chern class of (the tangent bundle on) X as

$$c(E) = \Pi_{i=1}^m \left(1 + D_i\right).$$

Namely these D_i's behave as stable Chern roots of the tangent bundle of X.

Therefore, by the multiplicative property of the L-class, we have

$$L(X) = \prod_{i=1}^m \left(1 - \textstyle\sum_{n=1}^\infty (-1)^n b_n D_i^{2n}\right).$$

Expanding the right-hand side of the above equality gives the equality asserted in the lemma. □

To prove Theorem 1.2 we need to give lower bounds on the intersection numbers

$$(-1)^p D_{i_1}^{2n_1} \cdots D_{i_p}^{2n_p}$$

that appear in the right-hand-side of Lemma 3.1, under our various hypotheses on the fan Δ. We begin by rewriting

$$(3) \qquad (-1)^p D_{i_1}^{2n_1} \cdots D_{i_p}^{2n_p} = \int_{D_{i_1} \cap \ldots \cap D_{i_p}} \left(-D_{i_1}\right)^{2n_1 - 1} \cdots \left(-D_{i_p}\right)^{2n_p - 1}.$$

This expression leads us to consider the restriction of the line bundles $O_X(-D_i)$ to the subvarieties $D_{i_1} \cap ... \cap D_{i_p}$. For any of the irreducible toric divisors D_i on X, let $O_{D_i}(-D_i)$ denote the restriction of $O_X(-D_i)$ to the toric subvariety D_i (this is the *conormal bundle* of D_i in X). Recall that for an invertible sheaf $O(E)$ on a d-dimensional orbifold X, one says that $O(E)$ is *big* if the corresponding divisor E satisfies $E^d > 0$. The key observation in obtaining the desired lower bounds is then

Lemma 3.2. *Let P be a simple d-dimensional polytope in $M_{\mathbb{R}}$, which is rational with respect to M, and Δ its normal fan in $N_{\mathbb{R}}$. Let D_i be any of the irreducible toric divisors on $X = X_\Delta$.*

(i) *If Δ is locally convex, then $O_{D_i}(-D_i)$ is generated by global sections.*
(ii) *If Δ is locally pointed convex, then $O_{D_i}(-D_i)$ is generated by global sections and big.*
(iii) *If Δ is locally strongly convex, then $O_{D_i}(-D_i)$ is ample.*

Assuming Lemma 3.2 for the moment, we finish the proof of Theorem 1.2.

Proof of Theorem 1.2 (i). Under the assumption that Δ is locally convex, we know that the restriction of $O_X\left(-D_{i_j}\right)$ to $D_{i_1}\cap ...\cap D_{i_p}$ is generated by global sections for $1\leq j\leq p$ by Lemma 3.2. This implies that the integral (3) equals the intersection number of such divisors on the toric subvariety $D_{i_1}\cap ...\cap D_{i_p}$ and therefore it is nonnegative [3], that is

$$(-1)^p\, D_{i_1}^{2n_1}\cdots D_{i_p}^{2n_p}\geq 0.$$

The non-negativity asserted in Theorem 1.2 (i) now follows term-by-term from the sum in Lemma 3.1. □

Proof of Theorem 1.2 (ii). If Δ is locally pointed convex, then $O_{D_i}\left(-D_i\right)$ is generated by global sections and big. The bigness of $O_{D_i}\left(-D_i\right)$ on D_i implies that

$$-D_i^d=\int_{D_i}\left(-D_i\right)^{d-1}$$

is strictly positive.

Claim. $-D_i^d\geq\frac{1}{m(P)^{d-1}}$.

To prove this, we proceed as in the *algebraic moving lemma* [18, §5.2, p. 107], making repeated use of the fact that if n_j is the first non-zero lattice point on the ray of Δ corresponding to D_j, then for any u in M, one has

$$\sum_j\langle u,n_j\rangle D_j=0 \tag{4}$$

in the Chow ring [18, Proposition, Part (ii), §5.2, p. 106]. This allows one to take intersection monomials that contain some divisor D_{j_0} raised to a power greater than 1, and replace one factor of D_{j_0} by a sum of other divisors. By repeating this process for all of the monomials in a total of $d-1$ stages, one can replace D_i^d by a sum of the form $\sum a_{j_1,\ldots,j_d}D_{j_1}\cdots D_{j_d}$ in which each term has $D_{j_1},\ldots,D_{j_d}$ *distinct* divisors which intersect at an isolated point of X, and each $a_{j_1,\ldots,j_d}$ is a rational number. We must keep careful track of the denominators of the coefficients introduced at each stage.

At the first stage, by choosing any u in M with $\langle u,n_i\rangle=1$, we can use (4) to replace one factor of D_i in D_i^d by a sum of other divisors D_j with *integer coefficients* (that is, introducing *no* denominators). However, in each of the next $d-2$ stages, when one wishes to use (4) to substitute for a divisor D_{j_0}, one must choose u in M constrained to vanish on normal vectors n_j for other divisors D_j in the monomial, and this may force the coefficient $\langle u,n_{j_0}\rangle$ of D_{j_0} to be larger than 1 in (4), although it will always be an integer factor of $m(P)$. Consequently, at each stage after the first, we may introduce factors into the denominators that divide into $m(P)$. Since there are $d-2$ stages after the first, we conclude that each $a_{j_1,\ldots,j_d}$ can be written with the denominator $m(P)^{d-2}$. Finally, each intersection product $D_{j_1}\cdots D_{j_d}$ is the reciprocal of the multiplicity at the corresponding point of X, which is the index $[N:N_\sigma]$ where σ is the d-dimensional cone of Δ corresponding to that point [18, §2.6]. Since each $[N:N_\sigma]$ divides $m(P)$, we conclude that $-D_i^d$ lies in $\frac{1}{m(P)^{d-1}}\mathbb{Z}$, and since it is positive, it is at least $\frac{1}{m(P)^{d-1}}$.

[3] This follows from the fact that the divisor class of a line bundle which is generated by global sections is a limit of $\mathbb{Q}$-divisors which are ample. Positivity of intersection numbers of ample divisors is well-known; see e.g. [19, Chapter 12]

