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Signed Permutation Statistics and Cycle Type

VICTOR REINER

We derive a multivariate generating function which counts signed permutations by their cycle type and two other descent statistics, analogous to a result of Gessel and Reutenauer [4, 5] for (unsigned) permutations. The derivation uses a bijection which is the hyperoctahedral analogue of Gessel's necklace bijection.

1. INTRODUCTION

The goal of this paper is to derive a multivariate generating function counting signed permutations π by three statistics: their *descent number* $d(\pi)$, *major index* $\mathrm{maj}(\pi)$ and *cycle type* $\lambda(\pi)$. This provides an analogue for the *hyperoctahedral group* B_n of a result of Gessel [4] for the *symmetric group* S_n. Our method is to develop a bijection for B_n analogous to Gessel's *necklace bijection* for S_n (described in [2, 5, 11]). We hope that this bijection may have applications to other problems, similar to applications of the necklace bijection [2, 6].

After giving the relevant definitions in Section 2, we describe the key bijection in Section 3, and use it to derive the main result in Section 4. Section 5 presents a few applications; namely, generating functions counting descents of *involutions* in B_n, *domino tableaux*, and *hyperoctahedral* and *cubical derangements*.

2. DEFINITIONS

The *hyperoctahedral group* B_n is the group of *signed permutations*, i.e. permutations and sign changes of the co-ordinates in $\mathbf{R}^n$. We will use the following two-line notation for signed permutations: letting e_i denote the ith standard basis vector in $\mathbf{R}^n$, then

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & & \pi_n \end{pmatrix}$$

(where $\pi_i \in \{\pm 1, \ldots, \pm n\}$) means that $\pi(e_i) = \mathrm{sgn}(\pi_i) e_{|\pi_i|}$.

Let Λ be the following linear order on $\mathbf{Z} - 0$:

$$+1 <_\Lambda +2 <_\Lambda \cdots + n <_\Lambda \cdots <_\Lambda -n <_\Lambda \cdots <_\Lambda -2 <_\Lambda -1.$$

The *descent set* of π is then defined to be

$$D(\pi) = \{i: 1 \leq i \leq n: \pi_i <_\Lambda \pi_{i+1}\}$$

(where by convention we set $\pi_{n+1} = n + 1$). We then define its *descent number* to be $d(\pi) = \# D(\pi) + 1$ and its *major index* to be $\mathrm{maj}(\pi) = \sum_{i \in D(\pi)} i$. For example, if

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ +2 & +7 & -1 & -5 & +8 & +3 & +6 & -4 \end{pmatrix}$$

then $D(\pi) = \{3, 4, 5, 8\}$, $d(\pi) = 4 + 1 = 5$, and $\mathrm{maj}(\pi) = 3 + 4 + 5 + 8 = 20$.

A signed permutation decomposes uniquely into a product of commuting cycles just as permutations do. Given such a cycle

$$C = \begin{pmatrix} i_1 & i_2 & \cdots & i_{m-1} & i_m \\ \varepsilon_1 i_2 & \varepsilon_2 i_3 & & \varepsilon_{m-1} i_m & \varepsilon_m i_1 \end{pmatrix}$$

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where $\varepsilon_i = \pm 1$ and $i_j > 0$, we say that C is an *even cycle* (of size m) if $\varepsilon_1\varepsilon_2 \cdots \varepsilon_m = +1$ and an *odd cycle* if $\varepsilon_1\varepsilon_2 \cdots \varepsilon_m = -1$. The *cycle type* of π is the pair of vectors $\lambda(\pi) = ((\lambda_1, \lambda_2, \ldots), (\mu_1, \mu_2, \ldots))$ where π has λ_i even cycles of size i and μ_i odd cycles of size i. For example, if π is the same as in our previous example, then

$$\pi = \begin{pmatrix} 1 & 2 & 7 & 6 & 3 \\ +2 & +7 & +6 & +3 & -1 \end{pmatrix}\begin{pmatrix} 4 & 5 & 8 \\ -5 & +8 & -4 \end{pmatrix},$$

so

$$\lambda(\pi) = ((0, 0, 1, 0, 0, \ldots), (0, 0, 0, 0, 1, 0, 0, \ldots)).$$

It is worth noting (although we will not need it) that the conjugacy class of π in B_n determined by its cycle type $\lambda(\pi)$.

A **Z**-*word of length m* is a vector $a = (a_1, \ldots, a_m) \in \mathbf{Z}^n$. Given such a word a, we define $|a| = |a_1| + \cdots + |a_m|$ and $\max(a) = \max\{|a_i|\}_{i=1}^m$. We will have use for two different group actions on **Z**-words of length m which we now describe. The *cyclic group C_{2m}* of order $2m$ acts by having its generator g act as a cyclic shift with sign change $g(a_1, a_2, \ldots, a_m) = (a_2, \ldots, a_m, a_{-1})$. The group $\mathbf{Z}_2 \times C_m$ of order $2m$ acts by having the generator r of C_m act as a cyclic shift $r(a_1, a_2, \ldots, a_n) = (a_2, \ldots, a_m, a_1)$, and by having the generator v of $\mathbf{Z}_2$ act as a global sign change $v(a_1, \ldots, a_n) = (-a_1, \ldots, -a_n)$.

