THE CYCLIC SIEVING PHENOMENON

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Abstract. The *cyclic sieving phenomenon* is defined for generating functions of a set affording a cyclic group action, generalizing Stembridge's $q = -1$ phenomenon. The phenomenon is shown to appear in various situations, involving q-binomial coefficients, Pólya-Redfield theory, polygon dissections, non-crossing partitions, finite reflection groups, and some finite field q-analogues.

1. Introduction

Stembridge's $q = -1$ *phenomenon* [42, 43, 44] has proven to be a useful tool in organizing various enumerative results. This paper introduces the more general *cyclic sieving phenomenon*, which we now define.

Let X be a finite set, with an action of a cyclic group C of order n. Elements within a C-orbit share the same stabilizer subgroup, whose cardinality we will call the *stabilizer-order* for the orbit. Let $X(q)$ be a polynomial in q having nonnegative integer coefficients, with the property that $X(1) = |X|$; we will think of $X(q)$ as a q-enumerator or generating function for X. Fix an isomorphism ω of C with the complex n^{th} roots of unity, that is, an embedding $\omega : C \hookrightarrow \mathbb{C}^\times$.

Definition-Proposition. *The following are equivalent conditions for a triple $(X, X(q), C)$ as above:*

(i) *For every $c \in C$,*

$$[X(q)]_{q=\omega(c)} = |\{x \in X : c(x) = x\}|.$$

(ii) *The coefficient a_ℓ defined uniquely by the expansion*

$$X(q) \equiv \sum_{\ell=0}^{n-1} a_\ell q^\ell \mod q^n - 1$$

Date: September 2003.

Key words and phrases. q-binomial coefficient, q-multinomial coefficient, roots-of-unity, principal specialization, Schur function, Kraśkiewicz-Weyman, Singer cycle, Springer regular element, polygon dissections, noncrossing partitions, hook formula, ordered tree.

First, second author supported by NSF grants DMS-0245379, DMS-0203282 respectively.

has the following interpretation: a_ℓ counts the number of C-orbits on X for which the stabilizer-order divides ℓ. In particular, a_0 counts the total number of C-orbits on X, and a_1 counts the number of free C-orbits on X.

When either of these two conditions holds, we say that $(X, X(q), C)$ exhibits the *cyclic sieving phenomenon.* The straightforward equivalence between conditions (i) and (ii) above is proven in Section 2, and related to a linear-algebraic/representation-theoretic paradigm for proving them.

When $|C| = 2$, condition (i) above is the $q = -1$ phenomenon. We observe many instances of the more general cyclic sieving phenomenon, beginning with the following result on q-binomial coefficients. It is well-known [24, Ex. I.2.3], [37, §7.8] that the q-binomial coefficients

$$\begin{bmatrix} N \\ k \end{bmatrix}_q \text{ and } \begin{bmatrix} N+k-1 \\ k \end{bmatrix}_q$$

are generating functions for k-subsets and k-multisubsets of $[N] := \{1, 2, \ldots, N\}$ counted according to certain natural q-weights. Say that the cyclic group C of order n acts *nearly freely* on $[N]$ if it is generated by an element $c \in \mathfrak{S}_N$ whose cycle type is either

- a cycles of size n, so that $N = an$ (and C_n acts freely), or
- a cycles of size n and one singleton cycle, so that $N = an + 1$

for some positive integer a.

Theorem 1.1. *Let the cyclic group C of order n act nearly freely on $[N]$.*

(a) *Let X be the set of k-multisubsets of $[N]$, and*

$$X(q) := \begin{bmatrix} N+k-1 \\ k \end{bmatrix}_q .$$

Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

(b) *Let X be the set of k-subsets of $[N]$, and*

$$X(q) := \begin{bmatrix} N \\ k \end{bmatrix}_q .$$

Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

Example 1.2. As an illustration, take $k = n = 6, N = 12$, and C the cyclic group generated by $c = (1, 3, 5, 7, 9, 11)(2, 4, 6, 8, 10, 12)$ acting on the subsets of $[12]$ of cardinality 6. Then

$$\begin{bmatrix} 12 \\ 6 \end{bmatrix}_q \equiv 160 + 150q + 156q^2 + 152q^3 + 156q^4 + 150q^5 \mod q^6 - 1$$

reflecting the fact that there are 150 free orbits, and, respectively, $6, 2, 2$ orbits having stabilizer-orders $2, 3, 6$. For an example involving multisets, take $k = n = 4$ and $N = 8$, and consider the cyclic group C_4 generated by $c = (1357)(2468)$ acting on the multisubsets of $[8]$ of cardinality 4. Then

$$\begin{bmatrix} 11 \\ 4 \end{bmatrix}_q \equiv 86 + 80q + 84q^2 + 80q^3 \mod q^4 - 1$$

reflecting the fact that there are 80 free orbits, and, respectively, $4, 2$ orbits having stabilizer-orders $2, 4$.

Theorem 1.1 is not hard; it is deduced using the representation theory paradigm in Section 3, or via calculation of explicit formulae for the coefficients a_ℓ in Section 4.

Special cases of this theorem provide combinatorial proofs for several results in the literature. For example, at the end of the paper [12], the authors ask for a combinatorial explanation of the following corollary to Theorem 1.1.

Corollary 1.3. *Let $a_\ell(n, k)$ denote the coefficient of q^ℓ in*

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \mod q^n - 1,$$

with the convention that $a_1(1, k) = 1$. Then

$$a_\ell(n,k) = \sum_{d|n,k,\ell} a_1\left(\frac{n}{d}, \frac{k}{d}\right). \tag{1.1}$$

This corollary is immediate from Theorem 1.1(a), with C the cyclic group of order n acting freely on $[n]$, and on its k-multisubsets: the left side of (1.1) counts orbits of multisets whose stabilizer-order is some d dividing ℓ; the number of such orbits for which the stabilizer-order is *exactly* d is the term $a_1(\frac{n}{d}, \frac{k}{d})$ on the right side of (1.1).

