SOME NOTES ON PÓLYA'S THEOREM, KOSTKA NUMBERS AND THE RSK CORRESPONDENCE

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1. Introduction

Let G be a permutation group acting on $\{1,\dots,n\}$. Then G acts on words $(i_1,\dots,i_n)$ by permuting positions, where the letters i_j are chosen from some set $\{1,2,\dots,N\}$. Note that G is a subgroup of the symmetric group S_n, and that this action on words is the usual action from Pólya enumeration theory.

Definition 1.1. Associated with the partition λ of n is an irreducible representation of S_n. Let χ^λ be the character of this representation. Define

$$K_{\lambda,G} := \langle \chi^\lambda, \mathrm{Ind}_G^{\mathfrak{S}_n} \mathbf{1}_G\rangle_{\mathfrak{S}_n} = \langle \mathrm{Res}_G^{\mathfrak{S}_n} \chi^\lambda, \mathbf{1}_G\rangle_G = \dim_{\mathbb{C}}(\chi^\lambda)^G$$

In other words, $K_{\lambda,G}$ is the dimension of the G-invariant subspace when one restricts the irreducible χ^λ from $\mathfrak{S}_n$ to G. Note that this immediately implies that for any permutation group G, one has

$$K_{\lambda,G} \le \dim_{\mathbb{C}} \chi^\lambda = f^\lambda,$$

where f^λ is the number of standard Young tableaux T of shape λ. Furthermore, when $G = S_\nu = S_{\nu_1}\times S_{\nu_2}\times\cdots\times S_{\nu_\ell}$, a Young subgroup of S_n, then $K_{\lambda,G} = K_{\lambda,\nu}$, the well-known *Kostka number* counting column-strict tableaux of shape λ and content ν, that is, having ν_1 ones, ν_2 twos, etc. So $K_{\lambda,G}$ generalizes Kostka numbers.

Problem 1.2. *For a permutation group G, interpret $K_{\lambda,G}$ as counting some subset of standard Young tableaux of shape λ.*

The second author considered the following expansion.

Proposition 1.3.

$$\#\{G\text{-orbits on words of content } \mu\} = \sum_{\lambda\vdash n} K_{\lambda,\mu} K_{\lambda,G}.$$

We include here two proofs.

Proof 1. Let C_ρ denote the conjugate class of elements of S_n of cycle type ρ. Recall that G acts on words, and in fact, G will act on words of content $\mu = (\mu_1,\mu_2,\dots\mu_N)$. Let Δ_μ be a set of orbit representatives for the action of G on words of content μ. Pólya's Theorem states that

$$\sum_\mu |\Delta_\mu| m_\mu = \frac{1}{|G|}\sum_\rho |C_\rho \cap G| p_\rho\,,$$

Date: April 7, 2020.

where p_ρ is the power sum symmetric function and m_μ is the monomial symmetric function. Expanding the power sums as Schur functions gives

$$\sum_\mu |\Delta_\mu| m_\mu = \frac{1}{|G|} \sum_\lambda \sum_\rho |C_\rho \cap G| \chi^\lambda_\rho s_\lambda = \sum_\lambda K_{\lambda,G} s_\lambda \,,$$

from the definition of $K_{\lambda,G}$. Expanding the Schur functions as monomial symmetric functions and equating coefficients of m_μ finishes the proof. □

Proof 2. Let $V = \mathbb{C}^N$ with basis $\{e_1, \dots, e_N\}$, and consider $V^{\otimes n}$ as a $GL(V) \times \mathfrak{S}_n$-representation as usual, in which g in $GL(V)$ acts diagonally

$$g(v_1 \otimes \cdots \otimes v_n) := g(v_1) \otimes \cdots \otimes g(v_n)$$

and σ in $\mathfrak{S}_n$ permutes tensor positions:

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma_1^{-1}} \otimes \cdots \otimes v_{\sigma_n^{-1}}.$$

Since G permutes words $\mathbf{i} := (i_1, \dots, i_n)$ in the same way that it permutes the set of basic tensors $e_{\mathbf{i}} := e_{i_1} \otimes \cdots e_{i_n}$, the G-invariant subspace $(V^{\otimes n})^G$ has a basis of G-orbit sums of basic tensors $\sum_{\mathbf{i}' \in G\mathbf{i}} e_{\mathbf{i}'}$. Since a typical diagonal matrix $\mathbf{x}$ having diagonal entries $(x_1, \dots, x_n)$ has $e_{\mathbf{i}}$ as a weight vector (i.e. eigenvector) with eigenvalue $x^{\mathrm{cont}(\mathbf{i})}$, each of these G-orbit sums $\sum_{\mathbf{i}' \in G\mathbf{i}} e_{\mathbf{i}'}$ is also weight vector with this same weight/eigenvalue. Hence the left side of the proposition is the dimension of the μ-weight space $(V^{\otimes n})^G_\mu$ for $GL(V)$ inside the G-invariant subspace $(V^{\otimes n})^G$.