An orbit or equivalence class P of **Z**-words of length m under the G_{2m}-action will be called a *twisted necklace* of size m, and an orbit D under the $\mathbf{Z}_2 \times C_m$-action will be called a *blinking necklace* of size m. Note that both group actions leave $|a|$ or $\max(a)$ invariant, so $|P|$, $\max(P)$, $|D|$ and $\max(D)$ all make sense for a twisted necklace P or blinking necklace D. We say that a twisted necklace P is *primitive* if its C_{2m}-action is free (i.e. no non-trivial group elements fixes any vector in the orbit P), and we say that a blinking necklace is *primitive* if its C_m-action is free (although its $\mathbf{Z}_2 \times C_m$-action need not be free). A *signed ornament f* is a *set* of primitive twisted necklaces along with a *multiset* of primitve blinking necklaces. We say that f has *type* $\lambda(f) = ((\lambda_1, \lambda_2, \ldots), (\mu_1, \mu_2, \ldots))$ if f consists of λ_i blinking necklaces of size i and μ_i twisted necklaces of size i. We also say that the size of f is the sum of the sizes of all of its blinking necklaces, and that

$$|f| = \sum_{\text{blinking necklaces } D \in f} |D| + \sum_{\text{twisted necklaces } P \in f} |P|,$$

$$\max(f) = \max\{\max(D), \max(P)\}_{\text{blinking necklaces } D,\text{ twisted necklaces } P \text{ in } f}.$$

See Figure 1 for some examples, and a way of visualizing twisted necklaces, blinking necklaces and signed ornaments.

Some last bits of terminology are as follows: given a subset $D \subseteq \{1, 2, \ldots, n\}$, we say that a vector $s = (s_1, \ldots, s_n) \in \mathbf{N}^n$ is *D-compatible* if:

(i) $s_1 \geq \cdots \geq s_n \geq s_{n+1} = 0$;
(ii) $s_i > s_{i+1}$ whenever $i \in D$.

3. The Signed Ornament Bijection

Theorem 3.1. *Let $\mathcal{O}_n$ be the set of all signed ornaments of size n, and $\mathcal{P}_n$ the set of all pairs (π, a), where $\pi \in B_n$ and $s \in \mathbf{N}^n$ is $D(\pi)$-compatible. Then there is a bijection ϕ: $\mathcal{O}_n \to \mathcal{P}_n$. Furthermore, if $\phi(f) = (\pi, s)$ then we have*:

(1) $\lambda(f) = \lambda(\pi)$;
(2) $|f| = |s|$;
(3) $\max(f) = \max(s)$.

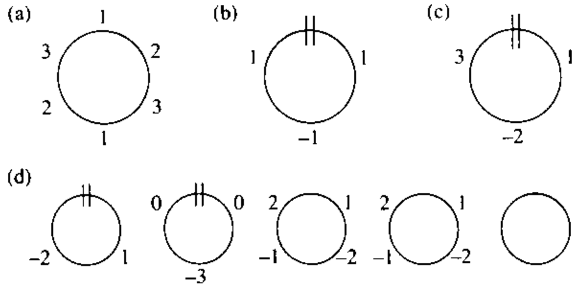

FIGURE 1. (a) A blinking necklace $D = \overline{(1, 2, 3, 1, 2, 3)}^{\mathbf{Z}_2 \times C_6}$ which is not primitive since $r^3(1, 2, 3, 1, 2, 3) = (1, 2, 3, 1, 2, 3)$. We visualize D as the sequence 1, 2, 3, 1, 2, 3 read clockwise around a ring. The generator r of C_m acts as a shift of the numbers counterclockwise by one position. (b) A twisted necklace $P = \overline{(1, -1, 1)}^{C_6}$ which is not primitive since $g^2(1, -1, 1) = (1, -1, 1)$. We visualize P as the sequence $1, -1, 1$ read clockwise around a ring, beginning to the right of the marker at 12 o'clock. The generator g of C_{2m} acts as a sign change of the number to the right of the marker and then a shift of the numbers counterclockwise. (c) A primitive twisted necklace $P = \overline{(1, -2, 3)}^{C_6}$. (d) A signed ornament $f = \{\overline{(1, 2)}^{C_4}, \overline{(0, -3, 0)}^{C_6}, \overline{(1, -2, -1, 2)}^{\mathbf{Z}_2 \times C_4}, \overline{(1, -2, -1, 2)}^{\mathbf{Z}_2 \times C_4}, \overline{(0)}^{\mathbf{Z}_2 \times C_1}\}$ of type $\lambda(f) = ((0, 1, 1, 0, 0, \ldots), (1, 0, 0, 2, 0, 0, \ldots))$, size 14, with $|f| = 18$ and $\max(f) = 3$.

SKETCH OF PROOF. The bijection is similar to Gessel's necklace bijection, described in [2, 5, 11]. We will describe the bijection ϕ and its inverse, omitting the straightforward verification that they are well-defined and have the image sets that we have claimed.

To describe ϕ, we start with a signed ornament f of size n. If $(f_1, \ldots, f_n)$ is the list of entries appearing in all the twisted necklaces and blinking necklaces of f in any order, let $s = (s_1, \ldots, s_n)$ be the weakly decreasing rearrangement of $(|f_1|, \ldots, |f_n|)$. For example, if f is the signed ornament of Figure 1(d), then $s = (3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0)$.