We have shown then that each term with $p=1$ on the right-hand side of Lemma 3.1 is at least $\frac{1}{m(P)^{d-1}}$, and the number of such terms is the number of codimension one faces of P, i.e. $f_{d-1}(P)$. Moreover we still have nonnegativity of the other terms $(-1)^p\, D_{i_1}^{2n_1}\cdots D_{i_p}^{2n_p}$ because $O_{D_i}(-D_i)$ is generated by global sections. Therefore, since $b_1=\frac{1}{3}$, we conclude from Lemma 3.1 that

$$(-1)^{d/2}\,\sigma(\Delta)\geq \frac{f_{d-1}}{3m(P)^{d-1}}.\quad\square$$

Proof of Theorem 1.2 (iii). If Δ is locally strongly convex, then $O_{D_i}(-D_i)$ is ample. By similar arguments as in assertions (i) and (ii), we have

$$(-1)^p\, D_{i_1}^{2n_1}\cdots D_{i_p}^{2n_p}\geq \frac{1}{m(P)^{d-1}}$$

provided that $D_{i_1}\cap\ldots\cap D_{i_p}$ is non-empty. By the simplicity of P, each of its codimension p faces can be expressed uniquely as the intersection of distinct codimension one faces, corresponding to the non-empty intersection of divisors $D_{i_1},\ldots,D_{i_p}$. Therefore, after choosing positive integers $n_1,\ldots,n_p$, the number of non-vanishing terms of the form $(-1)^p\, D_{i_1}^{2n_1}\cdots D_{i_p}^{2n_p}$ in the expansion of Lemma 3.1 is $f_{d-p}(P)$. Hence

$$\begin{aligned}
(-1)^{\frac{d}{2}}\sigma(X) &= (-1)^{\frac{d}{2}}\int_X L(X)\\
&= \sum_{p=1}^{d/2}(-1)^p \sum_{\substack{n_1+\ldots+n_p=d/2\\ n_i>0;\, i_1<\ldots<i_p}} b_{n_1}\cdots b_{n_p} D_{i_1}^{2n_1}\cdots D_{i_p}^{2n_p}\\
&\geq \sum_{p=1}^{d/2}\sum_{\substack{n_1+\ldots+n_p=d/2\\ n_i>0}} b_{n_1}\cdots b_{n_p}\frac{f_{d-p}(P)}{m(P)^{d-1}}\\
&= \sum_{p=1}^{d/2}\frac{f_{d-p}(P)}{m(P)^{d-1}}\left[\text{ coefficient of } x^d \text{ in } \left(\sum_{n\geq 1} b_n x^{2n}\right)^p\ \right].
\end{aligned}$$

Note that

$$\frac{\sqrt{x}}{\tanh\sqrt{x}} = 1-\sum_{n\geq 1}(-1)^n b_n x^n$$

implies

$$\sum_{n\geq 1} b_n x^{2n} = 1-\frac{x}{\tan(x)},$$

and note also that

$$\sum_{p\geq 1} f_{d-p}(P)\, t^p = t^d f(P,t^{-1}).$$

This allows us to rewrite the above inequality as in the assertion of Theorem 1.2 (iii). $\square$

Proof of Theorem 1.3 . Recall from the proof of Lemma 3.1 that the total L-class has expansion

$$L(X) = \sum_{p\geq 1}(-1)^p \sum_{\substack{(n_1,\ldots,n_p)\\ n_i>0;\, i_1<\ldots<i_p}} (-1)^{\sum n_i} b_{n_1}\cdots b_{n_p} D_{i_1}^{2n_1}\cdots D_{i_p}^{2n_p}.$$

Consequently,

$$(-1)^j L_j(X) = \sum_{p\geq 1}(-1)^p \sum_{\substack{n_1+\cdots+n_p=j\\ n_i>0;\, i_1<\ldots<i_p}} b_{n_1}\cdots b_{n_p} D_{i_1}^{2n_1}\cdots D_{i_p}^{2n_p}.$$

Therefore it suffices to show that each term

$$(-1)^p\, D_{i_1}^{2n_1}\cdots D_{i_p}^{2n_p}$$

is effective if Δ is locally strongly convex (the case where Δ is locally convex is similar). Here we use the fact from Lemma 3.2 that restriction of $O(-D_{i_k})$ to D_{i_k} is ample, and therefore also ample when further restricted to the transverse intersection $V = D_{i_1}\cap\cdots\cap D_{i_p}$. Consequently, the cycle class

$$(-D_{i_1})^{2n_1-1}\cdots(-D_{i_p})^{2n_p-1}$$

is effective in the Chow ring $CH(V)_{\mathbb{Q}}$ by Bertini's Theorem. Therefore $(-1)^p\, D_{i_1}^{2n_1}\cdot\cdot\cdot D_{i_p}^{2n_p}$ is effective in $CH(X)_{\mathbb{Q}}$. $\square$

Proof of Lemma 3.2 . We recall some facts about toric divisors contained generally in [18, §3.3, 3.4]. In general, any divisor E on X defines a continuous piecewise linear function Ψ_E^X on the support $|\Delta| = N_{\mathbb{R}}$. Every divisor E on X is linearly equivalent to a linear combination of irreducible toric divisors. If we write $E=\sum_{i=1}^m a_i D_i$, then Ψ_E^X is determined by the property that $\Psi_E^X(n_i) = -a_i$ where n_i is the first nonzero lattice point of N along ρ_i. In particular, $\Psi_{-D_i}^X : N_{\mathbb{R}} \to \mathbb{R}$ is determined by $\Psi_{-D_i}^X(n_j) = \delta_{ij}$. The ampleness of the line bundle $O_X(E)$ can be measured by the convexity of the piecewise linear function Ψ_E^X. More precisely, $O_X(E)$ is ample (resp. generated by global sections) if and only if Ψ_E^X is strictly convex (resp. convex).

We now discuss assertions (i), (iii) of the lemma, leaving (ii) for later. First we examine the particular case of the discussion in the previous paragraph where the bundle is $O_{D_i}(-D_i)$ on the toric subvariety D_i. Assume the divisor D_i corresponds to a ray ρ_i in the normal fan Δ. The fan Δ^{D_i} associated to D_i naturally lives in the quotient space $N_{\mathbb{R}}/\rho_i$ (here we are abusing notation by letting ρ_i denote both a ray and also the 1-dimensional subspace spanned by this ray) [18, §3.1]. Then every cone in Δ^{D_i} corresponds to a cone in Δ containing ρ_i as a face (and vice-versa), that is, a cone in $\mathrm{star}_\Delta(\rho_i)$. The boundary of $\mathrm{star}_\Delta(\rho_i)$ is $\mathrm{link}_\Delta(\rho_i)$, and here we use the fact, proven in the Appendix, that $\mathrm{link}_\Delta(\rho_i)$ is affinely equivalent to the graph of the continuous piecewise linear function $\Psi_{-D_i}^{D_i} : N_{\mathbb{R}}/\rho_i \to \mathbb{R}$. From the discussion in the previous paragraph, we conclude that $O_{D_i}(-D_i)$ is generated by global sections (resp. ample) if Δ is locally convex (resp. locally strongly convex).

Lastly we discuss asertion (ii) of the lemma. We want to prove that $O_{D_i}(-D_i)$ is big for every irreducible toric divisor D_i under the assumption that Δ is locally pointed convex. The fact that Δ is locally pointed convex says that the space $|\mathrm{star}_\Delta(\rho_i)|$ is a pointed convex polyhedral cone. There is then a unique fan Σ having the following properties:

- Σ is refined by $\mathrm{star}_\Delta(\rho_i)$, and they have the same support, that is,
$$|\Sigma| = |\mathrm{star}_\Delta(\rho_i)|,$$
- ρ_i is the only ray in the interior of Σ, and
- Σ is strongly convex in the sense that that every ray of Σ except for ρ_i is the intersection of $|\Sigma|$ with some supporting hyperplane.

