CYCLOTOMIC AND SIMPLICIAL MATROIDS

JEREMY L. MARTIN AND VICTOR REINER

Abstract. We show that two naturally occurring matroids representable over $\mathbb{Q}$ are equal: the **cyclotomic matroid** μ_n represented by the n^{th} roots of unity $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$ inside the cyclotomic extension $\mathbb{Q}(\zeta)$, and a direct sum of copies of a certain simplicial matroid, considered originally by Bolker in the context of transportation polytopes. A result of Adin leads to an upper bound for the number of $\mathbb{Q}$-bases for $\mathbb{Q}(\zeta)$ among the n^{th} roots of unity, which is tight if and only if n has at most two odd prime factors. In addition, we study the Tutte polynomial of μ_n in the case that n has two prime factors.

1. Introduction

This paper is about two matroids representable over $\mathbb{Q}$ that turn out, somewhat unexpectedly, to be **dual** (or **orthogonal**). Briefly, a matroid is a combinatorial abstraction of the linear dependence data associated to a (finite) set of vectors in a vector space. That is, the data for a matroid on ground set E records which subsets of E are dependent and independent, or the linear span and dimension of each subset, etc. The matroid is **representable** over a field $\mathbb{F}$ if the elements of E may be identified with vectors in an $\mathbb{F}$-vector space that achieve the matroid data. The **dual** of a matroid M on E is defined generally as the matroid M^* whose bases are the complements of bases in M; this abstracts the situation where M, M^* are matroids represented over $\mathbb{F}$ by the columns of two matrices whose row spaces are orthogonal complements of each other. For background, motivation, and matroid terminology left undefined here, see any of the standard references [6, 13, 15, 16, 17, 18].

The first matroid that we study relates to cyclotomic extensions. Let ζ_n be a primitive n^{th} root of unity, which we will abbreviate by ζ when no confusion can arise. Recall that the degree of the cyclotomic extension $\mathbb{Q}(\zeta)$ is given by Euler's ϕ-function $\phi(n)$, with the following formula: if $n = p_1^{m_1} \cdots p_r^{m_r}$ for distinct primes p_i, then

$$\phi(n) = p_1^{m_1-1} \cdots p_r^{m_r-1}(p_1 - 1) \cdots (p_r - 1).$$

2000 *Mathematics Subject Classification.* Primary 05B35; Secondary 11R18,55U10.

Key words and phrases. Cyclotomic extension, simplicial matroid, higher-dimensional tree, transportation polytope, Tutte polynomial.

First author supported by NSF Postdoctoral Fellowship. Second author supported by NSF grant DMS-0245379.

The $\mathbb{Q}$-vector space $\mathbb{Q}(\zeta)$ gives rise to a natural matroid μ_n, represented by the n vectors $Z_n = \{1, \zeta, \zeta^2, \ldots, \zeta^{n-1}\}$ and having rank $\phi(n)$. We call μ_n the **cyclotomic matroid of order** n.

The second matroid is a **simplicial matroid** [17, Chapter 6]; that is, it is represented by the columns of a simplicial boundary map (see, e.g., [12] for background on simplicial homology). Let $\Delta = \Delta^d$ be a d-dimensional simplicial complex[1] whose (reduced) simplicial homology with coefficients in $\mathbb{F}$ vanishes in codimension 1; that is, $\tilde{H}_{d-1}(\Delta, \mathbb{F}) = 0$. If Δ has exactly n facets (that is, faces of maximum dimension d), then the n columns in the matrix expressing the simplicial boundary map

$$\tilde{C}_d(\Delta^d, \mathbb{F}) \xrightarrow{\partial_d} \tilde{C}_{d-1}(\Delta^d, \mathbb{F})$$

represent a matroid over $\mathbb{F}$ that we will call the **simplicial matroid** $\mathcal{S}(\Delta^d, \mathbb{F})$. This is a matroid on n elements, with

$$\operatorname{rank} \mathcal{S}(\Delta^d, \mathbb{F}) \;=\; \dim_{\mathbb{F}} \operatorname{im} \partial_d \;=\; \dim_{\mathbb{F}} \ker \partial_{d-1}.$$