On the other hand Schur-Weyl duality gives a $GL(V) \times \mathfrak{S}_n$-isomorphism

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} V^\lambda \otimes \chi^\lambda,$$

where V^λ is the irreducible $GL(V)$-representation indexed by λ, where x acts with trace given by a Schur function $s_\lambda(\mathbf{x})$. Thus the μ-weight space has dimension

$$\sum_{\lambda \vdash n} \dim_{\mathbb{C}}(V^\lambda)_\mu \dim_{\mathbb{C}}(\chi^\lambda)^G = \sum_{\lambda \vdash n} K_{\lambda,\mu} K_{\lambda,G}. \quad \square$$

Problem 1.4. *For a permutation group G with a solution to Problem 1.2, find a bijection that proves the identity in Proposition 1.3.*

2. The case of a Young subgroup

Given a composition $\nu = (\nu_1, \dots, \nu_\ell)$ of n, let $\mathfrak{S}_\nu = \mathfrak{S}_{\nu_1} \times \cdots \times \mathfrak{S}_{\nu_\ell}$ be the associated Young subgroup, and $\mathfrak{A}_\nu \subset \mathfrak{S}_\nu$ its alternating subgroup. Letting $K_{\lambda,\nu}$ denote the Kostka number counting column-strict tableaux of shape λ and content ν, one has the following.

Proposition 2.1. *In this setting,*

$$K_{\lambda,\mathfrak{S}_\nu} = K_{\lambda,\nu}$$
$$K_{\lambda,\mathfrak{A}_\nu} = K_{\lambda,\nu} + K_{\lambda^t,\nu} \quad \textit{assuming } \nu \neq 1^n.$$

Proof. We'll apply the definiition $K_{\lambda,G} = \langle \chi^\lambda, \mathbf{1}_G \uparrow_G^{\mathfrak{S}_n} \rangle_{\mathfrak{S}_n}$ to both of $G = \mathfrak{S}_\nu$ and $G = \mathfrak{A}_\nu$. Denoting the induction product on characters χ, ψ of $\mathfrak{S}_a, \mathfrak{S}_b$ as

$$\chi * \psi := \chi \otimes \psi \uparrow_{\mathfrak{S}_a \times \mathfrak{S}_b}^{\mathfrak{S}_{a+b}},$$

one has that

$$\mathbf{1}_{\mathfrak{S}_\nu} \uparrow_{\mathfrak{S}_\nu}^{\mathfrak{S}_n} = \mathbf{1}_{\mathfrak{S}_{\nu_1}} * \cdots * \mathbf{1}_{\mathfrak{S}_{\nu_\ell}}$$

and transitivity of induction together with Frobenius reciprocity shows that

$$\begin{aligned}\mathbf{1}_{\mathfrak{A}_\nu} \uparrow_{\mathfrak{A}_\nu}^{\mathfrak{S}_n} &= \left(\mathbf{1}_{\mathfrak{A}_\nu} \uparrow_{\mathfrak{A}_\nu}^{\mathfrak{S}_\nu}\right) \uparrow_{\mathfrak{S}_\nu}^{\mathfrak{S}_n} \\ &= \left(\mathbf{1}_{\mathfrak{S}_\nu} + \mathbf{sgn}_{\mathfrak{S}_\nu}\right) \uparrow_{\mathfrak{S}_\nu}^{\mathfrak{S}_n} \\ &= \mathbf{1}_{\mathfrak{S}_{\nu_1}} * \cdots * \mathbf{1}_{\mathfrak{S}_{\nu_\ell}} + \mathbf{sgn}_{\mathfrak{S}_{\nu_1}} * \cdots * \mathbf{sgn}_{\mathfrak{S}_{\nu_\ell}}\end{aligned}$$