This cone Σ projects to a complete fan $\bar{\Delta}^{D_i}$ in $N_{\mathbb{R}}/\rho_i$, which is refined by Δ^{D_i}. Therefore we obtain a birational morphism [18, §1.4].
$$\pi : D_i = X_{\Delta^{D_i}} \to X_{\bar{\Delta}^{D_i}}.$$
Moreover $\Psi^{D_i}_{-D_i}$ still defines a continuous piecewise linear function on $N_{\mathbb{R}}/\rho_i$, the support of $\bar{\Delta}^{D_i}$, which is now strongly convex. Therefore it defines an ample Cartier divisor on $X_{\bar{\Delta}^{D_i}}$. Call this divisor C. Then it is not difficult to see that $O_{D_i}(-D_i)$ is just the pullback of $O_{X_{\bar{\Delta}^{D_i}}}(C)$. Moreover, their self-intersection numbers are equal, that is
$$(-D_i)^{d-1} = C^{d-1}.$$
Now C is an ample divisor on $X_{\bar{\Delta}^{D_i}}$ and therefore C^{d-1} is strictly positive. Hence the same is true for $-D_i$. That is, $O_{D_i}(-D_i)$ is a big line bundle on the toric subvariety D_i. This completes the proof of Lemma 3.2. □

The previous proof raises the following question: is the assumption of rationality for the simple polytope P really necessary in Theorem 1.2? In approaching this problem, it would be interesting if Lemma 3.1 and the intersection numbers that appear within it have some interpretation purely within the convexity framework used by McMullen [29].

4. Examples

In this section we discuss examples of simple polytopes P whose normal fans Δ satisfy the hypotheses of Theorem 1.2.

We begin with some properties of P that are Euclidean invariants, so we will assume that $M_{\mathbb{R}}$ is endowed with a (positive definite) inner product $\langle \cdot, \cdot \rangle$ which identifies $M_{\mathbb{R}}$ with its dual space $N_{\mathbb{R}}$. Thus we can think of both P and its normal fan Δ as living in $M_{\mathbb{R}}$.

Say that a polytope P is *non-acute in codimension* 1 (resp. *obtuse in codimension* 1) if every codimension 2 face of P has the property that the dihedral angle between the two codimension 1 faces meeting there is non-acute (resp. obtuse), that is, at least (resp. greater than) $\frac{\pi}{2}$. Say that P is *non-acute* (resp. *obtuse*) if P and every one of its faces of each dimension considered as polytopes in their own right are non-acute (resp. obtuse) in codimension 1. We have the following obvious implications:

obtuse $\Rightarrow$
obtuse in codimension 1 and non-acute $\Rightarrow$
non-acute .

The next proposition shows that these Euclideanly invariant conditions on P imply the affinely invariant conditions on Δ defined in the Section 3.

Proposition 4.1. *Let P be a simple d-dimensional polytope in $M_{\mathbb{R}}$, with normal fan Δ.*

(i) *If P is non-acute, then Δ is locally convex.*
(ii) *If P is obtuse in codimension 1 and non-acute, then Δ is locally pointed convex.*
(iii) *If P is obtuse, then Δ is locally strongly convex.*

The next corollary then follows immediately from the previous proposition and Theorem 1.2.

Corollary 4.2. *Let P be a simple, rational d-dimensional polytope in $M_{\mathbb{R}}$, with normal fan Δ.*

(i) *If P is non-acute, then*
$$(-1)^{\frac{d}{2}}\sigma(P) \geq 0.$$
(ii) *If P is obtuse in codimension 1, and non-acute, then*
$$(-1)^{\frac{d}{2}}\sigma(P) \geq \frac{f_{d-1}(P)}{3m(P)^{d-1}}.$$
(iii) *If P is obtuse, then*
$$(-1)^{\frac{d}{2}}\sigma(P) \geq \text{ coefficient of } x^d \text{ in } \left[\frac{t^d}{m(P)^{d-1}} f(P,t^{-1})\right]_{t\mapsto 1-\frac{x}{\tan(x)}}.$$

Remark 4.3.
It is easy to see that obtuse simple polytopes can always be made rational without changing their facial structure by a slight perturbation of their facets, so that one might think of removing the rationality assumption from part (iii) of the previous corollary. However, after this perturbation it is not clear what the lattice-invariant $m(P)$ will be, i.e. it could be any positive integer.

It is not obvious whether a non-acute, simple polytope always has the same facial structure as a rational, non-acute, simple polytope. This would follow if every non-acute, simple polytope had the same facial structure as an obtuse, simple polytope, but this is false. For example, a regular 3-dimensional cube is non-acute and simple, but no obtuse polytope can have the facial structure of a 3-cube.

Remark 4.4.
M. Davis has pointed out to us that the first assertion of Corollary 4.2 can be proven using facts from [6] and the mirror construction $M(P)$ of the next section, without any assumption that P is rational. We defer a sketch of this proof until the description of $M(P)$ at the end of that section.

Proof of Proposition 4.1. We begin by rephrasing some of our definitions about non-acuteness and obtuseness in terms of Δ. Obtuseness (resp. non-acuteness) in codimension 1 for P corresponds to the following property of Δ: any two vectors n, n' spanning the extremal rays of a 2-dimensional cone of Δ must have
$$\langle n, n'\rangle > 0 \qquad (\text{ resp. } \langle n, n'\rangle \geq 0).$$
Similarly, obtuseness (resp. non-acuteness) for P corresponds to the following property of Δ: any vectors $n_1, \ldots, n_t$ spanning the extremal rays of a t-dimensional (simplicial) cone of Δ must have
$$\langle \pi(n_1), \pi(n_2)\rangle > 0 \qquad (\text{ resp. } \langle \pi(n_1), \pi(n_2)\rangle \geq 0)$$
where π is the orthogonal projection onto the space perpendicular to the span of the vectors $n_3, n_4, \ldots, n_t$.

Having said this, observe that if P is non-acute in codimension 1, for any ray ρ in Δ (spanned by a vector which we name n), the hyperplane $\rho^{\perp}$ normal to ρ supports $\mathrm{star}_{\Delta}(\rho)$: if P is non-acute in codimension 1, we must have $\langle n, n' \rangle \geq 0$ for each vector n' spanning a ray in $\mathrm{star}_{\Delta}(\rho)$, and hence for every vector in $\mathrm{star}_{\Delta}(\rho)$. Similarly, if P is obtuse in codimension 1 then this hyperplane $\rho^{\perp}$ not only supports $\mathrm{star}_{\Delta}(\rho)$, but also intersects it only in the origin. Consequently, assertion (ii) of the lemma follows once we prove assertion (i).