One of our original motivations was to understand the following fact observed by Chapoton [6], which we deduce in Section 5 from either part (a) or (b) of Theorem 1.1.

Theorem 1.4. *The constant term a_0 in*

$$\begin{bmatrix} 2n-1 \\ n \end{bmatrix}_q \mod q^n - 1$$

counts ordered trees with n non-root vertices, up to cyclic rotation of subtrees about the root.

The two parts of Theorem 1.1 have interesting generalizations. Section 6 generalizes Theorem 1.1(a) to a result (Theorem 6.1) about principal specializations of weight enumerators from Pólya-Redfield theory. A sample application of this result is the following result of Read [26].

Corollary 1.5. *Let $X(q)$ be the generating function counting unlabelled (isomorphism classes of) simple graphs on n vertices according to their number of edges. Then $X(-1)$ counts the number of self-complementary unlabelled graphs on n vertices.*

Section 8 generalizes Theorem 1.1(b) to a statement about finite Coxeter groups, and more generally, unitary reflection groups. The symmetric group $\mathfrak{S}_N$ is a Coxeter group W of type A_{N-1}. The k-subsets of $[N]$ correspond to cosets W/W_J, where W_J is the parabolic subgroup $\mathfrak{S}_k \times \mathfrak{S}_{N-k}$. One then has

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = W^J(q) := \sum_{w \in W^J} q^{\ell(w)}$$

where W^J denotes the set of minimal length representatives for cosets of W_J. A cyclic subgroup C of W acts on the set of cosets W/W_J by left-multiplication. It turns out that C in $\mathfrak{S}_N$ acts nearly freely on $[N]$ exactly when it is generated by a *regular element* in the sense of Springer [35]. In Section 8, we deduce (a generalization of) the following statement, using Springer's theory of regular elements as rephrased by Kraśkiewicz and Weyman [21].

Theorem 1.6. *Let (W, S) be a finite Coxeter system and $J \subseteq S$. Let C be a cyclic subgroup generated by a regular element. Let X be the set of cosets W/W_J, and $X(q) := W^J(q)$.*

Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

Special cases of this result related to maximal parabolic subgroups for some of the other reflection groups are discussed in Section 8. It is also observed there that the longest element w_0 in W is always a regular element, and thus specializing Theorem 1.6 to $C = \langle w_0 \rangle$ gives the first case-free proof for all finite Coxeter groups of a theorem of Eng [13].

Section 7 proves two curious instances of the cyclic sieving phenomenon (Theorem 7.1), one involving dissections of a convex polygon, the other involving noncrossing partitions.

Section 9 deals with finite field q-analogues of Theorem 1.1(b). These relate to recent results of Drudge [11], counting orbits of k-dimensional subspaces in a finite vector space under the action of a Singer cycle, which is a q-version of an n-cycle. This turns out to be closely related to the invariant theory of $GL_n(\mathbb{F}_q)$ and its parabolic subgroups, leading naturally to the cyclic sieving phenomenon for flags of subspaces and a newly defined (q,t)-multinomial coefficient.

Section 10 proves that these q-analogues counting cyclic orbits of subspaces are polynomials in q with integer coefficients, and conjectures that they have nonnegative coefficients. Some easy cases of this conjecture are discussed, generalizing a result due independently to Andrews [1] and Haiman [17]. In the process of our analysis, we abstract two general principles, one for proving polynomiality of a rational function, and one for proving polynomials have nonnegative coefficients. The latter comes from the methods of Andrews [1], and is applied to prove some old and new positivity results.

Contents

2. Phenomen-ology

This section treats general facts about the cyclic sieving phenomenon. We begin by proving the equivalence of conditions (i) and (ii) in the Definition-Proposition of the introduction, relating them to a third, representation-theoretic condition. We then show how this third condition leads to a representation theory lemma (Lemma 2.4) useful for proving instances of the cyclic sieving phenomenon. For the purpose of stating this third equivalent condition, we introduce some notation.

As in the introduction, let C be a cyclic group of order n acting on a finite set X, and fix an embedding $\omega : C \hookrightarrow \mathbb{C}^\times$. Let a_ℓ be the coefficient uniquely defined by the expansion

$$X(q) \equiv \sum_{\ell=0}^{n-1} a_\ell q^\ell \mod q^n - 1.$$

Introduce a graded $\mathbb{C}$-vector space

$$A_X := \bigoplus_{i \geq 0} A_{X,i}$$

having $\sum_{i\geq 0} \dim_{\mathbb{C}} A_{X,i} q^i = X(q)$, and affording a representation of C in which each $c \in C$ acts on the graded component $A_{X,i}$ by the scalar $\omega(c)^i$.

The following encompasses the Definition-Proposition from the introduction.

Proposition 2.1. *With the above notation, the following are equivalent phrasings of the cyclic sieving phenomenon:*
(i) *For every $c \in C$,*

$$[X(q)]_{q=\omega(c)} = |\{x \in X : c(x) = x\}|.$$

(ii) *The coefficient a_ℓ counts the number of C-orbits on X for which the stabilizer-order divides ℓ.*

(iii) *As representations of C, the permutation representation $\mathbb{C}[X]$ and A_X are isomorphic.*

Proof. The equivalence of conditions (i) and (iii) is clear, as the left side of (i) is the character value for c in the representation A_X, while the right side of (i) is the character value for c in $\mathbb{C}[X]$.