Consequently, applying the Frobenius characteristic map, one has

$$\begin{aligned}K_{\lambda,\mathfrak{S}_\nu} &= \langle \chi^\lambda, \mathbf{1}_{\mathfrak{S}_\nu} \uparrow_{\mathfrak{S}_\nu}^{\mathfrak{S}_n} \rangle_{\mathfrak{S}_n} = \langle s_\lambda, h_\nu \rangle = K_{\lambda,\nu} \\ K_{\lambda,\mathfrak{A}_\nu} &= \langle \chi^\lambda, \mathbf{1}_{\mathfrak{A}_\nu} \uparrow_{\mathfrak{A}_\nu}^{\mathfrak{S}_n} \rangle_{\mathfrak{S}_n} = \langle s_\lambda, h_\nu + e_\nu \rangle = K_{\lambda,\nu} + K_{\lambda^t,\nu} \quad \square\end{aligned}$$

This solves Problem 1.2 in the case of $G = \mathfrak{S}_\nu$ or $G = \mathfrak{A}_\nu$, as follows. Recall standardizing column-strict tableaux to standard tableaux shows that, if

$$D(\nu) := \{\nu_1, \nu_1 + \nu_2, \ldots, \nu_1 + \nu_2 + \cdots + \nu_{\ell-1}\} \subset \{1, 2, \ldots, n-1\},$$

are the partial sums of ν, then

$$K_{\lambda,\nu} = \#\{\text{standard tableaux } T \text{ of shape } \lambda : \mathrm{Des}(T) \subseteq D(\nu)\}.$$

But then this also shows that

$$\begin{aligned}&K_{\lambda,\nu} + K_{\lambda^t,\nu} \\ &\quad = \#\{\text{standard tableaux } T \text{ of shape } \lambda : \text{ at least one of } \mathrm{Des}(T), \mathrm{Des}(T^t) \subseteq D(\nu)\}\end{aligned}$$

because no standard tableaux T can have *both* $\mathrm{Des}(T), \mathrm{Des}(T^t) \subseteq D(\nu)$: pick any value $i_0 \notin \mathrm{Des}(\nu)$ (such an i_0 exists, since without loss of generality $\nu \neq 1^n$), and then either T or T^t will have a descent at i_0. We next explain how it solves Problem 1.4 for $\mathfrak{S}_\nu$ using RSK, and for $\mathfrak{A}_\nu$ both RSK and dual RSK.

Proposition 2.2. *For any compositions $\mu = (\mu_1, \ldots, \mu_m)$ and $\nu = (\nu_1, \ldots, \nu_\ell)$ of n, one has bijections explaining why*

$$\begin{aligned}\#\{\mathfrak{S}_\nu\text{-orbits on words of content } \mu\} &= \sum_{\lambda \vdash n} K_{\lambda,\mu} K_{\lambda,\nu}, \\ \#\{\mathfrak{A}_\nu\text{-orbits on words of content } \mu\} &= \sum_{\lambda \vdash n} K_{\lambda,\mu} \left(K_{\lambda,\nu} + K_{\lambda^t,\nu}\right).\end{aligned}$$

Proof. There is a bijection between $\mathfrak{S}_\nu$-orbits on words of content μ and matrices $A = (a_{ij})$ in $\mathbb{N}^{m \times \ell}$ having row sum μ and column sum ν that goes as follows. Every orbit has a unique representative word whose letters in its j^{th}-interval of positions

$$I_j := [\nu_1 + \cdots + \nu_{j-1}, \nu_1 + \cdots + \nu_j]$$

appear in weakly increasing order, for each $j = 1, 2, \ldots, \ell$. This word is completely determined by the integers a_{ij} that specify how many times the letter i occurs within the interval I_j, so send this $\mathfrak{S}_\nu$-orbit to this matrix $A = (a_{ij})$. Then use Robinson-Schensted-Knuth row-insertion to send A to complete the bijection with pairs (P, Q) of tableaux counted by $\sum_{\lambda \vdash n} K_{\lambda,\mu} K_{\lambda,\nu}$, as usual.

On the other hand, one can slightly modify this bijection to obtain a bijection between $\mathfrak{A}_\nu$-orbits on words of content μ and *two disjoint sets* of matrices. Given such an $\mathfrak{A}_\nu$-orbit of words of content μ, classify it into two cases, as follows.

Case 1. There exists a representative of its $\mathfrak{A}_\nu$-orbit that is weakly increasing in each interval I_j, as before.