For assertions (i), (iii), we make use of the fact that strong or weak convexity of $\mathrm{star}_{\Delta}(\rho)$ can be checked *locally* in a certain way, similar to checking regularity of triangulations (see e.g. [16, §1.3]). Roughly speaking, each cone σ in the link of ρ must have the property that the union of cones containing σ within $\mathrm{link}_{\Delta}(n)$ "bend outwards" at σ away from ρ, rather than "bending inward" toward ρ. To be more formal, consider every minimal dependence of the form

$$\sum_{i \in F} \alpha_i n_i = \beta n + \sum_{j \in G} \beta_j m_j \tag{5}$$

where

- $\{n_i\}_{i \in F}$ are vectors spanning the extremal rays of some cone σ in $\mathrm{link}_{\Delta}(n)$,
- each m_j for f in G spans a ray in $\mathrm{link}_{\Delta}(\sigma)$,
- the coefficients α_i, β_j are all strictly positive.

Then $\mathrm{star}_{\Delta}(\rho)$ is strictly convex if and only in every such dependence we have $\beta < 0$. It is weakly convex if and only if in every such dependence we have $\beta \leq 0$.

As a step toward proving assertions (i), (iii), given a dependence as in (5) we apply the orthogonal projection π onto the space perpendicular to all of the $\{n_i\}_{i \in F}$, yielding the following equation

$$0 = \beta\, \pi(n) + \sum_{j \in G} \beta_j\, \pi(m_j),$$

and then taking the inner product with $\pi(n)$ on both sides yields

$$0 = \beta\, \langle \pi(n), \pi(n) \rangle + \sum_{j \in G} \beta_j\, \langle \pi(m_j), \pi(n) \rangle. \tag{6}$$

To prove (iii), we assume P is obtuse and that there is some choice of a dependence as in (5) such that $\beta \geq 0$. But then we reach a contradiction in Equation (6), because we assumed $\beta_j > 0$, we have $\langle \pi(m_j), \pi(n) \rangle > 0$ by virtue of the obtuseness of P, and $\langle \pi(n), \pi(n) \rangle$ is always non-negative.

To prove (i), we assume P is non-acute and that there is some choice of a dependence as in (5) such that $\beta > 0$. Then similar considerations in equation (6) imply that we must have $\langle \pi(n), \pi(n) \rangle = 0$, i.e. $\pi(n) = 0$. This would imply $\langle n_i, n \rangle = 0$ for each i in F. To reach a contradiction from this, take the inner product with n on both sides of equation (5), to obtain

$$0 = \beta\, \langle n, n \rangle + \sum_{j \in G} \beta_j\, \langle m_j, n \rangle.$$

Non-acuteness (even in codimension 1) of P implies $\langle m_j, n \rangle \geq 0$, and $\langle n, n \rangle$ is always positive, so this last equation is a contradiction to $\beta > 0$. $\square$

One source of non-acute simple polytopes are finite Coxeter groups (see [24, Chapter 1] for background). Recall that a finite *Coxeter group* is a finite group W acting on a Euclidean space and generated by reflections. Given a finite Coxeter group W, there is associated a set of (normalized) *roots* Φ by taking all the unit normals of reflecting hyperplanes. Let Z be the *zonotope* ([36, §7.3]) associated with Φ, that is,

$$Z = \left\{ \sum_{\alpha \in \Phi} c_\alpha \alpha : 0 \le c_\alpha \le 1 \right\}.$$

Proposition 4.5. *The zonotope Z associated to any finite Coxeter group W is non-acute and simple. Furthermore Z is obtuse in codimension 1 if W is irreducible.*

Proof. We refer to [24] for all facts about Coxeter groups used in this proof.

By general facts about zonotopes [36, §7.3], the normal fan Δ of Z is the complete fan cut out by the hyperplanes associated with reflections in W. The maximal cones in this fan are the *Weyl chambers* of W, which are all simplicial cones. Hence Z is a simple polytope.

To show that Z is non-acute, we must show that each of its faces is non-acute in codimension 1. However, these faces are always affine translations of Coxeter zonotopes corresponding to standard parabolic subgroups of W. So we only need to show Z itself is non-acute in codimension 1. This is equivalent to showing that every pair of rays in Δ which span a 2-dimensional cone form a non-obtuse angle.

Because W acts transitively on the Weyl chambers in Δ, we may assume that this pair of rays lie in the *fundamental Weyl chamber*, that is, we may assume that these rays come from the dual basis to some choice of simple roots $\alpha_1, \ldots, \alpha_d$. Since every choice of simple roots has the property that $\langle \alpha_i, \alpha_j \rangle \le 0$ for all $i \ne j$, the first assertion follows from the first part of Lemma 4.6 below.

The second assertion follows from Lemma 4.6 (ii) below. This is because the obtuseness graph for any choice of simple roots associated with a Coxeter group W is isomorphic to the (unlabelled) Coxeter graph, and the Coxeter graph is connected exactly when W is irreducible. $\square$

The following lemma was used in the preceding proof.

Lemma 4.6. *Let $\{\alpha_i\}_{i=1}^d$ be a basis for $\mathbb{R}^d$ with $\langle \alpha_i, \alpha_j \rangle \le 0$ for all $i \ne j$. Then the dual basis $\{\alpha_i^\vee\}_{i=1}^d$ satisfies*

(i) $\langle \alpha_i^\vee, \alpha_j^\vee \rangle \ge 0$ *for all $i \ne j$, and*

(ii) $\langle \alpha_i^\vee, \alpha_j^\vee \rangle > 0$ *for all $i \ne j$ if the "obtuseness graph" on $\{1, 2, \ldots, d\}$, having an edge $\{i, j\}$ whenever $\langle \alpha_i, \alpha_j \rangle < 0$, is connected.*

Proof. We prove assertion (i) by induction on d, with the cases $d = 1, 2$ being trivial. In the inductive step, assume $d \ge 3$. Without loss of generality, we must show $\langle \alpha_1, \alpha_2 \rangle \ge 0$.

Let $\pi : \mathbb{R}^d \to \alpha_d^\perp$ be orthogonal projection. Write

$$\alpha_i = \pi(\alpha_i) + c_i \alpha_d$$

for each $i \le d - 1$.

Our first claim is that $c_i \le 0$ for each $i \le d - 1$. To see this, note that

$$\begin{aligned} 0 &\ge \langle \alpha_i, \alpha_d \rangle \\ &= \langle \pi(\alpha_i), \alpha_d \rangle + c_i \langle \alpha_d, \alpha_d \rangle \\ &= c_i \langle \alpha_d, \alpha_d \rangle. \end{aligned}$$

Our second claim is that $\langle \pi(\alpha_i), \pi(\alpha_j)\rangle \leq 0$ for $1 \leq i \neq j \leq d-1$. To see this, note that

$$\begin{aligned} 0 &\geq \langle \alpha_i, \alpha_j \rangle \\ &= \langle \pi(\alpha_i), \pi(\alpha_j)\rangle + c_j\langle \alpha_d, \pi(\alpha_j)\rangle + c_i\langle \pi(\alpha_i), \alpha_d\rangle + c_ic_j\langle \alpha_d, \alpha_d\rangle \\ &= \langle \pi(\alpha_i), \pi(\alpha_j)\rangle + c_ic_j\langle \alpha_d, \alpha_d\rangle. \end{aligned}$$

and the last term in the last sum is non-negative by our first claim.