To prove the equivalence of (iii) and (ii), note that $\mathbb{C}[X]$ and A_X are isomorphic if and only if for each of the irreducible (degree one) representations $\{\rho^{(\ell)}\}_{\ell \in \mathbb{Z}/n\mathbb{Z}}$ of C, defined by $\rho^{(\ell)}(c) = \omega(c)^\ell$, one has equality of the intertwining numbers

$$\langle \rho^{(\ell)}, A_X \rangle_C = \langle \rho^{(\ell)}, \mathbb{C}[X] \rangle_C. \tag{2.1}$$

The coefficient a_ℓ of q^ℓ in $X(q) \mod q^n - 1$ has the following well-known expression in terms of n^{th} roots of unity:

$$a_\ell = \frac{1}{n} \sum_{\omega^n = 1} \omega^{-\ell} X(\omega). \tag{2.2}$$

Thus a_ℓ is exactly the left side of (2.1):

$$\langle \rho^{(\ell)}, A_X \rangle_C = \frac{1}{n} \sum_{c \in C} \omega(c)^{-\ell} X(\omega(c)) = a_\ell.$$

On the other hand, the right side of (2.1) can be computed using the fact that $\mathbb{C}[X]$ is a permutation representation of C which decomposes into a sum over C-orbits $\mathcal{O}$. Letting $C_{\mathcal{O}}$ denote the stabilizer subgroup within C of any element in the orbit $\mathcal{O}$, one has

$$\begin{aligned}
\langle \rho^{(\ell)}, \mathbb{C}[X]\rangle_C &= \sum_{C-\text{orbits } \mathcal{O}} \langle \rho^{(\ell)}, \mathrm{Ind}_{C_{\mathcal{O}}}^{C} \mathbf{1}\rangle_C \\
&= \sum_{C-\text{orbits } \mathcal{O}} \langle \mathrm{Res}_{C_{\mathcal{O}}}^{C} \rho^{(\ell)}, \mathbf{1}\rangle_{C_{\mathcal{O}}} \\
&= \sum_{C-\text{orbits } \mathcal{O}} \begin{Bmatrix} 1 & \text{if } |C_{\mathcal{O}}| \text{ divides } \ell \\ 0 & \text{otherwise.} \end{Bmatrix} \\
&= |\{C\text{-orbits on } X \text{ whose stabilizer-order divides } \ell\}|.
\end{aligned}$$

Thus the equality (2.1) holds exactly when condition (iii) holds. $\square$

Remark 2.2. The equivalence of conditions (i) and (ii) in Proposition 2.1 can also be proven easily via (number theoretic) Möbius inversion; see Proposition 4.1(ii).

Remark 2.3. It should be noted that condition (i) of the Definition-Proposition of the introduction immediately implies the following: if $(X, X(q), C)$ exhibits the cyclic sieving phenomenon, then $(X, X(q), C')$ exhibits the cyclic sieving phenomenon for any subgroup C' of C.

Condition (iii) in Proposition 2.1 leads to the following *representation theory paradigm*, generalizing a paradigm for proving $q = -1$ phenomena followed in [42, 43]. Given a cyclic group C acting (faithfully) on the set $[N]$, one obtains an embedding of C in $GL_N(\mathbb{C})$. Given a rational representation $\rho : GL_N(\mathbb{C}) \to GL(V)$, recall that its *character* $\chi_\rho(x_1, \dots, x_N)$ is defined to be the trace on V of any diagonalizable element in $GL_N(\mathbb{C})$ having eigenvalues $x_1, \dots, x_N$.

Lemma 2.4. *Let C be a cyclic group acting nearly freely on $[N]$. Let $\rho : GL_N(\mathbb{C}) \to GL(V)$ be a representation. Assume that there exists an integer m, and that V has a basis $\{v_x\}_{x \in X}$ which is permuted (up to scalars) by C in the following way:*

$$c(v_x) = \omega(c)^m \, v_{c(x)} \quad \textit{for all } c \in C, x \in X. \tag{2.3}$$

Then the induced C-action on X and the (twisted) principal specialization

$$X(q) := q^{-m} \chi_\rho(1, q, q^2, \dots, q^{N-1})$$

give rise to a triple $(X, X(q), C)$ that exhibits the cyclic sieving phenomenon.

Proof. First note that when a cyclic subgroup C of order n acts nearly freely on $[N]$, the eigenvalues of any c in C are given by the multiset

$$1, \omega(c), \omega(c)^2, \ldots, \omega(c)^{N-1}.$$

Consequently, the character value of c acting in the representation $\rho^{(-m)} \otimes \rho$ will be

$$\chi_{\rho^{(-m)}}(c) \cdot \chi_\rho(1, \omega(c), \omega(c)^2, \ldots, \omega(c)^{N-1}) = [X(q)]_{q=\omega(c)}.$$

In other words, $\rho^{(-m)} \otimes \rho \cong A_X$ as C-modules, as they have the same characters. On the other hand, the hypothesis (2.3) asserts an isomorphism $\rho^{(-m)} \otimes \rho \cong \mathbb{C}[X]$ of C-modules, so the result follows from condition (iii) of Proposition 2.1. ☐

Remark 2.5. The hypothesis of Lemma 2.4 can be rephrased. Whenever the C-action permutes basis elements $\{v_x\}_{x \in X}$ up to scalars, the representation will decompose into a direct sum indexed by the C-orbits on X. Each summand is a representation induced from a cyclic subgroup C_d, where $C_d = \langle c^{\frac{n}{d}} \rangle$ is the stabilizer within C of an orbit representative x. There will exist an integer m_x such that

$$c^{\frac{n}{d}}(v_x) = \omega(c^{\frac{n}{d}})^{m_x} v_x.$$

The hypothesis in (2.3) simply asserts that these integers m_x all coincide with a single integer m.

3. First proof of Theorem 1.1: representation theory

The goal of this section is to give our first proof of Theorem 1.1, using the representation theory paradigm, Lemma 2.4.

First proof of Theorem 1.1. Let $U = \mathbb{C}^N$ with standard basis $\{e_i\}_{i=1}^N$, affording the defining representation of $GL_N(\mathbb{C})$.