Defining π takes more work. We start by picking one representative $\mathbf{Z}$-word for each twisted necklace and blinking necklace in f. We also choose a linear order (to be used later for breaking ties) on each set of identical blinking necklaces in f. Given a position x on some twisted necklace or blinking necklace, we define an infinite word $w_x = (w_x^0, w_x^1, \ldots)$ as follows: w_x^0 is the value in position x in our chosen representative, while w_x^i is the value in the same position after applying the cyclic action i times (i.e. applying g^i if x is on a twisted necklace, r^i if on a blinking necklace). Alternatively, one can think of w_x as the infinite word obtained by starting at position x, reading clockwise and wrapping around the twisted necklace or blinking necklace that x lies on (with a sign change every time your go past a marker on a twisted necklace—see Figure 1). We define $\varepsilon_x = \pm 1$ as follows: $\varepsilon_x = -1$ if the first non-zero co-ordinate in $w_x = (w_x^0, w_x^1, \ldots)$ is negative, else $\varepsilon_x = +1$. Next we linearly order the positions $\{x\}$ by lexicographically ordering the words $\{\varepsilon_x w_x\}$ from largest to smallest, and break ties using the following procedure:

> If $\varepsilon_{x_1} w_{x_1} = \cdots = \varepsilon_{x_m} w_{x_m}$ for some set of positions $\{x_1, \ldots, x_n\}$, then one can check that at most one of them lies on a twisted necklace, and all the rest lie on copies of the same blinking necklace. We can relabel the x_i's to have $x_1, \ldots, x_{x-1}$ on distinct copies of the same blinking necklace with $\varepsilon_{x_1} = \cdots = \varepsilon_{x_k - 1} = +1$, x_k lying on a twisted necklace, and $x_{k+1}, \ldots, x_n$ lying on distinct copies of the same blinking necklace with $\varepsilon_{x_{k+1}} = \cdots = \varepsilon_{x_n} = -1$. We then linearly order them
>
> $$x_1 < \cdots < x_k < x_{k+1} < \cdots < x_n$$
>
> in such a way that $x_1 < \cdots < x_k$ *agrees* with the linear order chosen beforehand

on their identical blinking necklaces, and $x_{k+1} < \cdots < x_n$ is the *reverse* of the linear order chosen beforehand on their identical blinking necklaces.

Having linearly ordered the positions $\{x\}$, we let r_x be the rank of x in this order (e.g. if x comes first then $r_x = 1\}$. We then define π by saying that for each blinking necklace consisting of positions $(x_1, \ldots, x_n)$ in f, π has an even cycle

$$\begin{pmatrix} r_{x_1} & r_{x_2} & \cdots & r_{x_{m-1}} & r_{x_m} \\ \varepsilon_{x_1}\varepsilon_{x_2}r_{x_2} & \varepsilon_{x_2}\varepsilon_{x_3}r_{x_3} & & \varepsilon_{x_{m-1}}\varepsilon_{x_m}r_{x_m} & \varepsilon_{x_m}\varepsilon_{x_1}r_{x_1} \end{pmatrix},$$

and for each twisted necklace consisting of positions $(x_1, \ldots, x_n)$ in f, π has an odd cycle

$$\begin{pmatrix} r_{x_1} & r_{x_2} & \cdots & r_{x_{m-1}} & r_{x_m} \\ \varepsilon_{x_1}\varepsilon_{x_2}r_{x_2} & \varepsilon_{x_2}\varepsilon_{x_3}r_{x_3} & & \varepsilon_{x_{m-1}}\varepsilon_{x_m}r_{x_m} & -\varepsilon_{x_m}\varepsilon_{x_1}r_{x_1} \end{pmatrix}.$$

For example, let f be the signed ornament of Figure 1(d). Choosing representatives, and linearly ordering identical blinking necklaces arbitrarily we obtain

$$f = \overbrace{(1,-2)(0,-3,0)}^{\text{twisted necklaces}}\ \overbrace{(1,-2,-1,2)<(1,-2,-1,2)(0).}^{\text{blinking necklaces}}$$
$$x_1x_2 \quad x_3x_4x_5 \qquad x_6x_7x_8x_9 \qquad x_{10}x_{11}x_{12}x_{13}x_{14}$$

We then have

$w_{x_1} = 1, -2, -1, 2, \ldots$ $\quad \varepsilon_{x_1} = +1$ $\quad \varepsilon_{x_1}w_{x_1} = 1, -2, -1, 2, \ldots$

$w_{x_2} = -2, -1, 2, 1, \ldots$ $\quad \varepsilon_{x_2} = -1$ $\quad \varepsilon_{x_2}w_{x_2} = 2, 1, -2, -1, \ldots$

$w_{x_3} = 0, -3, 0, 0, 3, 0, \ldots$ $\quad \varepsilon_{x_3} = -1$ $\quad \varepsilon_{x_3}w_{x_3} = 0, 3, 0, 0, -3, 0, \ldots$

$w_{x_4} = -3, 0, 0, 3, 0, 0, \ldots$ $\quad \varepsilon_{x_4} = -1$ $\quad \varepsilon_{x_4}w_{x_4} = 3, 0, 0, -3, 0, 0, \ldots$

$w_{x_5} = 0, 0, 3, 0, 0, -3, \ldots$ $\quad \varepsilon_{x_5} = +1$ $\quad \varepsilon_{x_5}w_{x_5} = 0, 0, 3, 0, 0, -3, \ldots$

$w_{x_6} = w_{x_{12}} = 1, -2, -1, 2, \ldots$ $\quad \varepsilon_{x_6} = \varepsilon_{x_{12}} = +1$ $\quad \varepsilon_{x_6}w_{x_6} = \varepsilon_{x_{12}}w_{x_{12}} = 1, -2, -1, 2, \ldots$