Our third claim is that $\{\pi(\alpha_i)\}_{i=1}^{d-1}$ and $\{\alpha_i^\vee\}_{i=1}^{d-1}$ are dual bases inside $\alpha_d^\perp$. To see this, note that

$$\begin{aligned} \delta_{ij} &= \langle \alpha_i, \alpha_j^\vee \rangle \\ &= \langle \pi(\alpha_i), \alpha_j^\vee \rangle + c_i\langle \alpha_d, \alpha_j^\vee\rangle \\ &= \langle \pi(\alpha_i), \alpha_j^\vee \rangle. \end{aligned}$$

From the second and third claims, we can apply induction to conclude that $\langle \alpha_i^\vee, \alpha_j^\vee\rangle \geq 0$ for $1 \leq i \neq j \leq d-1$, and in particular this holds for $i = 1, j = 2$ as desired.

To prove assertion (ii), we use the same induction on d, with the cases $d = 1, 2$ still being trivial. We must in addition show that if $\{\alpha_i\}_{i=1}^{d}$ have connected obtuseness graph, then there is a re-indexing (that is a choice of α_d) so that $\{\pi(\alpha_i)\}_{i=1}^{d-1}$ also satisfies this hypothesis. To achieve this, let α_d correspond to a node d in the obtuseness graph whose removal does not disconnect it, e.g. choose d to be a leaf in some spanning tree for the graph. Then for $i \neq j$ with $i, j \leq d-1$ we have

$$\langle \pi(\alpha_i), \pi(\alpha_j)\rangle = \langle \alpha_i, \alpha_j\rangle - c_ic_j\langle \alpha_d, \alpha_d\rangle.$$

This implies $\pi(\alpha_i), \pi(\alpha_j)$ were obtuse whenever α_i, α_j were, so the obtuseness graph remains connected. □

Remark 4.7.
If the finite Coxeter group W is crystallographic (or a Weyl group) then a crystallographic root system associated with W gives a more natural choice of hyperplane normals to use than the unit normals in defining the Coxeter zonotope Z. With this choice, the normal fan Δ is not only rational with respect to the weight lattice N, but also $m(Z) = 1$ with respect to the dual lattice M. Hence Z is Delzant, so that the toric variety X_Δ is smooth.

For the classical Weyl groups W of types $A, B(= C), D$, there are known generating functions for the h-vectors of the associated Coxeter zonotopes Z, which specialize to give explicit generating functions for the signature $\sigma(Z)$. The h-vector in this case turns out to give the distribution of the elements of the Weyl group W according to their *descents*, i.e. the number of simple roots which they send to negative roots (see [5]). Generating functions for the descent distribution of all classical Weyl groups may be found in [32]. For example, it follows from these that if $Z_{A_{n-1}}$ is the Coxeter zonotope of type A_{n-1}, then we have the formula

$$\sum_{n\geq 0} \sigma(Z_{A_{n-1}})\frac{x^n}{n!} = \tanh(x)$$

which was computed in [17, Example p. 52] for somewhat different reasons.

The fact that Coxeter zonotopes have locally convex normal fans also follows because these normal fans come from *simplicial hyperplane arrangements* (we thank

M. Davis for suggesting this). Say that an arrangement of linear hyperplanes in $\mathbb{R}^d$ is *simplicial* if it decomposes $\mathbb{R}^d$ into a simplicial fan.

Proposition 4.8. *The fan Δ associated to a simplicial hyperplane arrangement is locally convex.*

Proof. For each ray ρ of Δ, we will express $\mathrm{star}_\Delta(\rho)$ as an intersection of closed half-spaces defined by a subset of the hyperplanes of $\mathcal{A}$, thereby showing that it is convex. To describe this intersection, note that since Δ is simplicial, given any chamber (d-dimensional cone) σ of Δ that contains ρ, there is a unique hyperplane H_σ bounding σ which does not contain ρ. Choose a linear functional u_σ which vanishes on H_σ and is positive on ρ, and then we claim that

$$|\mathrm{star}_\Delta(\rho)| = \bigcap_{\text{chambers } \sigma \supset \rho} \{u_\sigma \geq 0\}.$$

To see that the left-hand side is contained in the right, note that for any chamber σ containing ρ and any hyperplane H in $\mathcal{A}$ not containing ρ, we must have σ and ρ on the same side of H. Consequently, for every pair of chambers σ, σ' containing ρ we have $u_\sigma \geq 0$ on σ' (and symmetrically $u_{\sigma'} \geq 0$ on σ). This implies the desired inclusion.

To show that the right-hand side is contained in the left, since both sets are closed and d-dimensional, it suffices to show that every chamber in the left is contained in the right, or contrapositively, that every chamber not contained in the right is not in the left. Given a chamber σ not in the right, consider the unique chamber σ' containing ρ which is "perturbed in the direction of σ". In other words, σ' is chosen so that it contains a vector $v + \epsilon w$ where v is any non-zero vector in ρ, w is any vector in the interior of σ, and ϵ is a very small positive number. Since σ does not contain ρ, we know $\sigma \neq \sigma'$, and hence there is at least one hyperplane of $\mathcal{A}$ separating them. Since Δ is simplicial, every bounding hyperplane of σ' except for $H_{\sigma'}$ will contain r, and hence have σ and σ' on the same side (by construction of σ'). This means $H_{\sigma'}$ must separate σ and σ', so $u_{\sigma'} < 0$ on σ, implying σ is not in the left-hand side. ☐

The Coxeter zonotopes of type A are related to another infinite family of simple polytopes, the *associahedra*, which turn out to have locally convex normal fans. Recall [28] that the associahedron $\mathcal{A}_n$ is an $(n-3)$-dimensional polytope whose vertices correspond to all possible parenthesizations of a product $a_1 a_2 \cdots a_{n-1}$, and having an edge between two parenthesizations if they differ by a single "rebracketing". Equivalently, vertices of $\mathcal{A}_n$ correspond to triangulations of a convex n-gon, and there is an edge between two triangulations if they differ only by a "diagonal flip" within a single quadrilateral.

Proposition 4.9. *The associahedron $\mathcal{A}_n$ has a realization as a simple convex polytope whose normal fan Δ_n is locally convex.*

Proof. In [28, §3], the normal fan Δ_n is thought of as a simplicial complex, and more precisely, as the boundary of a simplicial polytope Q_n having the origin in its interior. There Q_n is constructed by a sequence of stellar subdivisions of certain faces of an $(n-3)$-simplex having vertices labelled $1, 2, \ldots, n-2$. Since the normal fan Δ_n is simplicial, the associahedron is simple (as is well-known).