For part (a), let V be the k^{th} symmetric power $\mathrm{Sym}^k(U)$. This has an obvious basis of symmetric tensors indexed by k-multisubsets of $[N]$. As this basis is permuted by the cyclic group C, Lemma 2.4 applies with $m = 0$ to

$$X(q) = \chi_{\mathrm{Sym}^k(V)}(1, q, q^2, \ldots, q^{N-1}) = \begin{bmatrix} N+k-1 \\ k \end{bmatrix}_q,$$

completing the proof.

For part (b), let V be the k^{th} exterior power $\bigwedge^k(U)$, having an obvious basis of antisymmetric tensors $v_S := \bigwedge_{i \in S} e_i$ indexed by k-subsets S of $[N]$. We claim that the C-action on these basis elements

v_S obeys the hypotheses of Lemma 2.4 with $m = \binom{k}{2}$. Once this is established, one can apply the lemma to

$$X(q) = q^{-\binom{k}{2}} \chi_{\bigwedge^k(V)}(1, q, q^2, \ldots, q^{N-1}) = \begin{bmatrix} N \\ k \end{bmatrix}_q,$$

completing the proof.

To verify this claim, in light of Remark 2.5, it remains to show that whenever S is a k-subset whose C-stabilizer is the cyclic group C_d, one has

$$c^{\frac{n}{d}}(v_S) = \omega(c^{\frac{n}{d}})^{\binom{k}{2}}\ v_S. \tag{3.1}$$

To check (3.1), uniquely decompose the subset $S = \sqcup_{i=0}^{b} S_i$ where $S_1, \ldots, S_b$ are the (non-empty) intersections of S with various n-cycles of a generator c for C in its action on $[N]$, and S_0 is either empty or possibly the singleton cycle of c. Note that either $k = b \cdot d$ or $b \cdot d + 1$. Then $c^{\frac{n}{d}}$ acts on v_S by the scalar $(-1)^{b(d-1)}$. Letting $\omega := \omega(c)$ be a primitive n^{th} root of unity, it then only remains to show that

$$\omega^{\frac{n}{d} \cdot \binom{k}{2}} = (-1)^{b(d-1)}.$$

Note that

$$\omega^{\frac{n}{d} \cdot \binom{k}{2}} = \omega^{\frac{n}{d} \cdot \binom{bd}{2}}$$

since $\binom{k}{2}$ is equal either to $\binom{bd}{2}$ or $\binom{bd}{2} + b \cdot d$, and $(\omega^{\frac{n}{d}})^{b \cdot d} = 1$. The result then follows from comparing the coefficient of x^0 on the two sides of the identity

$$\prod_{i=0}^{bd-1} \left(x - \omega^{\frac{n}{d} i}\right) = (x^d - 1)^b.$$

□

4. Second proof of Theorem 1.1: explicit formulae

One can sometimes verify directly that the cyclic sieving phenomenon holds for a triple $(X, X(q), C)$, by checking that both sides of condition (i) agree with an explicit formula. We use this method to give a second proof of Theorem 1.1.

Note that having done this, one gains explicit formulae for various other orbit-counts associated with the C-action on X. We collect some of these in Proposition 4.1 below, after recalling the *Ramanujan sum* [20, Th. 271, Th. 272]:

$$c_d(\ell) := \sum_{\substack{\omega \text{ a primitive} \\ d^{th} \text{ root of } 1}} \omega^{\ell} = \sum_{s | d, \ell} \mu\left(\frac{d}{s}\right) s, \tag{4.1}$$

where μ is the number-theoretic Möbius function. For example,

$$c_d(\ell) := \begin{cases} \phi(d) & \text{if } \gcd(d,\ell) = d, \\ \mu(d) & \text{if } \gcd(d,\ell) = 1, \end{cases}$$

where ϕ is the classical Euler-phi function.

The proof of the following proposition is a straightforward exercise in (number-theoretic) Möbius inversion, which we omit.

Proposition 4.1. *Given a cyclic group C of order n acting on a finite set X, let*

$$\beta(d) := |\{x \in X : x \text{ is fixed by at least the subgroup } C_d \text{ of } C\}|.$$

Then the number of C-orbits

(i) *in total is*
$$\frac{1}{n}\sum_{d|n} \phi(d)\beta(d).$$

(ii) *whose stabilizer-order divides ℓ is*
$$\frac{1}{n}\sum_{d|n} c_d(\ell)\beta(d).$$

(iii) *of size e, where $n = de$, is*
$$\frac{1}{e}\sum_{s:\, d|s|n} \mu\left(\frac{s}{d}\right)\beta(s).$$

The second proof of Theorem 1.1 requires some explicit evaluations of the q-binomial coefficient at roots for unity. All can be routinely deduced from setting q to a root of unity in a finite version of the q-binomial theorem; see [16, Exercise 1.2(vi)] or [36, Exercise 3.45(a,b)]. Some can also be found in the references[1] cited in the following proposition.

(**NB:** In the proposition below, and throughout the paper, a binomial coefficient containing any non-integer rational arguments is defined to be 0.)

Proposition 4.2. *Let n, k, a and d be positive integers, with $d|n$. Let ω be a primitive d^{th} root of unity. Then*

(i) *(see* [12, 15, 25, 45]*)*
$$\begin{bmatrix} an+k-1 \\ k \end{bmatrix}_{q=\omega} = \binom{\frac{an}{d} + \frac{k}{d} - 1}{\frac{k}{d}},$$

[1]The authors thanks Matthias Beck for providing some of the references via [5].