In this case, send this orbit to a matrix $A = (a_{ij})$ in $\mathbb{N}^{m\times\ell}$ having row sum μ and column sum ν just as before, and then use RSK row-insertion as before to send A to a pair (P, Q) counted by $\sum_{\lambda\vdash n} K_{\lambda,\mu}K_{\lambda,\nu}$.

Case 2. There exists no such representative of its $\mathfrak{A}_\nu$-orbit that is weakly increasing in each interval I_j.

In this case, one can see that, for each $j = 1, 2, \ldots, \ell$, the letters appearing within the interval I_j in these words must be *distinct*, that is, there are no repeats among the letters within the interval I_j. On the other hand, if one arbitrarily fixes a pair of adjacent positions $(i_0, i_0 + 1)$ that lie inside a single interval I_{j_0}, one can pick a unique representative that is *strictly* increasing inside every interval I_j with the exception of a strict decrease from positions i_0 to $i_0 + 1$. Encode this orbit via the matrix $A = (a_{ij})$ in $\{0, 1\}^{m\times\ell}$ where a_{ij} is the number of times that letter i occurs within the interval I_j. This is a $\{0, 1\}$ matrix having row sum μ and column sum ν, and so we can use *dual* RSK column-insertion to send it to a pair (P, Q) counted by $\sum_{\lambda\vdash n} K_{\lambda,\mu}K_{\lambda^t,\nu}$. □

3. The case of a cyclic group generated by a regular element

When $G = \langle c\rangle = \{1, c, c^2, \ldots, c^{d-1}\} \cong \mathbb{Z}/d\mathbb{Z}$ is the cyclic subgroup generated by a regular element in the sense of Springer [3], that is, c is a power of an n-cycle or an $(n-1)$-cycle inside $\mathfrak{S}_n$, one can at least solve Problem 1.2.

Proposition 3.1. *In the above setting, $K_{\lambda,G}$ is the number of standard Young tableaux T of shape λ having major index* $\mathrm{maj}(T) \equiv 0 \bmod d$.

Proof. Springer's theorem on regular elements from [3] can be phrased as saying that for a regular element c in a complex reflection group W, the character value $\chi^\lambda(c)$ is the evaluation $\left[f^\lambda(q)\right]_{q=\zeta}$, where ζ is a complex root-of-unity having the same multiplicative order as c, and where $f^\lambda(q)$ is the fake-degree polynomial for χ^λ. An old (easy) calculation of Lusztig, recorded in Stanley's survey on invariant theory [5, Prop. 4.11], asserts that when W is the symmetric group $\mathfrak{S}_n$,

$$f^\lambda(q) = \sum_T q^{\mathrm{maj}(T)} \tag{3.1}$$

where T runs through all standard Young tableaux of shape λ. Hence letting ζ be any primitive d^{th} root-of-unity, one has

$$K_{\lambda,G} = \langle \mathrm{Res}^{\mathfrak{S}_n}_G \chi^\lambda, \mathbf{1}_G\rangle_G = \frac{1}{|G|}\sum_{g\in G}\chi^\lambda(g) = \frac{1}{d}\sum_{i=0}^{d-1}\chi^\lambda(c^i) = \frac{1}{d}\sum_{i=0}^{d-1} f^\lambda(\zeta^i).$$

This last expression is the coefficient a_0 of q^0 when one uniquely expresses

$$f^\lambda(q) = \sum_{i=0}^{d-1} a_i q^i \mod q^d - 1,$$

which by (3.1) is the quantity asserted in the proposition. □

4. Behavior under products

We discuss here a reduction for Problem 1.2 when dealing with product groups. Suppose G acts on A and H acts on B, completely distinct from A. A well-known fact from Polya Theory (see [1]) is that the cycle index polynomial of $G\times H$ (acting on $A \cup B$) is the product of the cycle index polynomials for G and H. That is to say, writing the cycle index polynomials in the Schur basis,

$$\sum_{\lambda \vdash m+n} K_{\lambda, G\times H} s_\lambda = \left(\sum_{\mu \vdash m} K_{\mu,G} s_\mu \right) \left(\sum_{\nu \vdash n} K_{\nu,H} s_\nu \right) .$$

Equating coefficients of s_λ and using the Littlewood-Richardson rule, we get

$$K_{\lambda, G\times H} = \sum_{\substack{\mu \vdash m \\ \nu \vdash n}} K_{\mu,G} K_{\nu,H} g^{\lambda}_{\mu,\nu} . \tag{4.1}$$