$w_{x_7} = w_{x_{13}} = -2, -1, 2, 1, \ldots$ $\quad \varepsilon_{x_7} = \varepsilon_{x_{13}} = -1$ $\quad \varepsilon_{x_7}w_{x_7} = \varepsilon_{x_{13}}w_{x_{13}} = 2, 1, -2, -1, \ldots$

$w_{x_8} = w_{x_{10}} = -1, 2, 1, -2, \ldots$ $\quad \varepsilon_{x_8} = \varepsilon_{x_{10}} = -1$ $\quad \varepsilon_{x_8}w_{x_8} = \varepsilon_{x_{10}}w_{x_{10}} = 1, -2, -1, 2, \ldots$

$w_{x_9} = w_{x_{11}} = 2, 1, -2, -1, \ldots$ $\quad \varepsilon_{x_9} = \varepsilon_{x_{11}} = +1$ $\quad \varepsilon_{x_9}w_{x_9} = \varepsilon_{x_{11}}w_{x_{11}} = 2, 1, -2, -1, \ldots$

$w_{x_{14}} = 0, 0, \ldots$ $\quad \varepsilon_{x_{14}} = +1$ $\quad \varepsilon_{x_{14}}w_{x_{14}} = 0, 0, \ldots$

Ranking the x_i according to the lexicographic order on $\varepsilon_i x_i$ (and using our tie-breaking scheme) gives

position:	x_4	x_9	x_{11}	x_2	x_{13}	x_7	x_6	x_{12}	x_1	x_{10}	x_8	x_3	x_7	x_{14}
rank:	1	2	3	4	5	6	7	8	9	10	11	12	13	14

which then produces

$$\pi = \begin{pmatrix} 9 & 4 \\ -4 & +9 \end{pmatrix}\begin{pmatrix} 12 & 1 & 13 \\ +1 & -12 & +12 \end{pmatrix}\begin{pmatrix} 7 & 6 & 11 & 2 \\ -6 & +11 & -2 & +7 \end{pmatrix}\begin{pmatrix} 10 & 3 & 8 & 5 \\ -3 & +8 & -5 & +10 \end{pmatrix}\begin{pmatrix} 14 \\ +14 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ -13 & +7 & +8 & +9 & +10 & +11 & -6 & -5 & -4 & -3 & -2 & +1 & +12 & +14 \end{pmatrix}.$$

We have $D(\pi) = \{1, 11\}$ and so the sequence $s = (3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 0, 0, 0)$ is $D(\pi)$-compatible, as claimed.

It is easy to see that the map ϕ is well-defined (i.e. it does not depend on the orbit representatives or linear orders chosen). It requires a little more work (which we omit) to verify that s is always $D(\pi)$-compatible. The inverse of the map ϕ is not as hard to describe: for each cycle,

$$C=\begin{pmatrix} i_1 & i_2 & \cdots & i_{m-1} & i_m \\ \varepsilon_1 i_2 & \varepsilon_2 i_3 & & \varepsilon_{m-1} i_m & \varepsilon_m i_1 \end{pmatrix}$$

in π, let

$$v_C=(s_{i_1}, \operatorname{sgn}(\pi(i_1))s_{|\pi(i_1)|}, \operatorname{sgn}(\pi^2(i_1))s_{|\pi^2(i_1)|}, \ldots, \operatorname{sgn}(\pi^{m-1}(i_1))s_{|\pi^{m-1}(i_1)|})$$

and add to f the blinking necklace represented by v_C if C is even, or the twisted necklace represented by v_C if C is odd. It also requires a little work to verify that if s is $D(\pi)$ compatible, then every blinking necklace and twisted necklace in $\phi^{-1}(\pi, s)$ is primitive, and twisted necklaces appear no more than once. □

4. The Main Result

Given a pair of vectors $(\lambda, \mu)=((\lambda_1, \lambda_2, \ldots), (\mu_1, \mu_2, \ldots))$, let $\alpha^{(\lambda,\mu)}=\alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\cdots\beta_1^{\mu_1}\beta_2^{\mu_2}\cdots$. We adopt the convention that B_0 is the trivial group containing a single signed permutation π with $D(\pi)=\varnothing$ and $\lambda(\pi)=((0, 0, \ldots), (0, 0, \ldots))$. We then have the following:

Theorem 4.1 (cf. [4], equation (3)]). *Let*

$$(t;q)_n=(1-t)(1-tq)(1-tq^2)\cdots(1-tq^{n-1}).$$

Then

$$\sum_{n\geqslant 0}\frac{x^n}{(t;q)_{n+1}}\sum_{\pi\in B_n} t^{d(\pi)}q^{\operatorname{maj}(\pi)}\alpha^{\lambda(\pi)}=\sum_{s\geqslant 0}t^s\prod_{m\geqslant 1}\prod_{i=0}^{s(m-1)}(1-\alpha_m x^m q^i)^{-D_{m,i}^{(s)}}(1+\beta_m x^m q^i)^{P_{m,i}^{(s)}}$$

where $D_{m,i}^{(s)}$ is the number of primitive blinking necklaces D of size m with $|D|=i$ and $\max(D)\leqslant s$, *and $P_{m,i}^{(s)}$ is the number of primitive twisted necklaces P of size m with $|P|=i$ and* $\max(P)\leqslant s$.

Remark. More explicit formulas for $D_{m,i}^{(s)}$ and $P_{m,i}^{(s)}$ will be given in the next theorem.