(ii)
$$\begin{bmatrix} an \\ k \end{bmatrix}_{q=\omega} = \binom{\frac{an}{d}}{\frac{k}{d}},$$

(iii) *(see* [5, Th. 3]*)*
$$\begin{bmatrix} an+k \\ k \end{bmatrix}_{q=\omega} = \binom{\frac{an}{d} + \lfloor \frac{k}{d} \rfloor}{\lfloor \frac{k}{d} \rfloor},$$

(iv)
$$\begin{bmatrix} an+1 \\ k \end{bmatrix}_{q=\omega} = \binom{\frac{an}{d}}{\frac{k}{d}} + \binom{\frac{an}{d}}{\frac{k-1}{d}}.$$

Second proof of Theorem 1.1. Consider first the case where C acts freely on $[N]$ for $N = an$. Then C acts also on the collection Ω of all k-multisets of $[N]$. Note that if $c \in C$ has multiplicative order d, then it fixes exactly
$$\beta(d) = \binom{\frac{an}{d} + \frac{k}{d} - 1}{\frac{k}{d}}$$
k-multisubsets of $[N]$; those fixed by c come from a $\frac{k}{d}$-multiset of $\left[\frac{an}{d}\right]$ by replacing occurrences of a single i with the set of d values in the same cycle of c as i. Comparing this with Proposition 4.2(i) verifies condition (i) of the Definition-Proposition from the introduction.

The other cases of the theorem are proven analogously; in each case condition (i) of the Definition-Proposition is easily verified since $\beta(d)$ can be computed explicitly, and one then compares this with Proposition 4.2(ii), (iii), or (iv). $\square$

Although not necessary for what follows, it is interesting to see how Proposition 4.2(i) and (ii) generalize to principal specializations of Schur functions. The generalization involves the notion of *t-cores* and *t-quotients* of a partition λ [24, Ex. I.1.8], [37, Exercise 7.59]. In a special case, it is implicit in [23, p. 128]; see also [24, Ex. I.3.17, Ex. I.5.24, Ex. I.7.8].

Theorem 4.3. *Let $t|N$, and let q be a primitive t^{th} root of unity. Then $s_\lambda(1, q, \cdots, q^{N-1})$ is zero unless the t-core of λ is empty, in which case*
$$s_\lambda(1, q, \cdots, q^{N-1}) = \operatorname{sgn}(\chi^\lambda(t^k)) \prod_{i=0}^{t-1} s_{\lambda^{(i)}}(\underbrace{1, 1, \cdots, 1}_{\frac{N}{t}}),$$
where $k = \frac{|\lambda|}{t}$, the t-quotient of λ is $(\lambda^{(0)}, \lambda^{(1)}, \cdots, \lambda^{(t-1)})$, and χ^λ is the irreducible character of the symmetric group $\mathfrak{S}_{kt}$ indexed by λ.

Sketch of proof. The Schur function may be expanded in terms of the power sum symmetric functions p_ρ using characters of the symmetric group [24, §I.7]

$$s_\lambda(x) = \sum_\rho \frac{\chi^\lambda(\rho)}{z_\rho} p_\rho(x). \tag{4.2}$$

Choosing $x = (1, q, \cdots, q^{N-1})$, where q is a primitive t^{th} root of unity gives $p_j(x) = 0$ unless $t|j$, in which case $p_j = N$. Thus in (4.2) we may assume that t divides each part of ρ. By the Murnaghan-Nakayama rule [24, Ex. I.7.4], if λ has non-empty t-core, then $\chi^\lambda(\rho) = 0$, and thus $s_\lambda(x) = 0$. If λ does have empty t-core note that (4.2) implies the polynomial expansion

$$s_\lambda(1, q, \cdots, q^{N-1}) = N^k \frac{\chi^\lambda(t^k)}{t^k k!} + O(N^{k-1}). \tag{4.3}$$

We compare this with a different polynomial expansion in N. Suppose that λ has t-quotient $(\lambda^{(0)}, \lambda^{(1)}, \cdots, \lambda^{(t-1)})$ and empty t-core. By the hook-content formula [37, p. 374]

$$s_\lambda(1, q, \cdots, q^{N-1}) = q^{n(\lambda)} \prod_{x \in \lambda} \frac{1 - q^{N+c(x)}}{1 - q^{h(x)}}. \tag{4.4}$$

The hooks $h(x)$ in λ which are divisible by t are t times the hooks of the t-quotients [24, Ex. I.1.8(d)], as are the contents.

Consequently

$$s_\lambda(1, q, \cdots, q^{N-1}) = R \cdot \prod_{i=0}^{t-1} s_{\lambda^{(i)}}(\underbrace{1, 1, \cdots, 1}_{\frac{N}{t}})$$

where

$$R := q^{n(\lambda)} \prod_{\substack{x \in \lambda \\ c(x) \not\equiv 0 \mod t}} (1 - q^{c(x)}) / \prod_{\substack{x \in \lambda \\ h(x) \not\equiv 0 \mod t}} (1 - q^{h(x)})$$

is a quantity independent of N. Comparing this with (4.3) gives

$$R = \chi^\lambda(t^k) \ \frac{\prod_{i=0}^{t-1} \prod_{x \in \lambda^{(i)}} h(x)}{k!}.$$

From the results of [40], one has

$$\chi^\lambda(t^k) = \mathrm{sgn}(\chi^\lambda(t^k)) \ \frac{k!}{\prod_{i=0}^{t-1} \prod_{x \in \lambda^{(i)}} h(x)}.$$

Thus $R = \mathrm{sgn}(\chi^\lambda(t^k))$. $\square$

The deduction of Proposition 4.2(i) from Theorem 4.3 with $\lambda = k$ is immediate. To deduce Proposition 4.2(ii) from the case where $\lambda = 1^k$ one needs

$$\operatorname{sgn}\left(\chi^{1^k}(t^{k/t})\right) = (-1)^{\frac{(t-1)k}{t}}.$$

We remark that Proposition 4.2(ii) generalizes easily to

$$\begin{bmatrix} an \\ k_1, \cdots, k_m \end{bmatrix}_{q=\omega} = \begin{pmatrix} \frac{an}{d} \\ \frac{k_1}{d}, \cdots, \frac{k_m}{d} \end{pmatrix}. \tag{4.5}$$

when ω is a primitive d^{th} root of unity and $d|n$. This then leads to a similar direct proof of the following proposition. When C acts on $[N]$, it induces an action on $\mathbf{k}$-*flags of subsets*

$$\emptyset \subset S^{k_1} \subset S^{k_1+k_2} \subset \cdots \subset S^{k_1+\cdots+k_{m-1}} \subset S^{k_1+\cdots+k_{m-1}+k_m} = [N]$$

where $|S^k| = k$.