In this paper, we consider the $(r-1)$-dimensional simplicial complex $\Delta^{r-1}_{n_1,\ldots,n_r}$ defined as the **simplicial join** [12, §62] of 0-dimensional complexes $\Delta^0_{n_1}, \ldots, \Delta^0_{n_r}$, where $\Delta^0_{n_i}$ consists of n_i disjoint vertices. That is, a face of $\Delta^{r-1}_{n_1,\ldots,n_r}$ contains at most one vertex from each $\Delta^0_{n_i}$. Our main result expresses the connection between cyclotomic matroids and these particular simplicial matroids.

Theorem 1. *Let $n = p_1^{m_1} \cdots p_r^{m_r}$, with $p_1, \ldots, p_r$ distinct primes and $m_1, \ldots, m_r$ positive integers.*

Then the following two matroids representable over $\mathbb{Q}$ are dual:

- *The cyclotomic matroid μ_n.*
- *The direct sum of $p_1^{m_1-1} \cdots p_r^{m_r-1}$ copies of $\mathcal{S}(\Delta^{r-1}_{p_1,\ldots,p_r}, \mathbb{Q})$.*

Bolker [5] was the first to study the simplicial matroid $\mathcal{S}(\Delta^{r-1}_{n_1,\ldots,n_r}, \mathbb{Q})$, where the n_i are positive integers, not necessarily prime. He proposed the bases of this matroid as higher-dimensional analogues of spanning trees[2] in bipartite graphs, and studied their relation to vertices of certain transportation polytopes. Each such basis T of $\mathcal{S}(\Delta^{r-1}_{n_1,\ldots,n_r}, \mathbb{Q})$ gives rise to a $\mathbb{Q}$-acyclic pure $(r-1)$-dimensional simplicial complex Δ_T, obtained by attaching the $(r-1)$-simplices indexed by T to the $(r-2)$-skeleton of $\Delta^{r-1}_{n_1,\ldots,n_r}$. He also suggested

$$\prod_{i=1}^{r} n_i^{\prod_{j \neq i}(n_j - 1)} \tag{1}$$

[1]We adopt the following notational convention throughout. When we wish to emphasize that a simplicial complex Δ has dimension d, we will denote it by the symbol Δ^d; otherwise, we will frequently omit the superscript to simplify notation.

[2]This is not the only way to generalize the notion of spanning tree to 2- or higher-dimensional simplicial complexes; see, e.g., [4], [7], [8]. These other generalizations, however, play no role in our present study.

as an upper bound for the number of bases, and showed that this formula gives the exact number of bases if and only if at most two of the n_i exceed 2. In particular, Bolker showed [5, Theorems 27, 28] that this occurs exactly when for each basis T, the (pure torsion) integer homology group $\tilde{H}_{r-2}(\Delta_T, \mathbb{Z})$ is trivial.

A result of Adin [3] (generalizing work of Kalai in [10]) implies that (1) is indeed an upper bound for the number of bases of $\Delta^{r-1}_{n_1,\dots,n_r}$. The method of Adin and Kalai is to generalize the classical Kirchhoff Matrix-Tree Theorem to simplicial complexes, using the Binet-Cauchy determinant formula: this yields the result

$$\sum_{\substack{\text{bases } T \text{ of}\\ \mathcal{S}(\Delta^{r-1}_{n_1,\dots,n_r},\mathbb{Q})}} \left|\tilde{H}_{r-2}(\Delta_T, \mathbb{Z})\right|^2 = \prod_{i=1}^{r} n_i^{\prod_{j\neq i}(n_j-1)}. \tag{2}$$

That is, Bolker's formula (1) is an *exact* count for such bases with the following weighting: instead of a basis T contributing 1 to the count, its contribution is the square of the order of its integral homology group in codimension 1.

Combining the results of Adin and Bolker with Theorem 1, and the fact that dual matroids have the same number of bases, one immediately obtains the following corollary.