Our strategy for showing Δ_n is locally convex is to relate it to the Weyl chambers of type A_{n-3}. If we assume that the $(n-3)$-simplex above is regular, and take its barycenter as the origin in $\mathbb{R}^{n-3}$, then the barycentric subdivision of its boundary is a simplicial polytope isomorphic to the Coxeter complex for type A_{n-3}. Hence the normal fan Δ_n of $\mathcal{A}_n$ refines the fan of Weyl chambers for type A_{n-3}. Note that an alternate description of this Weyl chamber fan is that it is the set of all chambers cut out by the hyperplanes $x_i = x_j$, that is, its (open) chambers are defined by inequalities of the form $x_{\pi_1} > x_{\pi_2} > \cdots > x_{\pi_{n-2}}$ for permutations π of $\{1, 2, \ldots, n-2\}$.

To show Δ_n is locally convex, we must first identify the rays ρ of Δ_n, and then show that $\mathrm{star}_{\Delta_n}(\rho)$ is a pointed convex cone. According to the construction of [28, §3], a ray ρ_{ij} of Δ_n corresponds to the barycenter of a face of the $(n-3)$-simplex which is spanned by a set of vertices labelled by a *contiguous* sequence $i, i+1, \ldots, j-1, j$ with $1 \leq i \leq j \leq n-2$, with $(i,j) \neq (1, n-2)$. It is then not hard to check from the construction that $\mathrm{star}_{\Delta}(\rho_{ij})$ consists of the union of all (closed) chambers for type A_{n-3} which satisfy the inequalities

$$x_i, x_{i+1}, \ldots, x_{j-1}, x_j \geq x_{i-1}, x_{j+1}$$

(where here we omit the inequalities involving x_{i-1} if $i = 1$, and similarly for x_{j+1} if $j = n+2$). It is clear that these inequalities describe a convex cone, and hence Δ_n is locally convex. $\square$

It follows then from this Proposition and Theorem 1.2(ii) that $(-1)^{\frac{n-3}{2}}\sigma(\mathcal{A}_n) \geq 0$ for n odd (and of course, $\sigma(\mathcal{A}_n) = 0$ for n even). However, as in the case of Coxeter zonotopes of type A, we can compute $\sigma(\mathcal{A}_n)$ explicitly using the formulas for the f-vector or h-vector of $\mathcal{A}_n$ given in [28, Theorem 3]. Specifically, these formulas imply that for $n \geq 3$ we have

$$\begin{aligned}
\sigma(\mathcal{A}_n) &= \sum_{i=0}^{n-3} (-1)^i \frac{1}{n-1} \binom{n-3}{i} \binom{n-1}{i+1} \\
&= {}_2F_1 \left(\begin{matrix} 3-n & 2-n \\ & 2 \end{matrix} \middle| -1 \right) \\
&= \begin{cases} (-1)^{\frac{n-3}{2}} C_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}
\end{aligned}$$

where C_n denotes the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$. Here the ${}_2F_1$ is hypergeometric series notation, and the last equality uses Kummer's summation of a well-poised ${}_2F_1$ at -1 (see e.g. [2, p. 9]).

Returning to the discusion of non-acute and obtuse polytopes, it is worth noting the following facts, pointed out to us by M. Davis. Recall that a simplicial complex K is called *flag* if every set of vertices $v_1, \ldots, v_r$ which pairwise span edges of K also jointly span an $(r-1)$-simplex of K.

Proposition 4.10. *A polytope P is non-acute (resp. obtuse) if and only if each of its 2-dimensional faces are non-acute (resp. obtuse).*

Furthermore, non-acuteness of any polytope P implies that P is simple

Proof. The first assertion for non-acute polytopes follows from an easy lemma due to Moussong [6, Lemma 2.4.1]. In the notation of [6], saying that every 2-dimensional face of P is non-acute (in codimension 1) is equivalent to saying that for every vertex v of P, the spherical simplex $\sigma = Lk(v, P)$ has size $\geq \frac{\pi}{2}$. Then [6, Lemma 2.4.1] asserts that every link $Lk(\tau, \sigma)$ of a face of this spherical simplex also has size $\geq \frac{\pi}{2}$. But a face F of P containing v has $Lk(v, F)$ of the form $Lk(\tau, \sigma)$ for some τ, and hence F is non-acute in codimension 1 when considered as a polytope in its own right. That is, P is non-acute. An easy adaptation of this argument to the obtuse case proves the first assertion of the proposition for obtuse polytopes.

The fact that non-acuteness implies simplicity again comes from considering the spherical simplex $\sigma = Lk(v, P)$ for any vertex v, which will have size $\geq \frac{\pi}{2}$. Then its polar dual spherical convex polytope σ^* will have all of its dihedral angles less than or equal to $\frac{\pi}{2}$. This forces σ^* to be a spherical simplex, by [35, p. 44], and hence σ itself must be a spherical simplex. This implies v has exactly d neighbors, so P is simple. □

Obtuse polytopes turn out to be relatively scarce in comparison with non-acute polytopes, For example, it is easily seen that Coxeter zonotopes, although always non-acute by Proposition 4.5, are not in general obtuse in dimensions 3 and higher. It is easy to find obtuse polytopes in dimensions up to 4:

- in dimension 2, the regular n-gons for $n \geq 5$,
- in dimension 3, the dodecahedron,
- in dimension 4 the "120-cell" (see [7, pp. 292-293])

However M. Davis has pointed out to us that in dimensions 5 higher, there are no obtuse polytopes, due to a result of Kalai [25, Theorem 1] (see also [35, p. 68] for the case of simple polytopes): every d-dimensional convex polytope for $d \geq 5$ contains either a triangular or quadrangular 2-dimensional face.

5. Relation to conjectures of Hopf and of Charney and Davis

In this section we discuss the relation of Theorem 1.2(i) to the conjectures of Hopf and of Charney and Davis mentioned in the Introduction.

Let M^d be a compact d-dimensional closed Riemannian manifold. When d is odd, Poincaré duality implies that the Euler characteristic $\chi(M)$ vanishes. When d is even, a conjecture of H. Hopf (see e.g. [6]) asserts that if M^d has non-positive sectional curvature, the Euler characteristic $\chi(M^d)$ satsfies

$$(-1)^{\frac{d}{2}}\chi(M^d) \geq 0.$$

This result is known for $d = 2, 4$ by Chern's Gauss-Bonnet formula, but open for general d; see [6, §0] for some history.