Proposition 4.4. *Let $C = C_n$ act freely on $[an]$. Let X be the set of all $\mathbf{k}$-flags of subsets with the induced C-action, and*

$$X(q) = \begin{bmatrix} an \\ k_1, \cdots, k_m \end{bmatrix}_q.$$

Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

The previous proposition is the special case of Theorem 1.6 where W is a type A Weyl group $W(A_{an-1}) = \mathfrak{S}_{an}$. It turns out that Theorem 1.6 shows that the conclusion of Proposition 4.4 also holds when C acts only nearly freely but not freely; see the discussion following Theorem 8.2.

A finite field analogue of Proposition 4.4 will be given in Theorem 9.4 below.

Remark 4.5. In Proposition 4.2(i),(ii), if $\gcd(n, k) = 1$, then the coefficients modulo $q^n - 1$ in the sieved sums are constant (see the discussion following Conjecture 10.3 below). Necessary and sufficient conditions for a general q-binomial coefficient $\begin{bmatrix} N \\ k \end{bmatrix}_q$ to have constant coefficients mod $q^m - 1$ have been given in [46], where a bijection between these classes is also given. An analogous bijection for the *major index* statistic on permutations [10] has been given in [4].

5. Cyclically equivalent ordered trees

We return to Chapoton's observation (Theorem 1.4) from the introduction, and strengthen it in Corollary 5.1 below.

An *ordered tree* [41, §3.1] can be defined recursively as a root vertex r, connected by edges to the roots of a linearly ordered sequence $T_1, \ldots, T_m$ of ordered trees. We think of such trees as embedded in the plane, in which the linear order on $T_1, \ldots, T_m$ is their order from left to right. We will consider the set of ordered trees with n non-root vertices, which is well-known to have cardinality $\frac{1}{n+1}\binom{2n}{n}$, the Catalan number.

Given an ordered tree $(r, T_1, \ldots, T_m)$, we will say that it is *cyclically equivalent* to the ordered tree $(r, T_2, T_3, \ldots, T_m, T_1)$. (Note that we do not allow cyclic rotations about non-root vertices.) One can also speak of the symmetry group of such an equivalence class, namely the group of cyclic rotations of subtrees of the root which stabilizes any representative tree from the class; this will be a cyclic group whose order divides m.

Corollary 5.1. *The coefficient of q^ℓ in*

$$\begin{bmatrix} 2n-1 \\ n \end{bmatrix}_q \mod q^n - 1$$

is the number of cyclic equivalence classes of ordered trees with n non-root vertices whose symmetry group has order dividing ℓ.

For example, if $n = 3$, we have

$$\begin{bmatrix} 5 \\ 3 \end{bmatrix}_q \equiv 4 + 3q + 3q^2 \mod q^3 - 1.$$

The q^0 coefficient of 4 above reflects the fact that among the 5 ordered trees with 3 non-root vertices, there are 4 cyclic equivalence classes, as two of the trees are cyclically equivalent. Furthermore, exactly one of these classes has a non-trivial C_3 symmetry group.

Proof. We describe an encoding of ordered trees with n non-root vertices by n-multisubsets of $[n]$. Given such an ordered tree, record a depth-first search by writing down a word w with n A's and n B's, recording moves along an edge away from (resp. toward) the root with an A (resp. B). Then let the multisubset of $[n]$ have k_i occurrences of the letter i if there are k_i occurrences of the letter B between the i^{th} and $(i+1)^{st}$ occurrences of A.

It is easy to see that given an n-multisubset of $[n]$, the corresponding word w encodes such a depth-first search of an ordered tree if and only if w is a *ballot sequence*, that is, it has at least as many A's as B's in any initial segment. Furthermore, the ballot sequences corresponding to n-multisubsets of $[n]$ in the same C_n-orbit correspond to cyclic rotations of the subtrees of the root.

Consequently, the result follows from Theorem 1.1(a) with $k = n$ and $N = 2n$. $\square$

6. The phenomenon in Pólya-Redfield theory

The goal of this section is to prove Theorem 6.1, an instance of the cyclic sieving phenomenon in Pólya-Redfield theory that generalizes Theorem 1.1(a). The proof is a quick application of Lemma 2.4, generalizing the first proof of Theorem 1.1(a). We begin with some review of Pólya-Redfield theory; see e.g. [37, §7.24].

Let G be a group of permutations acting on a finite set S. The *Pólya-Redfield cycle indicator* for this action is the symmetric function

$$P_G(x_1, x_2, \ldots) = \frac{1}{|G|} \sum_{g \in G} p_{\lambda(g)}$$

where $\lambda(g)$ denotes the cycle type of g as it permutes S, and p_λ is the *power-sum* symmetric function corresponding to λ.

The action of G on S induces an action on the set $[N]^S$ of N-colorings of S. Letting $X = [N]^S/G$ denote the set of G-orbits of N-colorings, the cycle indicator has the following interpretation as the *pattern inventory* for these G-orbits:

$$P_G(x_1, x_2, \ldots, x_N) = \sum_{G-\text{orbits } \mathcal{O}} x^{\mathcal{O}} \tag{6.1}$$

where $x^{\mathcal{O}} := \prod_{i=1}^N x_i^{m_i}$ if m_i denotes the number of occurrences of the color i in any representative of the orbit $\mathcal{O}$.