Corollary 2. *Let $n = p_1^{m_1} \cdots p_r^{m_r}$, with $p_1, \dots, p_r$ distinct primes and $m_1, \dots, m_r$ positive integers, and let ζ be a primitive n^{th} root of unity.*

Then the number of subsets of $Z_n = \{1, \zeta, \zeta^2, \dots, \zeta^{n-1}\}$ that are bases for $\mathbb{Q}(\zeta)$ is bounded above by

$$\left(\prod_{i=1}^{r} p_i^{\prod_{j\neq i}(p_j-1)}\right)^{p_1^{m_1-1}\cdots p_r^{m_r-1}},$$

with equality if and only if $n = 2^a p^b q^c$ for nonnegative integers a, b, c.

Section 2 contains the proof of Theorem 1. In Section 3, we study the case that $n = p_1^{m_1} p_2^{m_2}$ has only two prime factors. Here the simplicial complex $\Delta^1_{p_1,p_2}$ is just the complete bipartite graph K_{p_1,p_2}, and the matroid $\mathcal{S}(\Delta^1_{p_1,p_2}, \mathbb{Q})$ is the **cycle matroid** (or **graphic matroid**) of K_{p_1,p_2}, whose bases are spanning trees. We study enumerative invariants finer than the number of bases for these particular graphic matroids, such as their Tutte polynomials, and a weighted enumeration of their spanning forests.

2. Proof of Theorem 1

We first state a well-known general fact about matroids, to be used in the proof.

Lemma 3. *Let M be a matroid of rank ρ on ground set E. Suppose there exists a (disjoint) decomposition $E = \bigsqcup_{i=1}^{t} E_i$ such that $\operatorname{rank}(M) = \sum_{i=1}^{t} \operatorname{rank}(M|_{E_i})$.*

Then $M \cong \bigoplus_{i=1}^{t} M|_{E_i}$.

Proof. Because of the assumption on ranks, a basis for M must be a disjoint union of bases of the matroids $M|_{E_i}$. □

Proof of Theorem 1.
We first reduce to the case where n is square-free. Let $s = p_1 \cdots p_r$ denote the square-free part of n, and let

$$t = \frac{n}{s} = p_1^{m_1-1} \cdots p_r^{m_r-1}.$$

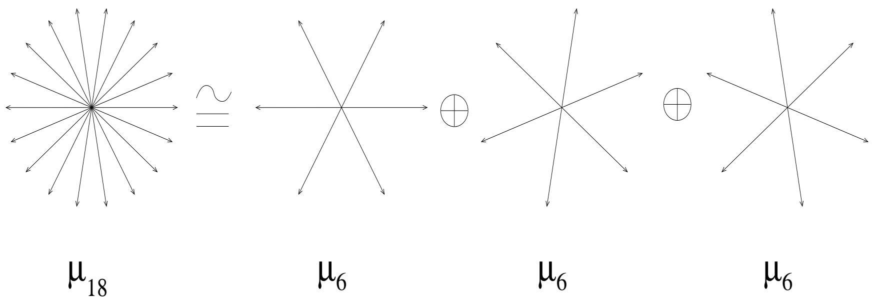
Note that

$$\mathrm{rank}\ \mu_n = \phi(n) = t \cdot \phi(s) = t \cdot \mathrm{rank}\ \mu_s.$$

For $0 \leq j \leq t-1$, let $E_j := \{\zeta^j, \zeta^{j+t}, \ldots, \zeta^{j+(s-1)t}\}$. Then

$$Z_n = \bigsqcup_{j=0}^{t-1} E_j,$$

and the submatroid $\mu_n|_{E_j}$ obtained by restricting μ_n to the ground set E_j is isomorphic to μ_s. By Lemma 3, it follows that μ_n is isomorphic to a direct sum of t copies of μ_s. The following figure illustrates the decomposition for $n = 18$, $s = 6$ and $t = 3$.



Since duality commutes with direct sums, we have reduced to the case that $n = p_1 \cdots p_r$ is square-free, and assume this for the remainder of the proof.