Proof. Let $\mathcal{O}$ be the set of all signed ornaments. We use the method of [3] and count

$$\sum_{f\in\mathcal{O}} t^{\max(f)+1}q^{|f|}\alpha^{\lambda(f)}x^{\operatorname{size}(f)}$$

in two ways; one direct, and the other using the signed ornament bijection.

For the direct approach, we have

$$\begin{aligned}\sum_{f\in\mathcal{O}} t^{\max(f)+1}q^{|f|}\alpha^{\lambda(f)}x^{\operatorname{size}(f)} &= (1-t)\sum_{s\geqslant 0}t^s\sum_{\substack{f\in\mathcal{O}\\ \max(f)+1\leqslant s}} q^{|f|}\alpha^{\lambda(f)}x^{\operatorname{size}(f)}\\ &= (1-t)\sum_{s\geqslant 0}t^s\prod_{\substack{\text{prim. blinking necklaces } D\\ \max(D)+1\leqslant s}}(1-q^{|D|}\alpha_{\lambda(D)}x^{\operatorname{size}(D)})^{-1}\\ &\quad\times\prod_{\substack{\text{prim. twisted necklaces } P\\ \max(P)+1\leqslant s}}(1+q^{|P|}\beta_{\lambda(P)}x^{\operatorname{size}(P)})\\ &= (1-t)\sum_{s\geqslant 0}t^s\prod_{m\geqslant 1}\prod_{i=0}^{(s-1)m}(1-\alpha_m x^m q^i)^{-D_{m,i}^{(s)}}(1+\beta_m x^m q^i)^{P_{m,i}^{(s)}}. \qquad (1)\end{aligned}$$

For the other approach, using the bijection of Theorem 3.1 we have

$$\sum_{f\in\mathcal{O}} t^{\max(f)+1} q^{|f|} \alpha^{\lambda(f)} x^{\mathrm{size}(f)} = \sum_{n\geq 0} \sum_{\substack{(\pi,s):\pi\in B_n\\ D(\pi)\text{-compatible } s}} t^{\max(s)+1} q^{|s|} \alpha^{\lambda(\pi)} x^n$$
$$= \sum_{n\geq 0} \sum_{\pi\in B_n} \alpha^{\lambda(\pi)} x^n \sum_{D(\pi)\text{-compatible } s} t^{\max(s)+1} q^{|s|}. \qquad (2)$$

For a fixed π, we can encode a $D(\pi)$-compatible sequence $s=(s_1,\ldots,s_n)$ as a partition $\rho=(\rho_1\geq\cdots\geq\rho_n\geq 0)$ with at most n parts in the following way: let

$$\rho_i = s_i - \#\{j\in D(\pi): j\geq i\}$$

It is easy to see that this encoding is reversible and that it has two key properties: (i) $\max(s)=\max(\rho)+\#D(\pi)$ and (ii) $|s|=|\rho|+\max(\pi)$. So for a fixed $\pi\in B_n$, we have

$$\sum_{D(\pi)\text{-compatible } s} t^{\max(s)+1} q^{|s|} = t^{d(\pi)} q^{\mathrm{maj}(\pi)} \sum_{\substack{\text{partitions } \rho\\ \text{with } \leq n \text{ parts}}} t^{\max(\rho)} q^{|\rho|}$$
$$= t^{d(\pi)} q^{\max(\pi)} \frac{1}{(tq;q)_n},$$

where the last equality comes from a standard argument in partitions. Plugging this last equation into equation (2) yields

$$\sum_{f\in\mathcal{O}} t^{\max(f)+1} q^{|f|} \alpha^{\lambda(f)} x^{\mathrm{size}(f)} = \sum_{n\geq 0} \frac{\sum_{\pi\in B_n} t^{d(\pi)} q^{\mathrm{maj}(\pi)} \alpha^{\lambda(\pi)} x^n}{(tq;q)_n},$$

and setting this equal to the direct count (equation (1)) yields the theorem. □

One can make this theorem more explicit by computing generating functions for $D^{(s)}_{m,i}$ and $P^{(s)}_{m,i}$:

THEOREM 4.2. *Let*

$$D^{(s)}_m(q) = \sum_{i=0}^{(s-1)m} D^{(s)}_{m,i} q^i$$

and

$$P^{(s)}_m(q) = \sum_{i=0}^{(s-1)m} P^{(s)}_{m,i} q^i$$

for $m\geq 1$, $s\geq 0$. *Then we have*

$$P^{(s)}_m(q) = \begin{matrix} \dfrac{1}{2m}\displaystyle\sum_{\substack{d|m\\ d \text{ odd}}} \mu(d)([2s-1]_d^{m/d}-1), & s>0,\\ 0, & s=0,\end{matrix}$$

$$D^{(s)}_m(q) = \begin{cases} \dfrac{1}{2m}\displaystyle\sum_{\substack{d|m\\ d \text{ odd}}} \mu(d)([2s-1]_d^{m/d}-1), & s>0,\ m>1\\ \frac{1}{2}([2s-1]_1+1) & s>0,\ m=1\\ 0 & s=0\end{cases}$$

where $\mu(d)$ *is the number-theoretic* Möbius function (*see, e.g.*, [7]) *and*

$$[2s-1]_d = q^{d(s-1)}+\cdots+q^{2d}+q^d+1+q^d+q^{2d}+\cdots+q^{d(s-1)}$$

is a q-analogue *of the number* $2s-1$ (*i.e. when* $q=1$ *we have* $[2s-1]_d=2s-1$).