Replacing G and H by identity groups gives this well-known identity,

$$f_\lambda = \sum_{\substack{\mu \vdash m \\ \nu \vdash n}} f_\mu f_\nu g^{\lambda}_{\mu,\nu} . \tag{4.2}$$

Equation 4.2 is proved bijectively (see [6]) as follows. Let (T, S, R) be a triple of tableaux. T is a SYT of shape μ, using, say, a red alphabet $[n]$. S is a SYT of shape ν, using a green alphabet $[m]$. R is a SSYT of shape λ/μ of content ν whose reading word is a lattice word. We construct SYT U of shape λ. Let the μ portion of U (red alphabet) be T. For the λ/μ portion, use the RSK algorithm to construct the (green) permutation w whose insertion tableau is S and whose recording tableau is the SYT which corresponds to the reading word of R. Green permutation w will fit the shape λ/μ, thus completing the construction.

The reverse of this construction is as follows. Starting with a SYT U of shape λ, let T be the portion of U which includes the first m letters (which will occupy shape μ). Color T red. Relabel the rest of U (which we call V) by replacing $m+1$ with a green 1, $m+2$ with a green 2, and so on. Then S is the insertion tableau obtained by applying the RSK algorithm to the reading word of V. The reading word of R is the lattice word which corresponds to the recording tableau, and necessarily fits the shape λ/μ. This proves the following.

Corollary 4.1. *Assume we have a solution to Problem 1.2 for G, H interpreting $K_{\mu,G}$ and $K_{\nu,H}$ for each μ, ν as counting certain sets of tableaux.*

Then the above construction solves Problem 1.2 for $G \times H$, yielding a set of tableaux counted by $K_{\lambda,G\times H}$ for each λ.

Example 4.2. Let us examine the specific case of two cyclic groups, $G = Z_m$ and $H = Z_n$. The tableaux constructed in $K_{\mu,G}$ and $K_{\nu,H}$ depend on divisibility conditions within the alphabets for actions of G and H respectively. So we will assume that G acts on $[n]$ and H acts on $[m]$. But for the purposes of the construction of the tableau U counted by $K_{\lambda,G\times H}$, the alphabet of the second set is larger than the alphabet of the first set. To reflect this, we color the alphabet under the G action red and color the alphabet under the H action green, with red smaller than green.

Under the construction described above, it is clear that in U (the tableau of shape λ), the red portion will have the property that maj will be divisible by m. But it will also be true in the green portion that maj will be divisible by n. This is because the insertion tableau that corresponds (S in the above discussion) under RSK is the same as the rectification (using *jeu de taquin*) of the green portion of U. Since the descent set is exactly preserved by *jeu de taquin* (see [4]), both the green portion of U and S will have exactly the same descent sets, and therefore the same maj.

Thus $K_{\lambda,Z_m\times Z_n}$ counts the standard tableaux of shape λ having maj in their red portion divisible by n and maj in their green portion divisible by n.

5. A related pair of invariant theory problems

Again in the setting of a permutation group G inside $\mathfrak{S}_n$, Garsia and Stanton [2] proposed a method to solve the following problem about the G-invariant ring S^G inside the polynomial ring $S := \mathbb{Q}[x_1,\dots,x_n]$. Invariant theory tells us that S^G is a free module over the $\mathfrak{S}_n$-invariant (i.e. symmetric) polynomials $S^{\mathfrak{S}_n} = \mathbb{Q}[e_1(\mathbf{x}),\dots,e_n(\mathbf{x})]$, which form a polynomial subalgebra generated by the elementary symmetric functions $e_i(\mathbf{x})$.

Problem 5.1. *Find an explicit set of monomials* $\{\mathbf{x}^a\}$ *whose orbit sums* $\{\sum_{g\in G} g(\mathbf{x}^a)\}$ *give a free* $S^{\mathfrak{S}_n}$*-basis for* S^G*.*

Their method was to show that it suffices to solve the following analogous problem, replacing S by the *Stanley-Reisner ring* of the *Boolean algebra* of rank n

$$R := \mathbb{Q}[\ y_S\ ]_{S\subset[n]}/(\ y_Sy_T\ )_{S,T \text{ not nested}}.$$

Again it is known that R^G is a free module over the $\mathfrak{S}_n$-invariants $R^{\mathfrak{S}_n} = \mathbb{Q}[\theta_1,\dots,\theta_n]$, which is again a polynomial subalgebra generated by the elements $\theta_i := \sum_{S\in\binom{[n]}{i}} y_S$.