For each prime p dividing n, consider the short exact sequence of $\mathbb{Q}$-vector spaces

$$0 \to \mathbb{Q} \xrightarrow{i_p} \mathbb{Q}^p \xrightarrow{\pi_p} \mathbb{Q}(\zeta_p) \to 0$$

in which π_p maps the j^{th} standard basis element e_j in $\mathbb{Q}^p$ to ζ_p^j, and the image of i_p is the line $\mathbb{Q}(e_1 + \cdots + e_n)$. Rephrasing this, the cochain complex $\mathcal{C}(p)$ defined by

$$\begin{array}{ccccccc} 0 & \to & \mathcal{C}^0(p) & \longrightarrow & \mathcal{C}^1(p) & \to & 0 \\ 0 & \to & \mathbb{Q} & \xrightarrow{i_p} & \mathbb{Q}^p & \to & 0 \end{array}$$

has

$$H^0(\mathcal{C}(p)) = 0,$$
$$H^1(\mathcal{C}(p)) = \mathbb{Q}(\zeta_p) \ (\cong \mathbb{Q}^{p-1}).$$

Note that the complex $\mathcal{C}(p)$ coincides, up to a shift in homological degree, with the augmented simplicial cochain complex that computes the cohomology of the 0-dimensional complex Δ_p^0. Similarly, the tensor product of complexes

$$\mathcal{C} := \bigotimes_{j=1}^{r} \mathcal{C}(p_j)$$

coincides with the augmented simplicial cochain complex for the simplicial join $\Delta_{p_1,\ldots,p_r} \cong \Delta_{p_1}^0 * \cdots * \Delta_{p_r}^0$. The Künneth formula over $\mathbb{Q}$ then implies that the complex $\mathcal{C}$ has no homology except in the last position. The last cochain group in $\mathcal{C}$ maps surjectively onto the homology as follows:

$$\mathcal{C}^r \ := \ \bigotimes_{j=1}^{r} \mathcal{C}^1(p_j) \ \cong \ \bigotimes_{j=1}^{r} \mathbb{Q}^{p_j} \ \xrightarrow{\otimes_j \pi_{p_j}} \ \bigotimes_{j=1}^{r} \mathbb{Q}(\zeta_{p_j}). \tag{3}$$

The Chinese Remainder Theorem allows us to identify the last map in (3) with the surjection $\mathbb{Q}^n \xrightarrow{\pi_n} \mathbb{Q}(\zeta)$ sending e_j to ζ^j, that is, the map whose matrix has columns representing μ_n. Because $\mathcal{C}$ has no other homology, the last coboundary map $\delta \colon \mathcal{C}^{r-1} \to \mathcal{C}^r$ has columns spanning the kernel of π_n. Its transpose

$$C_{r-1}(\Delta_{p_1,\ldots,p_r};\mathbb{Q}) \ \xrightarrow{\delta^T} \ C_{r-2}(\Delta_{p_1,\ldots,p_r};\mathbb{Q})$$

has columns which represent the simplicial matroid $\mathcal{S}(\Delta_{p_1,\ldots,p_r},\mathbb{Q})$, hence this matroid is dual to μ_n. $\square$

We close this section with a few remarks.

Remark 4. To give some further context for Theorem 1, it should be noted that matroid duality has played a key role in the study of simplicial matroids from the beginning. Crapo and Rota studied the full simplicial matroid $\mathcal{S}_{m-k}^m(\mathbb{F})$ corresponding to the $(k-1)$-skeleton of the full simplex with m vertices, and observed that Alexander duality implies that $\mathcal{S}_k^m(\mathbb{F})$ and $\mathcal{S}_{m-k}^m(\mathbb{F})$ are dual matroids [6, Proposition 11.4]; see also [17, Theorem 6.2.1]. Lindström [11] observed another interesting example of such a duality, this time involving two natural matroids (one of them simplicial), both of whose ground sets are indexed by the bases of some (other) simple matroid; see [17, §6.5]. $\square$