Theorem 6.1. *With notation as above, assume C is a cyclic group acting nearly freely on the color set $[N]$, and define*

$$X(q) := P_G(1, q, q^2, \ldots, q^{N-1}).$$

Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

Note that when $|S| = k$ and G is the full symmetric group S_k, this theorem specializes to Theorem 1.1(a).

Proof. As in the proof of Theorem 1.1(a), let $U := \mathbb{C}^N$ afford the defining representation of $GL_N(\mathbb{C})$. Consider the $|S|$-fold tensor power $U^{\otimes|S|}$, with G acting on the tensor positions and $GL_N(\mathbb{C})$ acting diagonally. Let $V = \left(U^{\otimes|S|}\right)^G$ be the G-invariant subspace for this action, which still affords a representation ρ of $GL_N(\mathbb{C})$. Then V has an obvious basis $e_{\mathcal{O}}$ of G-symmetrized basic tensors, indexed by the set of $X = [N]^S/G$ of G-orbits $\mathcal{O}$ of N-colorings. Since such a basis element $e_{\mathcal{O}}$ is a $GL_N(\mathbb{C})$-weight vector of weight $x^{\mathcal{O}}$, equation (6.1) implies ρ has character $\chi_\rho = P_G$. Since the cyclic group C permutes this basis, Lemma 2.4 with $m = 0$ implies the result. $\square$

The $N = 2$ special case of the previous theorem is an interesting result of de Bruijn.

Corollary 6.2. *(de Bruijn [9]) Let G be a group permuting a finite set S, and $X := 2^S/G$ the set of G-orbits on the subsets of S, Let $X(q)$ be the generating function counting these G-orbits according to cardinality of any representative subset.*

Then $X(-1)$ counts the self-complementary G-orbits of subsets.

Proof. Identify a 2-coloring of S with the subset that receives the second color. Then $P_G(1, q) = X(q)$ as defined in the corollary. Letting the cyclic group C of order 2 swap the colors amounts to swapping a subset $S' \subset S$ for its complement $S - S'$. $\square$

Note that this corollary reduces to Theorem 1.5 from the introduction when $S = \binom{[n]}{2}$ is the set of potential edges in a graph on vertex set $[n]$, with S carrying an action of $G = \mathfrak{S}_n$ induced from the action on the vertex set $[n]$. Here G-orbits of subsets of edges are identified with unlabelled graphs on n vertices.

Remark 6.3. Corollary 6.2 has a flavor very similar to that of Stembridge's original examples [42]. In fact, the special case of Corollary 6.2 where $S = [ab]$ and $G = \mathfrak{S}_a \wr \mathfrak{S}_b$ coincides with the (easy and well-known) $c = 1$ special case of Stembridge's observation concerning plane partitions inside an $a \times b \times c$ box [42, §2.1].

7. The phenomenon in polygons: two instances

In this section we observe two instances of the cyclic sieving phenomenon involving convex polygons with cyclic actions by rotation; one for dissections of the polygon, the other for noncrossing partitions.

7.1. **Polygon dissections.** Consider dissections of a convex n-gon using k noncrossing diagonals. The formula for the number of such dissections has a long history; see [39]. It is given by

$$\begin{aligned} f(n,k) &= \frac{1}{n+k}\binom{n+k}{k+1}\binom{n-3}{k} \\ &= \frac{(n+k-1)!}{k!(k+1)!(n-k-3)!(n-1)(n-2)}. \end{aligned}$$

It was noted by K. O'Hara and A. Zelevinsky that this last expression coincides with the number of standard Young tableaux of the shape $(k+1)^2\,1^{n-k-3}$, using the celebrated hook formula of Frame, Robinson and Thrall [37, Corollary 7.21.6].

A natural q-analogue of $f(n,k)$ comes from replacing integers and factorials by their q-analogues

$$\begin{aligned} [n]_q &:= 1+q+q^2+\cdots+q^{n-1}, \\ [n]!_q &:= [n]_q[n-1]_q\cdots[2]_q[1]_q. \end{aligned}$$

One can then define

$$\begin{aligned} f(n,k;q) &:= \frac{1}{[n+k]_q}\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q \begin{bmatrix} n-3 \\ k \end{bmatrix}_q \\ &= \frac{[n+k-1]!_q}{[k]!_q[k+1]!_q[n-k-3]!_q[n-1]_q[n-2]_q} \qquad (7.1) \\ &= q^{-\binom{n-k-1}{2}-k} \sum_{\substack{\text{standard Young tableaux } T \\ \text{of shape } (k+1)^2\,1^{n-k-3}}} q^{\mathrm{maj}(T)} \end{aligned}$$

where the statistic $\mathrm{maj}(T)$ is the *major index* of T, defined to be the sum of values i for which $i+1$ appears in a lower row of T than i. The last equality in (7.1) is a special case of a q-hook formula of Stanley [37, Corollary 7.21.5].

Theorem 7.1. *Let X be the set of dissections of a convex n-gon using k noncrossing diagonals. Let the cyclic group C of order n act on X by cyclic rotations of the n-gon. Let $X(q) := f(n,k;q)$.*

Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

Proof. We verify by direct calculation condition (i) of the Definition-Proposition from the introduction. Starting with the left side of condition (i), one must evaluate $f(n,k;q)$ at $q=\omega$, where ω is a primitive d^{th} root-of-unity for some divisor d of n. To do this, one examines the numerator and denominator factors in (7.1), and uses these facts:

- The factor $[m]_q$ has a (simple) zero at $q=\omega$ if and only if $d|m$, $d \neq 1$, and
- when $m_1 \equiv m_2 \mod d$,
$$\lim_{q \to \omega} \frac{[m_1]_q}{[m_2]_q} = \begin{cases} \frac{m_1}{m_2} & \text{if } m_1 \equiv m_2 \equiv 0 \mod d \\ 1 & \text{if } m_1 \equiv m_2 \not\equiv 0 \mod d. \end{cases}$$