Problem 5.2. *Find an explicit set of monomials* $\{\mathbf{y}^a\}$ *whose orbit sums* $\{\sum_{g\in G} g(\mathbf{y}^a)\}$ *give a free* $R^{\mathfrak{S}_n}$ *basis for* R^G*.*

They showed that if one produced such a set of monomials $\{\mathbf{y}^a\}$, then when one applies the following $\mathfrak{S}_n$-equivariant $\mathbb{Q}$-linear isomorphism (but not a ring map) called the *transfer map*

$$\begin{array}{rcl} R & \xrightarrow{T} & \mathbb{Q}[x_1,\dots,x_n] \\ y_{S_1}\cdots y_{S_\ell} & \longmapsto & \mathbf{x}^{S_1}\cdots\mathbf{x}^{S_\ell} \end{array}$$

with $\mathbf{x}^S := \prod_{i\in S} x_i$, it will transfer the solution $\{\mathbf{y}^a\}$ to the latter problem for R^G into a solution $\{\mathbf{x}^a\} = \{T(\mathbf{y}^a)\}$ of the former problem for S^G.

So how to find these $\{\mathbf{y}^a\}$? Firstly, they explain why one should be able to find them among the squarefree chain monomials

$$y_{S_1}\cdots y_{S_{\ell-1}} \tag{5.1}$$

where

$$\varnothing = S_0 \subset S_1 \subset \cdots \subset S_{\ell-1} \subset S_\ell = [n].$$

Secondly they give the numerology about which multidegrees the $\{\mathbf{y}^a\}$ should live in, if one uses the fine $\mathbb{N}^n$-grading that gives all variables y_S with $|S| = i$ the multidegree which is the i^{th}-standard basis vector in $\mathbb{N}^n$. One defines two quantities that are related by inclusion-exclusion, and might nowadays be called the *fine f-vector* and *fine h-vector* of the quotient by G of the Coxeter complex of type A_{n-1}: for a composition $\alpha = (\alpha_1,\dots,\alpha_\ell)$ of n, let

$$\begin{aligned} f^G_\alpha &:= \#\text{ of double cosets } G\backslash\mathfrak{S}_n/\mathfrak{S}_\alpha \\ h^G_\alpha &:= \sum_{\beta \text{ coarsening } \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} f^G_\beta \end{aligned}$$

Then h^G_α predicts exactly how many chain monomials $\{\mathbf{y}^a\}$ as in (5.1) should have the cardinalities of the subsets $(S_1, \ldots, S_{\ell-1})$ equal to the partial sums of α.

We claim that for the usual cyclic groups G, the Springer result (in the guise of one of our earliest CSP results) is again giving us a simple prediction for those h^G_α. Say that a statistic $\mathrm{stat} : \mathfrak{S}_n \to \mathbb{N}$ is *strongly Mahonian* if it has the property that its distribution over inverse descent classes is the same as maj or inv.

Proposition 5.3. *Let $G = \langle c \rangle = \{1, c, c^2, \ldots, c^{d-1}\} \cong \mathbb{Z}/d\mathbb{Z}$ be a cyclic subgroup of $\mathfrak{S}_n$ generated by a regular element, and let* stat *be any strongly Mahonian statistic* stat*. Then h^G_α counts the w in $\mathfrak{S}_n$ having* $\mathrm{stat}(w) \equiv 0 \bmod d$ *and whose inverse descent composition is α, i.e.* $\mathrm{Des}(w^{-1})$ *is the set of partial sums of α.*

Proof. It is equivalent by inclusion-exclusion to prove that f^G_α counts the w in $\mathfrak{S}_n$ having $\mathrm{stat}(w) \equiv 0 \bmod d$ and whose inverse descent composition *coarsens* α. Our strongly Mahonian assumption says that $\begin{bmatrix} n \\ \alpha \end{bmatrix}_q = \sum_w q^{\mathrm{stat}(w)}$ where the sum ranges over w for which $\mathrm{Des}(w^{-1})$ coarsens α. Hence the quantity that we want to equal f^G_α is the coefficient a_0 of q^0 when one uniquely expands the q-multinomial

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_q \equiv \sum_{i=0}^{d-1} a_i q^i \bmod q^d - 1.$$

On the other hand, one of our favorite CSP's has this q-multinomial as $X(q)$ and the cyclic G-action by left translation on $X = \mathfrak{S}_n / \mathfrak{S}_\alpha$. This implies that this coefficient a_0 counts the G-orbits on X, that is $\# G \backslash \mathfrak{S}_n / \mathfrak{S}_\alpha = f^G_\alpha$. □

Problem 5.4. *For such regular element cyclic groups G, find a strongly Mahonian statistic* stat *such that one can read off a good set of monomials $\{\mathbf{y}^a\}$ from the inverse descent sets $D(w^{-1})$ of those w in $\mathfrak{S}_n$ having* $\mathrm{stat}(w) \equiv 0 \bmod d$.