Remark 5. It has been observed (e.g., by Johnsen [9]) that the primitive n^{th} roots of unity form a $\mathbb{Q}$-basis for $\mathbb{Q}(\zeta)$ if and only if n is square-free. Thus if $n = p_1 \cdots p_r$ is square-free, one might ask which particular basis of the simplicial matroid $\mathcal{S}(\Delta_{p_1,\ldots,p_r}^{r-1},\mathbb{Q})$ (that is, which higher-dimensional tree T of $\Delta_{p_1,\ldots,p_r}^{r-1}$) corresponds to the set of primitive n^{th} roots. Passing through matroid duality, and the Chinese Remainder Theorem identification in the proof of the theorem, one can check that T is the union of the stars within $\Delta_{p_1,\ldots,p_r}^{r-1}$ of the vertices

$v_1, \ldots, v_r$, where v_i is the unique vertex of $\Delta^0_{p_i}$ labeled by 0 modulo p_i. Applying a simple nerve argument to this cover of T by stars, one finds that T is not only $\mathbb{Q}$-acyclic as expected, but in fact contractible. Hence it has vanishing homology group $\tilde{H}_{r-2}(\Delta_T, \mathbb{Z})$, and contributes exactly 1 in the summation of equation (2). □

Remark 6. If n is divisible by at most two primes (that is, $r \leq 2$ above), then the complex $\Delta^{r-1}_{p_1,p_2}$ is a graph. In this case, the matroid μ_n is **cographic** (see, e.g., [17, p. 35], and Section 3 below).

Furthermore, if n is odd, then the cyclotomic matroid μ_{2n} is the parallel extension of μ_n in which one creates one parallel copy for each ground set element. For instance, μ_n is again cographic for $n = 2pq$, where p, q are odd primes: it is the cographic matroid of the graph obtained by duplicating every edge of the complete bipartite graph $K_{p,q}$. □

3. The case of two prime factors

The goal of this section is to compute some enumerative invariants of the cyclotomic matroid μ_n, finer than the number of bases, in the special case where $n = p_1^{m_1} p_2^{m_2}$ has only two prime factors.

The **Tutte polynomial** $T_M(x, y)$ is one of the most important isomorphism invariants of a matroid M; see the excellent survey by Brylawski and Oxley [18, Chapter 6] for background. We begin by reviewing briefly the definition of the Tutte polynomial, as well as some elementary properties relevant for our calculations.

Let M be a matroid with ground set E, and let r be the rank function on subsets A of E. When M is representable, $r(A)$ is the dimension of the linear span of the vectors in A. The **Tutte polynomial** $T_M(x, y)$ may be defined as the corank-nullity generating function

$$T_M(x, y) := \sum_{A \subset E} (x-1)^{r(M)-r_M(A)} (y-1)^{|A|-r_M(A)} \tag{4}$$

[18, §6.2]. Two easy consequences of (4) are as follows. First, if M^* is the dual matroid to M, then

$$T_{M^*}(x, y) = T_M(y, x). \tag{5}$$

Second, setting $y = 1$ and replacing x with $x+1$ in (4) gives a generating function for independent subsets of M according to their cardinality:

$$T_M(x+1, 1) \;=\; \sum_{\text{independent } I \subset E} x^{r(M)-|I|}. \tag{6}$$

Let $G = (V, E)$ be a graph, and let $M = M(G)$ be the corresponding graphic matroid on ground set E, whose bases are the spanning trees of G and whose independent sets are acyclic subgraphs (= forests). Then there is a substitution of variables in $T_G := T_{M(G)}$ which gives Crapo's **coboundary polynomial** $\bar{\chi}_G(q,t)$ [18, §6.3F]:

$$\begin{aligned}\bar{\chi}_G(q,t) \;&:=\; q^{-1} \sum_{\substack{\text{vertex colorings}\\ f\colon V\to\{1,2,\ldots,q\}}} t^{|\{\{v_1,v_2\}\in E\colon\, f(v_1)=f(v_2)\}|} \\ &=\; (t-1)^{|V|-c(G)}\, T_G\left(\frac{q+t-1}{t-1},\; t\right),\end{aligned} \tag{7}$$

where $c(G)$ denotes the number of connected components of G. Note that $T_G(x,y)$ may be recovered from $\bar{\chi}_G(x,y)$ via the substitution $q=(x-1)(y-1), t=y$, and that the usual chromatic polynomial $\chi_G(q)$ is the specialization $q\,\bar{\chi}_G(q,0)$.