PROOF. The special cases when either m or s is small are easy to check, so assume $m>1$, $s>1$.

Computing $P_m^{(s)}(q)$ is easier. For each cyclic subgroup C_d of order d in C_{2m}, let $P_{C_d}=\sum_p q^{|p|}$, where the sum ranges over all $\mathbf{Z}$-words of size m with $\max(p)\leqslant s-1$ whose symmetry group intersects C_{2m} in *exactly* C_d, and let $P_{\supseteq C_d}$ be the same except that we sum over words whose symmetry group *contains* C_d. We then have trivially that

$$P_{\supseteq C_e}=\sum_{e|d} P_{=C_d}$$

and hence

$$P_{=C_e}=\sum_{e|d}\mu\left(\frac{d}{e}\right)P_{\supseteq C_d}$$

by Möbius inversion [(7, Section 16.3]). Note that since each primitive twisted necklace is represented by exactly $2m$ distinct words, we have

$$P_m^{(s)}(q)=\frac{1}{2m}P_{=C_1}=\frac{1}{2m}\sum_{d|2m}\mu(d)P_{\supseteq C_d}.$$

One can check that if $d\mid 2m$ and d is even, then the only $\mathbf{Z}$-word the symmetries of which contain C_d is $(0,0,\ldots,0)$. If $d\mid 2m$ and d is odd, such a $\mathbf{Z}$-word must be of the form

$$\overbrace{p=w,\ -w,\ w,\ -w,\ \ldots,\ -w,\ w,}^{d}$$

where w is an arbitrary $\mathbf{Z}$-word of length m/d. Hence $P_{\supseteq C_d}=[2s-1]_d^{m/d}$. This gives us

$$P_m^{(s)}(q)=\frac{1}{2m}\Bigg(\sum_{\substack{d|2m\\ d\text{ even}}}\mu(d)+\sum_{\substack{d|2m\\ d\text{ odd}}}\mu(d)[2s-1]_d^{m/d}\Bigg)$$

and since

$$\sum_{\substack{d|2m\\ d\text{ even}}}\mu(d)=-\sum_{\substack{d|2m\\ d\text{ odd}}}\mu(d)$$

we have

$$P_m^{(s)}(q)=\frac{1}{2m}\sum_{\substack{d|m\\ d\text{ odd}}}\mu(d)([2s-1]_d^{m/d}-1).$$

We undertake a similar argument for $D_m^{(s)}(q)$. Say that a $\mathbf{Z}$-word p of size m with $\max(p)\leqslant s-1$ is *totally free* if $\mathbf{Z}_2\times C_m$ acts freely on its orbit, *cyclically free* if C_m acts freely, and a *half-moon* if C_m acts freely but $\mathbf{Z}_2\times C_m$ does not. We let *TF, CF* and *HM* denote the sum $\sum_p q^{|p|}$ as p ranges over each of the previous three sets, respectively. Given a primitive blinking necklace D, as an orbit of words it either consists of $2m$ totally free words, or m half-moons, so we conclude that

$$D_m^{(s)}(q)=\frac{1}{2m}TF+\frac{1}{m}HM.$$

On the other hand, a cyclically free word is either totally free or a half-moon so $CF=TF+HM$ and hence

$$D_m^{(s)}(q)=\frac{1}{2m}(CF-HM)+\frac{1}{m}HM=\frac{1}{2m}(CF+HM).$$

We can compute CF and HM separately, each by a Möbius inversion similar to our analysis of $P_m^{(s)}(q)$, yielding

$$CF = \sum_{d|m} \mu(d)[2s-1]_d^{m/d},$$

$$HM = \begin{cases} \sum_{\substack{d|m \\ d \text{ odd}}} \mu(d)([2s-1]_{2d}^{m/2d} - 1), & m \text{ even}, \\ 0, & m \text{ odd}. \end{cases}$$

Combining these with the last equation, and a little manipulation, gives the expression stated in the theorem. □

THEOREM 4.3 (cf. [4, equations (4), (5)]).

$$\sum_{n\geq 0} \frac{B_n(t,q,\alpha)}{(t;q)_{n+1}} = \sum_{s\geq 0} t^s \exp\Big(\sum_{m\geq 1}\sum_{k\geq 1} \frac{x^{mk}}{k}[\alpha_m^k D_m^{(s)}(q^k) - (-\beta_m)^k P_m^{(s)}(q^k)]\Big)$$

$$= 1 + \frac{t}{1-\alpha_1 x} + \tag{3}$$

$$\frac{1}{1-\alpha_1 x}\sum_{s\geq 2} t^s \exp\Big(\sum_{m\geq 1}\sum_{k\geq 1}\frac{x^{mk}}{2mk}(\alpha_m^k - (-\beta_m)^k)\frac{1}{2m}\sum_{\substack{d|m\\ d \text{ odd}}}\mu(d)([2s-1]_{dk}^{m/d} - 1\Big)$$

$$= 1 + \frac{t}{1-\alpha_1 x} + \tag{4}$$

$$\frac{1}{1-\alpha_1 x}\sum_{s\geq 2} t^s \exp\Big(\sum_{m\geq 1}\frac{x^m}{2m}\sum_{d|m}([2s-1]_d^{m/d} - 1)\sum_{\substack{k|d\\ d/k \text{ odd}}}\mu(d/k)(\alpha_{m/k}^k - (-\beta_{m/k})^k)\Big).$$

(5)