For example, one would like to do something analogous to the case where $G = \{e\}$ is the trivial group, in which one just takes all permutations $w = (w_1, \ldots, w_n)$ in $\mathfrak{S}_n$ and reads off the *descent monomial*

$$\mathbf{y}_w := \prod_{i \in \mathrm{Des}(w)} y_{(w_1, w_2, \ldots, w_i)}.$$

Fortunately, Garsia and Stanton also gave us a good linear-algebraic criterion for testing whether some particular collection of chain-monomials as in (5.1), which have the correct multidegree distribution predicted by h^G_α, will give us a $R^{\mathfrak{S}_n}$-basis for R^G or not. One simply checks invertibility over $\mathbb{Q}$ of a certain $\frac{n!}{|G|} \times \frac{n!}{|G|}$ square $\{0,1\}$-incidence matrix whose

- rows are indexed by the (G-orbits of) the chains of subsets $S_1 \subset \cdots \subset S_{\ell-1}$ indexing the candidate monomials $\{\mathbf{y}^a\}$, and
- columns are indexed by the (G-orbits of) maximal chains of subsets

$$\varnothing \subset \{w_1\} \subset \{w_1, w_2\} \subset \cdots \subset \{w_1, \ldots, w_{n-1}\} \subset [n]. \tag{5.2}$$

The $\{0,1\}$-matrix entry is a 1 if some representative of the G-orbit of chain of subsets is contained in a representative of the G-orbit fo the maximal chain, and a 0 otherwise.

Garsia and Stanton work out in full detail a nice example [2, Chap.7, Example 3, p. 150] relevant for our special case, namely $G = C_4$ generated by a 4-cycle (1234) inside $\mathfrak{S}_4$. In fact, this example foiled Vic's first few naive attempts to write down some $\{\mathbf{y}^a\}$ that worked using inverse descent sets. But see the next section ...

6. What might one hope/ask for?

This raises (at least) two questions, given some strongly Mahonian statistic stat$(-)$, and d a divisor of $n-1$ or n. Let G be the subgroup of $\mathfrak{S}_n$ generated by a regular element of order d, and let Δ_n/G denote the quotient Boolean complex of the Coxeter complex of type A_{n-1} by G, with fine f-vector entries f_α and fine h-vector entries h_α

Letting A_n denote the subset of w in $\mathfrak{S}_n$ for which $\mathrm{stat}(w^{-1}) \equiv 0 \bmod d$, by switching $w \leftrightarrow w^{-1}$ in Proposition 5.3, we know that if $D(\alpha)$ denotes the partial sums of α, then $h^G_\alpha = |\{w \in A_n : \mathrm{Des}(w) = \alpha\}$ for every composition α of n.

Question 6.1. Does A_n give us a Garsia-Stanton style *partitioning* of the faces of Δ_n/G as follows?:

$$\Delta_n/G = \bigsqcup_{w \in A_n} [w|_{\mathrm{Des}(w)}, w] \tag{6.1}$$

where here for $w = (w_1, \ldots, w_n)$, the w in the second position represents the complete flag of G-orbits of faces as in (5.2), while the $w|_{\mathrm{Des}(w)}$ in the first position is the subflag where one only takes the subsets whose cardinality lies in $\mathrm{Des}(w)$.

Question 6.2. Does A_n give us a Garsia-Stanton style solution to the invariant theory problem, in that the $\{0,1\}$-incidence $\frac{n!}{|G|} \times \frac{n!}{|G|}$ matrix with rows and columns indexed by A_n whose (w, w') entry is

$$\begin{cases} 1 & \text{if the } G\text{-orbit of the flag } w|_{\mathrm{Des}(w)} \text{ is contained in that of } w, \\ 0 & \text{otherwise.} \end{cases}$$

is invertible at primes that do not divide $|G|$?

Let's explore this a bit for the major index, that is, $\mathrm{stat}(w) := \mathrm{maj}(w)$, and examine the special case where $d = n$, so G is the cyclic group generated by the n-cycle $c = (1, 2, \ldots, n0$.