In this section, we consider the special case that $n=p_1^{m_1}p_2^{m_2}$. By Theorem 1, the cyclotomic matroid μ_n is dual to the direct sum of $p_1^{m_1-1}p_2^{m_2-1}$ copies of the simplicial matroid $\mathcal{S}(\Delta^1_{p_1,p_2};\mathbb{Q})$. As mentioned previously, the 1-dimensional complex $\Delta^1_{p_1,p_2}$ is simply the complete bipartite graph K_{p_1,p_2}, and the simplicial matroid is the usual graphic matroid $M(K_{p_1,p_2})$.

We begin by obtaining an exponential generating function for the coboundary polynomials of these cycle matroids, mimicking Ardila's method in [1, Theorem 2.4.1]. Let K_{p_1,p_2} have bipartite vertex set $V_1 \sqcup V_2$ with $|V_i|=p_i$. Every vertex-coloring $f\colon V\to\{1,2,\ldots,q\}$ decomposes the partite sets V_1,V_2 into (possibly empty) color classes $V_1^{(i)}$, $V_2^{(i)}$:

$$V_1=\bigsqcup_{i=1}^{q} V_1^{(i)}, \qquad\qquad V_2=\bigsqcup_{i=1}^{q} V_2^{(i)}$$

with

$$|\{\{v_1,v_2\}\in E\colon f(v_1)=f(v_2)\}| \;=\; \sum_{i=1}^{q} |V_1^{(i)}||V_2^{(i)}|.$$

Now exponential generating function manipulation (see [14, Prop. 5.1.3]) gives the following formula.

Proposition 7. *Denote by $\bar{\chi}_{p_1,p_2}(q,t)$ the coboundary polynomial of the graphic matroid of the complete bipartite graph K_{p_1,p_2}. Then*

$$1+q\left(\sum_{\substack{(p_1,p_2)\in\\ \mathbb{N}^2-\{(0,0)\}}} \bar{\chi}_{p_1,p_2}(q,t)\,\frac{x_1^{p_1}x_2^{p_2}}{p_1!\,p_2!}\right) \;=\; \left(\sum_{(m_1,m_2)\in\mathbb{N}^2} t^{m_1m_2}\,\frac{x_1^{m_1}x_2^{m_2}}{m_1!\,m_2!}\right)^{q}.$$

This formula does not appear to generalize to the case that n has more than two prime factors.

Remark 8. Setting $t = 0$ yields the following simple exponential generating function for the chromatic polynomial $\chi_{p_1,p_2}(q)$ of K_{p_1,p_2}:

$$1 + \sum_{(p_1,p_2)\in\mathbb{N}^2-\{(0,0)\}} \chi_{p_1,p_2}(q)\, \frac{x_1^{p_1} x_2^{p_2}}{p_1!\, p_2!} \quad = \quad (e^{x_1} + e^{x_2} - 1)^q\,. \tag{8}$$

□

We now apply the generating function for coboundary polynomials to enumerate the $\mathbb{Q}$-linearly independent subsets of the n^{th} roots of unity. Setting $t = y+1$ and $x_i = z_i/y$ in the formula of Proposition 7 and applying (7) gives

$$\begin{aligned} y^{-1} \sum_{(p_1,p_2)\in\mathbb{N}^2-\{(0,0)\}} T_{K_{p_1,p_2}}\left(\frac{q+y}{y},\, y+1\right) \frac{z_1^{p_1} z_2^{p_2}}{p_1!\, p_2!} \quad &= \\ q^{-1}\left[\left(\sum_{(m_1,m_2)\in\mathbb{N}^2} \frac{(y+1)^{m_1 m_2}}{y^{m_1+m_2}}\, \frac{z_1^{m_1} z_2^{m_2}}{m_1!\, m_2!}\right)^q - 1\right]. \end{aligned} \tag{9}$$