Charney and Davis [6] explored a combinatorial analogue of this conjecture, and we refer the reader to their paper for terms which are not defined precisely here. Let M^d be a compact d-dimensional closed manifold which has the structure of a (locally finite) *Euclidean cell complex*, that is, it is formed by gluing together convex polytopes via isometries of their faces. One can endow such a cell complex with a metric space structure that is Euclidean within each polytopal cell, making it a *geodesic space*. Gromov has defined a notion of when a geodesic space is *nonpositively curved*, and Charney and Davis made the following conjecture:

Conjecture 5.1. [6, Conjecture A] *If M^d is a non-positively curved, piecewise Euclidean, closed manifold with d even, then*

$$(-1)^{\frac{d}{2}}\chi(M^d) \geq 0.$$

For piecewise Euclidean cell complexes, nonpositive curvature turns out to be equivalent to a local condition at each vertex. Specifically, at each vertex v of

M^d, one has a piecewise spherical cell complex $Lk(v, M^d)$ called the *link* of v in M^d, which is homeomorphic to a *generalized homology* $(d-1)$*-sphere*, and inherits its own geodesic space structure. Nonpositive curvature of M^d turns out to be a metric condition on each of these complexes $Lk(v, M^d)$. Charney and Davis show [6, (3.4.3)] that the Euler characteristic $\chi(M^d)$ can be written as the sum of certain local quantities $\kappa(Lk(v, M^d))$ defined in terms of the metric structure on $Lk(v, M^d)$:

$$\chi(M^d) = \sum_v \kappa(Lk(v, M^d)). \tag{7}$$

In the special case where the polytopes in the Euclidean cell decomposition of M^d are all right-angled cubes, the links $Lk(v, M^d)$ are all simplicial complexes, and the quantity $\kappa(Lk(v, M^d))$ has a simple combinatorial expression purely in terms of the numbers of simplices of each dimension in these complexes (that is, independent of their metric structure). Furthermore, in this case, non-positive curvature corresponds to the combinatorial condition that each link is a *flag complex*, that is, the minimal subsets of vertices in $Lk(v, M^d)$ which do not span a simplex always have cardinality two. They then noted that in this special case, Conjecture 5.1 would follow via Equation (7) from

Conjecture 5.2. [6, Conjecture D] *If Δ is a flag simplicial complex triangulating a generalized homology $(d-1)$-sphere with d even, then*

$$(-1)^{\frac{d}{2}}\kappa(\Delta) \geq 0.$$

This *Charney-Davis conjecture* is trivial for $d = 2$, has recently been proven by Davis and Okun [15] for $d = 4$ using L_2-homology of Coxeter groups, and is also known (by an observation of Babson and a result of Stanley; see [6, §7]) for the special class of flag simplicial complexes which are barycentric subdivisions of boundaries of convex polytopes.

Local convexity of simplicial fans turns out to be stronger than flagness:

Proposition 5.3. *A locally convex complete simplicial fan Δ in $\mathbb{R}^d$ is flag, when considered as a simplicial complex triangulating a $(d-1)$-sphere.*

Proof. Assume that Δ is not flag, so that there exist rays $\rho_1, \ldots, \rho_k$ whose convex hull $\sigma := \text{conv}\ (\rho_1, \ldots, \rho_k)$ is *not* a cone of Δ, but $\text{conv}\ (\rho_i, \rho_j)$ *is* a cone of Δ for each i, j. Choose such a collection of rays of minimum cardinality k, so that $\text{conv}\ (\rho_1, \ldots, \hat{\rho_i}, \ldots, \rho_k)$ is a cone of Δ for each i (in other words, the boundary complex $\partial\sigma$ is a subcomplex of Δ).

We wish to show that $\text{star}_\Delta(\rho_1)$ is not convex. To see this, consider $\sigma \cap \text{star}_\Delta(\rho_1)$, that is, the collection of cones

$$\{\sigma' \cap \sigma : \sigma' \in \text{star}_\Delta(\rho_1)\}.$$

Since σ is not a cone of Δ but $|\Delta| = \mathbb{R}^d$, this collection must contain at least one 2-dimensional cone of the form $\sigma' \cap \sigma = \text{conv}\ (\rho_1, \rho)$, where ρ is a ray of σ but $\rho \notin \{\rho_2, \ldots, \rho_k\}$. Since ρ lies inside σ and $\partial\sigma$ is a subcomplex of Δ, ρ cannot lie in $\partial\sigma$ (else some cone of $\partial\Delta$ would be further subdivided, and not be a cone of Δ). Consequently ρ lies in the interior of σ. Then the ray $\rho' := \rho - \epsilon\rho_1$ for very small $\epsilon > 0$ has the following properties:

- ρ' lies in σ, because ρ was in the interior of σ,

- ρ' therefore lies in the convex hull of $\mathrm{star}_\Delta(\rho_1)$, since σ does (as its extreme rays $\rho_1, \ldots, \rho_k$ of σ are all in $\mathrm{star}_\Delta(\rho_1)$),
- ρ' does not lie in $\mathrm{star}_\Delta(\rho_1)$, else it would lie in a cone σ' of Δ containing ρ_1, and then σ' would contain ρ in the relative interior of one of its faces, a contradiction.

Therefore $\mathrm{star}_\Delta(\rho_1)$ is not convex. □

In light of the preceeding proposition, one might ask if every flag simplicial sphere has a realization as a locally convex complete simplicial fan. We thank X. Dong for the following argument showing that an even weaker statement is false. One can show that complete simplicial fans always give rise to PL-spheres. Therefore if one takes the barycentric subdivision of any regular cellular sphere which is not PL (such as the double suspension of Poincaré's famous homology sphere), this will give a flag simplicial sphere which is not PL and therefore has no realization as a complete simplicial fan (let alone one which is locally convex).

Our results were motivated by the Charney-Davis conjecture and the following fact: when P is a simple d-polytope and Δ is its normal fan considered as a $(d-1)$-dimensional simplicial complex, one can check that

$$\sigma(P) = 2^d \kappa(\Delta). \tag{8}$$

As a consequence, we deduce the following from Proposition 5.3 and Theorem 1.2 (i).

Corollary 5.4. *Let P be rational simple polytope, Δ its normal fan. If Δ is locally convex, then it is flag and satisfies the Charney-Davis conjecture.*

In particular by Corollary 4.2, if P is a non-acute simple rational polytope then its normal fan Δ is flag and satisfies the Charney-Davis conjecture.

It is worth mentioning that the special case of Conjecture 5.1 considered in [6] where M^d is decomposed into right-angled cubes is "polar dual" to another special case that fits nicely with our results. Say that M^d has a *corner decomposition* if the local structure at every vertex in the decomposition is combinatorially isomorphic to the coordinate orthants in $\mathbb{R}^d$, that is, each link $Lk(v, M^d)$ has the combinatorial structure of the boundary complex of a d-dimensional *cross-polytope* or *hyperoctahedron*. (Note that this condition immediately implies that each of the d-dimensional polytopes in the decomposition must be simple). A straightforward counting argument (essentially equivalent to the calculation proving [6, (3.5.2)]) shows that for a manifold M^d with corner decomposition into simple polytopes $P_1, \ldots, P_N$ one has

$$\chi(M^d) = \frac{1}{2^d} \sum_{i=1}^{N} \sigma(P_i). \tag{9}$$

The following corollary is then immediate from this relation and Theorem 1.2.

Corollary 5.5. *Let M^d be an d-dimensional manifold with d even, having a corner decomposition.*

If each of the simple d-polytopes in the corner decomposition is rational and has normal fan which is locally convex, then

$$(-1)^{\frac{d}{2}} \chi(M^d) \geq 0.$$

In particular, this holds if each of the simple d-polytopes is non-acute.