Example 6.3. When $n = 3$, no problem for both questions:

$w^{-1} : \mathrm{maj}(w^{-1}) \equiv 0 \bmod 3$	w	$w\vert_{\mathrm{Des}(w)}$
123	123	$\varnothing$
321	$3 \cdot 2 \cdot 1$	$(3 \subset 23) \equiv_G (1 \subset 13)$

One can check that this gives a partitioning. And this matrix is invertible:

$$\begin{array}{c} \\ \varnothing \\ 1 \subset 13 \end{array} \begin{array}{c} \begin{array}{cc} 1 \subset 12 & 1 \subset 13 \end{array} \\ \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \end{array}$$

Example 6.4. When $n = 4$, both problems have negative answers:

$w^{-1} : \mathrm{maj}(w^{-1}) \equiv 0 \bmod 4$	w	$w\vert_{\mathrm{Des}(w)}$
1234	1234	$\varnothing$
2143	$2 \cdot 14 \cdot 3$	$(2 \subset 124) \equiv_G (1 \subset 134)$
3142	$24 \cdot 13$	$(24) \equiv_G (13)$
3241	$4 \cdot 2 \cdot 13$	$(4) \subset (24) \equiv_G (1) \subset (13)$
4132	$24 \cdot 3 \cdot 1$	$(24) \subset (234) \equiv_G (13) \subset (123)$
4231	$4 \cdot 23 \cdot 1$	$(4 \subset 234) \equiv_G (1 \subset 134)$

Note that the second and last rows look the same here, so this can't be a partitioning, and the $\{0,1\}$ incidence matrix will have two equal rows also, so it can't be invertible. One also runs into an obvious problem with the putative partitioning (6.1), in that the w in A_n lying in rows 3 and 6 lie in the same G-orbit: $4231 = c^2 2413$. Similarly those lying in rows 4 and 5 lie in the same G-orbit: $4213 = c^2 2431$.

However, there are two somewhat promising pieces of news when n is a prime p, and G is the cyclic group generated by the p-cycle $c = (1, 2, \ldots, p)$ in $\mathfrak{S}_p$. First of all, the maximal flags w at the *tops* of the intervals $[w|_{\mathrm{Des}(w)}, w]$ in the putative partitioning (6.1) really do lie in different G-orbits.

Proposition 6.5. *When G is the cyclic group generated by the p-cycle $c = (1, 2, \ldots, p)$ in $\mathfrak{S}_p$ with p prime, the elements w in A_n all lie in different G-orbits, that is, one cannot have two elements w_1, w_2 with $\mathrm{maj}(w_i^{-1}) \equiv 0 \bmod p$ and $w_2 = c^m w_1$ for some integer m.*

Proof. We claim that, even if n is not prime, for $c = (1, 2, \ldots, n)$ one has

$$\mathrm{maj}((cw)^{-1}) \equiv \mathrm{maj}(w^{-1}) + k(w) \bmod n$$

where $k(w)$ is the number of *cyclic inverse descents* of w, that is, the number of values $i = 1, 2, \ldots, n$ such that $i + 1$ (taken modulo n, so $n + 1 = 1$) appears to the left of i within $w = (w_1, \ldots, w_n)$. The reason is that $\mathrm{maj}(w^{-1})$ can be thought of as summing the values i in the inverse descent set of w, but when taken mod n, it can also be thought of as summing the values in the cyclic inverse descent set– this can only differ by n from the other former. And it is easy to see that the values in the cyclic inverse descent set each go up by 1 (modulo n) in passing from w to cw.

It should also be clear that this value $k = k(w)$ is constant on the G-orbits $\{w, cw, c^2w, \ldots, c^{n-1}w\}$, and hence $\mathrm{maj}((c^m w)^{-1}) \equiv \mathrm{maj}(w^{-1}) + mk \bmod n$.

Now note that this value k is at most $n - 1$, and hence when $n = p$ is prime, k is a cyclic generator for $\mathbb{Z}/p\mathbb{Z}$. Therefore the elements of $\{w, cw, c^2w, \ldots, c^{p-1}w\}$ will take on all the possible different residue classes in $\mathbb{Z}/p\mathbb{Z}$ for the major index of their inverse, and only one achieves the zero residue class. $\square$

Secondly, here are the first few determinants for the $\{0,1\}$-incidence matrices:

p	2	3	5	7
det	1	1	5^3	7^{15}

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