Multiplying through by y and taking the limit as q approaches 0 (via L'Hôpital's Rule), we obtain

$$\begin{aligned} \sum_{(p_1,p_2)\in\mathbb{N}^2-\{(0,0)\}} T_{K_{p_1,p_2}}(1, y+1)\, \frac{z_1^{p_1} z_2^{p_2}}{p_1!\, p_2!} \quad &= \\ z \log\left(\sum_{(m_1,m_2)\in\mathbb{N}^2} \frac{(y+1)^{m_1 m_2}}{y^{m_1+m_2}}\, \frac{z_1^{m_1} z_2^{m_2}}{m_1!\, m_2!}\right). \end{aligned} \tag{10}$$

Once again, recall that for $n = p_1^{m_1} p_2^{m_2}$, Theorem 1 implies that the cyclotomic matroid μ_n is dual to the direct sum of $p_1^{m_1-1} p_2^{m_2-1}$ copies of the graphic matroid $M(K_{p_1,p_2})$. Therefore, (5) and (6) imply the following result.

Corollary 9. *Let $n = p_1^{m_1} p_2^{m_2}$, where p_1, p_2 are distinct primes.*
Then the generating function

$$\sum_{\substack{I\subset Z_n \\ \mathbb{Q}-\text{linearly independent}}} y^{\phi(n)-|I|}$$

equals the $(p_1^{m_1-1} p_2^{m_2-1})^{th}$ power of the coefficient of $\frac{z_1^{p_1} z_2^{p_2}}{p_1!\, p_2!}$ in

$$y \log\left(\sum_{(m_1,m_2)\in\mathbb{N}^2} \frac{(y+1)^{m_1 m_2}}{y^{m_1+m_2}}\, \frac{z_1^{m_1} z_2^{m_2}}{m_1!\, m_2!}\right).$$

Remark 10. One can deduce from equation (8) the generating function counting acyclic orientations of K_{p_1,p_2}; see [14, Ex. 5.6]. Ardila [2, Prop. 1] has recently generalized this acyclic orientation calculation in a direction different from ours, by computing the generating function for faces of various dimensions in the hyperplane arrangement associated to K_{p_1,p_2}. $\square$

We conclude this section with an observation about the independent set polynomial $T_M(x+1,1)$ (see (6)) for $M = M(K_{p,q})$. Note that $T_M(x+1,1)$ enumerates spanning subsets of Z_{pq} by cardinality when p, q are distinct primes. Computations for small values of p, q suggest that for $p \leq q$ (not necessarily prime), $T_{M(K_{p,q})}(x+1,1)$ is divisible by $(x+p)^{q-p+1}$. This follows from a more refined statement (Proposition 11 below) on the enumeration of spanning forests in $K_{p,q}$. Let $K_{p,q}$ have disjoint vertex sets $V = \{v_1, \ldots, v_p\}$ and $W = \{w_1, \ldots, w_q\}$. For a subset F of its edges and u a vertex, let $\deg_F(u)$ denote the degree of u in the edge-subgraph corresponding to F. Define a forest enumerator

$$A_{p,q} \ = \ A_{p,q}(x_1, \ldots, x_p) \ := \sum_{\text{forests } F} x_1^{\deg_F(v_1)} \ldots x_p^{\deg_F(v_p)}.$$

For example, it can be calculated that

$$\begin{aligned} A_{1,q} &= (1+x_1)^q, \\ A_{2,q} &= (1+x_1+x_2)^{q-1}(1+x_1+x_2+qx_1x_2). \end{aligned}$$

Note that $A_{p,q}$ specializes to the independent set polynomial of $K_{p,q}$ as follows. Since $K_{p,q}$ has pq edges, each of which is incident to exactly one v_i, we obtain

$$T_{K_{p,q}}(x+1,1) \ = \ x^{pq} A_{p,q}(x^{-1}, \ldots, x^{-1}).$$

Proposition 11. *Let $q \geq p \geq 1$ be integers. Then*

$$A_{p,q} = \sum_{j=0}^{p-1} (1+x_1+\cdots+x_p)^{q-j} \tilde{A}_{p,j}$$

where $\tilde{A}_{p,q} := \sum_F x_1^{\deg_F(v_1)} \ldots x_p^{\deg_F(v_p)}$, in which the summation is over spanning forests F in $K_{p,q}$ in which every vertex w_i has degree at least 2.