PROOF. The second equality follows from the first using the expressions for $D_m^{(s)}(q)$ and $P_m^{(s)}(q)$ and $P_m^{(s)}(q)$ given by the previous theorem. The third equality follows from the second upon replacing mk by m and dk by k. Thus it suffices to show the first equality above. By Theorem 4.1, we need to show that

$$\prod_{m\geq 1}\prod_{i=0}^{s(m-1)} (1-\alpha_m x^m q^i)^{-D_{m,i}^{(s)}}(1+\beta_m x^m q^i)^{P_{m,i}^{(s)}}$$

$$= \exp\Big(\sum_{m\geq 1}\sum_{k\geq 1}\frac{x^{mk}}{k}[\alpha_m^k D_m^{(s)}(q^k) - (-\beta_m)^k P_m^{(s)}(q^k)]\Big).$$

This is achieved by taking the natural logarithm of both sides and using the Taylor series for $\log(1-\alpha_m x^m q^i)$ and $\log(1+\beta_m x^m q^i)$ on the left-hand side. □

5. APPLICATIONS

In this section we give corollaries to Theorems 4.1 and 4.3 obtained by specializing the various variables. Most of the proofs will be omitted, as they require only standard manipulations of generating functions and a few simple calculations with the number-theoretic Möbius function.

By setting $\alpha_i = \beta_i = 1$ for all i in Theorem 4.3, formula (5), we recover a result equivalent to the specialization $p = a = 1$ in ([8, Section 3]):

COROLLARY 5.1.

$$\sum_{n\geq 0} \frac{x^n}{(t;q)_{n+1}} \sum_{\pi \in B_n} t^{d(\pi)} q^{\mathrm{maj}(\pi)} = \sum_{s \geq 0} \frac{t^s}{1 - x[2s-1]_1}.$$

By eliminating t in Theorem 4.3, formula (4) we obtain the following:

COROLLARY 5.2.

$$\sum_{n\geq 0} \frac{x^n}{(q;q)_n} \sum_{\pi \in B_n} q^{\mathrm{maj}(\pi)} \alpha^{\lambda(\pi)}$$
$$= \frac{1}{1-\alpha_1 x} \exp\Big(\sum_{m\geq 1} \sum_{k \geq 1} \frac{x^{mk}}{2mk} (\alpha_m^k - (-\beta_m)^k) \frac{1}{2m} \sum_{\substack{d|m \\ d \text{ odd}}} \mu(d) \Big(\Big(\frac{1+q^{dk}}{1-q^{dk}} \Big)^{m/d} - 1 \Big) \Big).$$

By eliminating q in this last corollary, we obtain the generating function for the *cycle indicators* of B_n, the hyperoctahedral analogue of a result of Touchard [10] (this result may also be derived in a much more direct fashion):

COROLLARY 5.3.

$$\sum_{n \geq 0} \frac{x^n}{n!} \sum_{\pi \in B_n} \alpha^{\lambda(\pi)} = \exp\Big(\sum_{m \geq 1} \frac{\alpha_m + \beta_m}{2m} (2x)^m \Big).$$

Rather than looking at statistics over the whole group B_n, one can count $d(\pi)$ (and/or $\mathrm{maj}(\pi)$) as π ranges over classes of signed permutations that are characterized by their cycle type. For example, an element $\pi \in B_n$ is an *involution* (i.e. $\pi^2 = 1$) iff it has no even cycles of size greater than two and no odd cycles of size larger than one. Hence by setting $\alpha_i = 0$ for $i \geq 3$ and $\beta_i = 0$ for $i \geq 2$ (and $q = 1$ for convenience) in Theorem 4.1, we obtain the following:

COROLLARY 5.4 (cf. [4, equation (11)]).

$$\sum_{n \geq 0} \frac{x^n}{(1-t)^{n+1}} \sum_{\substack{\pi \in B_n \\ \pi^2 = 1}} t^{d(\pi)} \alpha^{\lambda(\pi)} = 1 + \sum_{s \geq 1} t^s (1-\alpha_1 x)^{-s} (1+\beta_1 x)^{s-1} (1-\alpha_2 x^2)^{s(1-s)}.$$

This last result has a second interpretation involving *domino tableaux*. A domino tableau of size n is a plane partition whose multiset of entries is $\{1, 1, 2, 2, \ldots, n, n\}$, weakly decreasing along rows and columns, with the two occurrences of i adjacent either horizontally or vertically for all i (see Figure 2). Stanton and White [9] give a generalization of the Robinson–Schensted correspondence which, as a special case, produces a bijection between involutions in B_n and

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -4 & +5 & +3 & -1 & +2 & -6 \end{pmatrix}$$

$$T = \begin{matrix} 1 & 1 & 3 & 4 & 5 \\ 2 & 2 & 3 & 4 & 5 \\ 6 & 6 & & & \end{matrix}$$

(a)

$$\begin{matrix} 7 & 6 & 4 & 3 & 2 \\ 6 & 5 & 3 & 2 & 1 \\ 2 & 1 & & & \end{matrix}$$

(b)

FIGURE 2. (a) An involution $\pi \in B_9$ and its corresponding domino tableau T. Note that $D(\pi) = D(T) = \{1, 2, 4, 6\}$. (b) The hook lengths for the shape of T from (a). From this we can see that it has one distinguished marked row (row 1) and one distinguished unmarked row (row 3), which agrees with the fact that $\lambda(\pi) = ((1, 2, 0, 0, \ldots), (1, 0, 0, \ldots))$.