Several interesting examples of manifolds with corner decompositions into simple polytopes that are either Coxeter zonotopes (hence non-acute) or associahedra (hence locally convex) may be found in [14].

There is also an important general construction of such manifolds called *mirroring* which we now discuss. This construction (or its polar dual) appears repeatedly in the work of Davis [10, 11, 13, 14], and was used in [6, §6] to show that the case of their Conjecture 5.1 for manifolds decomposed into right-angled cubes is equivalent to their Conjecture 5.2. In a special case, this construction begins with a generalized homology $(d-1)$-sphere L with n vertices and produces a cubical orientable generalized homology d-manifold ML having 2^n vertices, with the link at each of these vertices isomorphic to L. Hence we have

$$\chi(ML) = 2^n \cdot \kappa(L).$$

We wish to make use of the polar dual of this construction, which applies to an arbitrary simple d-dimensional polytope P, yielding an orientable d-manifold $M(P)$ with a corner decomposition having every d-dimensional cell isometric to P. The construction is as follows: denote the $(d-1)$-dimensional faces of P by $F_1, F_2, \ldots, F_n$, and let $M(P)$ be the quotient of 2^n disjoint copies $\{P_\epsilon\}_{\epsilon \in \{+,-\}^n}$ of P, in which two copies $P_\epsilon, P_{\epsilon'}$ are identified along their face F_i whenever ϵ, ϵ' differ in the i^{th} coordinate and nowhere else. As a consequence of equation (9) we have

$$\chi(M(P)) = 2^{n-d} \cdot \sigma(P), \tag{10}$$

which shows that the "non-acute" assertions in Corollaries 5.5 and 5.4 are equivalent.

We can now use the mirror construction to complete the proof of an assertion from the previous section. We are indebted to M. Davis for the statement and proof of this assertion.

Proof of Corollary 4.2 (i) without assuming rationality of P (as referred to in Remark 4.4): Assume that P is a simple non-acute d-dimensional polytope with d even. We wish to show that $(-1)^{\frac{d}{2}}\sigma(P) \geq 0$.

Construct $M(P)$ as above, a manifold with corner decomposition into non-acute simple polytopes having $\chi(M(P)) = 2^{n-d}\sigma(P)$ if P had n codimension 1 faces. In the notation of [6], this means that all the links $Lk(v, M(P))$ have size $\geq \frac{\pi}{2}$ and are combinatorially isomorphic to boundaries of cross-polytopes. This implies that these links' underlying simplicial complexes are flag complexes satisfying [6, Conjecture D'], and then [6, Proposition 5.7] implies that each of these links $Lk(v, M(P))$ satisfies [6, Conjecture C']. This implies that $(-1)^{\frac{d}{2}}\kappa(Lk(v, M(P))) \geq 0$. Combining this with Equation (7), we conclude that $(-1)^{\frac{d}{2}}\chi(M(P)) \geq 0$, and finally via Equation (10), that $(-1)^{\frac{d}{2}}\sigma(P) \geq 0$.

We note that a similar argument (involving an adaptation of [6, Lemma 2.4.1]) proves $(-1)^{\frac{d}{2}}\sigma(P) > 0$ when P is obtuse, but does not yield in any obvious way the stronger assertion of Corollary 4.2 (iii). □

6. Appendix: the conormal bundle of a toric divisor

In this appendix we describe the conormal bundle of a toric divisor on $X = X_\Delta$ when the fan Δ is complete and simplicial. Denote the collection of toric divisors on X by $D_1, \ldots, D_m$. As mentioned in Section 3, the conormal bundle of a divisor, say D_1, can be identified as the restriction of $O_X(-D_1)$ to D_1, which we renamed $O_{D_1}(-D_1)$. It corresponds to a continuous piecewise linear function:

$$\Psi^{D_1}_{-D_1} : N_{\mathbb{R}}/\rho_1 \to \mathbb{R}$$

as in the discussion of Section 3. Here ρ_1 is the ray in the fan Δ corresponding to the divisor D_1. We wish to identify the graph of $\Psi^{D_1}_{-D_1}$ with $\mathrm{link}_\Delta(\rho_1)$, which we recall is the boundary of $\mathrm{star}_\Delta(\rho_1)$, the latter being the union of all cones of Δ containing ρ_1.

Proposition 6.1. *Let D_1 be a toric divisor of a toric variety $X = X_\Delta$ with Δ simplicial. Then the graph of the piecewise linear function for $O_{D_1}(D_1)$ is affinely equivalent to the boundary $\mathrm{link}_\Delta(\rho_1)$ of $\mathrm{star}_\Delta(\rho_1)$, where ρ_1 is the ray corresponding to D_1.*

Proof. We can index the toric divisors $D_1, \ldots, D_m$ of X in such a way that $D = D_1$ and $D_2, ..., D_l$ are those which are adjacent to D_1. Let n_i be the first nonzero lattice point along the ray ρ_i corresponding to D_i. We choose a decomposition of N into a direct sum of $\mathbb{Z}n_1$ with another lattice N' which is isomorphic to N/ρ_1 (here we are abusing notation by referring to the quotient lattice $N/\mathbb{Z}n_1$ as N/ρ_1). Then we can write $n_i = b_i n_1 + c_i n_i'$ where $n_i' \in N' \cong N/\rho_1$ is indecomposable (i.e. not of the form kn_i'' for some integer k with $|k| \geq 2$ and $n_i'' \in N'$), and c_i is some nonnegative integer.

Now we choose the linear functional u on N such that $\langle u, n_1 \rangle = 1$ and its restriction to N' is zero. Then in the Chow group of X we have the following relation (see [18, p.106]):

$$\sum_{i=1}^{m} \langle u, n_i \rangle D_i = 0.$$

When we restrict this relation to the toric subvariety D_1 then those terms involving D_i with $i > l$ will disappear because they are disjoint from D_1, and using the formula on [18, p. 108], we have

$$\bigotimes_{i=1}^{l} O_{D_1}\left(\frac{\langle u, n_i \rangle}{c_i} D_i\right) = O_{D_1}.$$

Or equivalently, since $\langle u, n_1 \rangle = 1$ and $\langle u_i, n_1 \rangle = b_i$, we have

$$O_{D_1}(-D_1) = \bigotimes_{i=2}^{l} O_{D_1}\left(\frac{b_i}{c_i} D_i\right).$$

Now under the identification $N' \cong N/\rho_1$, the restriction of the divisor D_i to D_1 corresponds to the ray in N/ρ_1 spanned by n_i' when $2 \leq i \leq l$. Therefore the piecewise linear function $\Psi^{D_1}_{-D_1} : N_{\mathbb{R}}/\rho_1 \to \mathbb{R}$ is determined by $\Psi^{D_1}_{-D_1}(n_i') = \frac{b_i}{c_i}$. This implies the assertion of the proposition. ◻

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