In particular, $A_{p,q}$ is divisible by

$$(1+x_1+\cdots+x_p)^{q-p+1}.$$

Proof. Given F a spanning forest of $K_{p,q}$, let $W' \subset W$ be the set of w_i for which $\deg_F(w_j) > 1$. We claim that $|W'| < p$, else F contains at least $2p$ edges among the $2p$ vertices $|W' \cup V|$, which contradicts the assumption that F is acyclic.

After choosing the restriction of F to the vertices $W' \cup V$, the remaining edges of F consist of at most one edge incident to each vertex in $W - W'$. Classifying F according to the cardinality $|W'| = j$, the assertion follows. $\square$

The quotient $\frac{A_{p,q}}{(1+x_1+\cdots+x_p)^{q-p+1}}$ is a symmetric polynomial of total degree $2(p-1)$ in the variables $x_1, \ldots, x_p$. It appears not to factor further in general. One might also hope for a factorization of a forest enumerator that keeps track of the degrees in V as well as those in W, but this also appears not to factor further.

References

[1] F. Ardila, Enumerative and algebraic aspects of matroids and hyperplane arrangements. Ph.D. thesis, MIT, 2003.

[2] F. Ardila, A tropical morphism related to the hyperplane arrangement of the complete bipartite graph. Preprint, `arXiv:math.CO/0404287`, 2004.

[3] R.M. Adin, Counting colorful multi-dimensional trees. *Combinatorica* **12** (1992), 247–260.

[4] L.W. Beineke and R.E. Pippert, Properties and characterizations of k-trees. *Mathematika* **18** (1971), 141–151.

[5] E.D. Bolker, Simplicial geometry and transportation polytopes. *Trans. Amer. Math. Soc.* **217** (1976), 121–142.

[6] H.H. Crapo and G.-C. Rota, On the foundations of combinatorial theory: Combinatorial geometries. Preliminary edition. The M.I.T. Press, Cambridge, Mass.-London, 1970.

[7] S. Faridi, The facet ideal of a simplicial complex. *Manuscripta Math.* **109** (2002), no. 2, 159–174.

[8] F. Harary and E.M. Palmer, On acyclic simplicial complexes. *Mathematika* **15** (1968), 115–122.

[9] K. Johnsen, Lineare Abhängigkeiten von Einheitswurzeln. *Elem. Math.* **40** (1985), 57–59.

[10] G. Kalai, Enumeration of $\mathbb{Q}$-acyclic simplicial complexes. *Israel J. Math.* **45** (1983), 337–351.

[11] B. Lindström, Matroids on the bases of simple matroids. *Europ. J. Combin.* **2** (1981), 61–63.

[12] J.R. Munkres, Elements of algebraic topology. Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[13] J.G. Oxley, Matroid theory. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.

[14] R.P. Stanley, Enumerative Combinatorics, vol. 2. Cambridge University Press, Cambridge, 1999.

[15] D.J.A. Welsh, Matroid theory. London Math. Soc. Monographs **8**. Academic Press, London-New York, 1976.

[16] N. White, Theory of matroids. *Encyclopedia of Mathematics and its Applications* **26**. Cambridge University Press, Cambridge, 1986.

[17] N. White, Combinatorial geometries. *Encyclopedia of Mathematics and its Applications* **29**. Cambridge University Press, Cambridge, 1987.

[18] N. White, Matroid applications. *Encyclopedia of Mathematics and its Applications* **40**. Cambridge University Press, Cambridge, 1992.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: `martin@math.umn.edu`

School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: `reiner@math.umn.edu`