domino tableaux of size n. It turns out that one can read off the descent set and cycle type of the involution from its corresponding domino tableau fairly easily, by a method which we now describe. Given a domino tableau T of size n and $i \leqslant n$, the *head* of i is the entry of T containing i that lies farthest to the northeast. If the head of i is the (x, y)-entry T, then the *orientation* of i is $O(i) = (-1)^{xy}$. We then define the descent set of T by

$$D(T) = \left\{ \begin{matrix} i: O(i) = O(i+1) = +1 \\ \text{head of } i \text{ is strictly west of head of } i+1 \end{matrix} \right\}$$
$$\cup \left\{ \begin{matrix} i: O(i) = O(i+1) = -1 \\ \text{head of } i \text{ is not strictly west of head of } i+1 \end{matrix} \right\}$$
$$\cup \{ i\, O(i) = -1,\ O(i+1) = +1 \}$$

where by convention we say that $O(n+1) = +1$ (see Figure 2). It is not hard to show, using the results of [9], that if the involution π corresponds to the tableau T, then $D(\pi) = D(T)$. To recover the cycle structure of π, we need only look at the shape of T, but first we need some definitions Given a cell in the shape of T, its *hook length* is one plus the number of cells strictly to the right in the same row or strictly below it in the same column. A row of T (or its shape) is *distinguished* if among the cells of that row there are an odd number which have even hook lengths. Finally, a row is *marked* if $(-1)^{x+y} = +1$, where (x, y) is the last cell in the row; else it is *unmarked*. It is again not hard to show using results from [9] that if an involution π with $\lambda(\pi) = ((\lambda_1, \lambda_2, 0, 0, \ldots), (\mu_1, 0, 0, \ldots))$ corresponds to the domino tableau T, then λ_1 is the number of distinguished marked rows of T's shape, and μ_1 is the number of distinguished unmarked rows (and of course $\lambda_2 = \frac{1}{2}(n - \lambda_1 - \lambda_2)$). In light of these facts, the left-hand side of the previous corollary may reinterpreted as a generating function counting domino tableaux by their descents, and distinguished marked and unmarked rows.

Another property determined by the cycle type of a signed permutation π is its number of *fixed points* $\lambda_1(\pi) = \#\{i: \pi_i = i\}$ and *negated points* $\mu_1(\pi) = \#\{i: \pi_i = -i\}$. Specializing Theorem 4.3, equation (5) gives the folowing:

Corollary 5.5 (cf. [4, equation (15)].

$$\sum_{n \geqslant 0} \frac{x^n}{(t;q)_{n+1}} \sum_{\pi \in B_n} t^{d(\pi)} q^{\mathrm{maj}(\pi)} a^{\lambda_1(\pi)} b^{\mu_1(\pi)}$$
$$= 1 + \frac{t}{1-ax} + \sum_{s \geqslant 2} \frac{t^s}{1 - x[2s-1]_1} \frac{(-bq;q)_{s-1}(xq;q)_s}{(axq;q)s(-xq;q)_{s-1}}$$

If one views B_n as the symmetry group of the n-dimensional hyperoctahedron (the convex hull in $\mathbf{R}^n$ of the points $\{\pm e_1, \ldots, e_n\}$), then fixed points of π correspond to pairs $\{\pm e_i\}$ of vertices of the hyperoctahedron which are fixed by the action of π. If π has no fixed vertices, i.e. $\lambda_1(\pi) = 0$, we will say that π is a *hyperoctahedral derangement*. By setting $a = 0$, $b = 1$ in the previous corollary, we obtain the following:

Corollary 5.6.

$$\sum_{n \geqslant 0} \frac{x^n}{(t;q)_{n+1}} \sum_{\pi \in HD_n} t^{d(\pi)} q^{\mathrm{maj}(\pi)} = \sum_{s \geqslant 0} \frac{t^s}{1 - x[2s-1]_1} (x;q)_s,$$

where HD_n is the set of all hyperoctahedral derangements in B_n.

One may also view B_n as the symmetry group of the n-dimensional cube (the convex hull in $\mathscr{R}^n$ of all points of the form $(\varepsilon_1, \ldots, \varepsilon_n)$ where $\varepsilon_i = \pm 1$). In this context, a signed permutation is a *cubical derangement* if it leaves no vertex of the n-cube fixed, and a *cubical rearrangement* otherwise. Chen and Stanley [1] observe that π is a cubical rearrangement iff π has no odd cycles. This leads to the following result:

COROLLARY 5.7.

$$\sum_{n\geq 0} \frac{x^n}{(t;q)_{n+1}} \sum_{\pi \in CR_n} t^{d(\pi)} q^{\mathrm{maj}(\pi)} = 1 + \frac{1}{1-x} \sum_{s\geq 1} t^s \exp\Bigg(\sum_{m\geq 1} \frac{x^m}{2m} \sum_{\substack{d \mid m \\ d \text{ a power of } 2}} ([2s-1]_d^{m/d} - 1) \Bigg)$$

where CR_n is the set of all cubical rearrangements in B_n.

Either by eliminating t and q in the previous corollary or, more directly by setting $\alpha_i = 1$ and $\beta_i = 0$ in Corollary 5.3, we recover a result from [1]:

COROLLARY 5.8.

$$\sum_{n\geq 0} \#CR_n \frac{x^n}{n!} = \frac{1}{\sqrt{1-2x}}.$$

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VICTOR REINER
Department of Mathematics,
University of Minnesota,
Minneapolis, Minnesota 55455, U.S.A.