The following computation is then a straightforward exercise (albeit somewhat tedious). For $d \geq 2$ a divisor of n,

$$[f(n,k;q)]_{q=\omega} = \begin{cases} \frac{\lfloor \frac{n+k-1}{2} \rfloor !}{\lfloor \frac{n-k-3}{2} \rfloor ! \lfloor \frac{k}{2} \rfloor ! \lfloor \frac{k+1}{2} \rfloor ! \frac{n-2}{2}} & \text{if } d=2, \\ \frac{\left(\frac{n+k}{d}-1\right)!}{\left(\frac{n-k}{d}-1\right)!\left(\frac{k}{d}\right)!^2} & \text{if } d \geq 3 \text{ and } d|k, \\ 0 & \text{otherwise.} \end{cases} \tag{7.2}$$

For the right side of condition (i), we must compute the number of dissections of the n-gon using k diagonals which are invariant under d-fold rotation. For this we utilize a result of Simion [33, Proposition 1] counting the number $a_{n,p}$ of subdivisions of an n-gon, with n even, which are centrally symmetric, using p *antipodal pairs* of diagonals (and where a diameter of the n-gon is counted as *one* antipodal pair):

$$a_{n,p} = \binom{\frac{n}{2}-1}{p}\binom{\frac{n}{2}+p-1}{p}. \tag{7.3}$$

As one might expect from the formulae in (7.2), the calculation will proceed in two cases, $d=2$ and $d \geq 3$, with the case $d=2$ broken into two subcases depending upon the parity of the number k of diagonals.

Case 1(a): $d=2$ and k odd. In this case, a centrally symmetric subdivision with k diagonals will contain a unique diameter. This diameter can be chosen in $\frac{n}{2}$ ways, and the rest of the subdivision is completely determined by the subdivision of the $(\frac{n}{2}+1)$-gon using $\frac{k-1}{2}$ diagonals on either side of the diameter. Hence the number of such subdivisions is
$$\frac{n}{2} \cdot f\left(\frac{n}{2}+1, \frac{k-1}{2}\right) = \frac{\frac{n+k-1}{2}!}{\frac{n-k-3}{2}!\frac{k-1}{2}!\frac{k+1}{2}!\frac{n-2}{2}},$$
which agrees with (7.2) for $d=2$ and k odd.

Case 1(b): $d=2$ and k even. Here the set of subdivisions counted by $a_{n,\frac{k}{2}}$ in (7.3) contains not only the subdivisions that we wish to count, but also those which have $k-1$ diagonals; since $k-1$ is odd, the latter can be enumerated as in Case 1(a). Hence the desired number of subdivisions with k diagonals is

$$\begin{aligned}
&a_{n,\frac{k}{2}} - \frac{n}{2}\cdot f\left(\frac{n}{2}+1, \frac{k-2}{2}\right) \\
&= \binom{\frac{n}{2}-1}{\frac{k}{2}}\binom{\frac{n}{2}+\frac{k}{2}-1}{\frac{k}{2}} - \frac{\frac{n+k-2}{2}!}{\left(\frac{n-k}{2}-1\right)!\left(\frac{k}{2}-1\right)!\frac{k}{2}!\frac{n-2}{2}} \\
&= \frac{\frac{n+k-2}{2}!}{\frac{n-k-4}{2}!\left(\frac{k}{2}!\right)^2\frac{n-2}{2}},
\end{aligned}$$

which agrees with (7.2) for $d = 2$ and k even.

Case 2: $d \geq 3$. In the case $d \geq 3$, it is easily seen that any diagonal in a subdivision with d-fold rotational symmetry lies in a free orbit under this action of C_d. Consequently k must be divisible by d, in agreement with (7.2).

When d divides k, the d-fold rotationally symmetric subdivisions using k diagonals decompose into two sets (similar in spirit to the cases of k odd and k even for $d = 2$): there are those for which the central polygon in the subdivision is a d-gon, and those for which it is not. In the former set, one can choose this central d-gon in $\frac{n}{d}$ ways, and then the rest is completely determined by the subdivision of the $(\frac{n}{d}+1)$-gon using $\frac{k}{d}-1$ diagonals that lies in each $\frac{2\pi}{d}$-sector outside an edge of the central d-gon. Such subdivisions are therefore counted by $\frac{n}{d}\cdot f\left(\frac{n}{d}+1, \frac{k}{d}-1\right)$. In the latter set, one can employ an obvious bijection (described in the proof of [29, Proposition 4.2]) between subdivisions with d-fold rotational symmetry and those with central symmetry. This bijects the collection of d-fold symmetric subdivisions of an n-gon using k diagonals and no central d-gon to the collection of centrally symmetric subdivisions of a $\frac{2n}{d}$-gon using $\frac{2k}{d}$ diagonals. The latter were counted in Case 1(b). Totalling the cardinalities of these two sets gives

$$\begin{aligned}
&\frac{n}{d}\cdot f\left(\frac{n}{d}+1, \frac{k}{d}-1\right) + \frac{\frac{2n/d+2k/d-2}{2}!}{\left(\frac{2n/d-2k/d-4}{2}\right)!\left(\frac{2k/d}{2}!\right)^2\frac{2n/d-2}{2}} \\
&= \frac{\left(\frac{n+k}{d}-1\right)!}{\left(\frac{n-k}{d}-1\right)!\frac{k}{d}!\left(\frac{k}{d}-1\right)!\left(\frac{n}{d}-1\right)} + \frac{\left(\frac{n+k}{d}-1\right)!}{\left(\frac{n-k}{d}-2\right)!\left(\frac{k}{d}!\right)^2\left(\frac{n}{d}-1\right)} \\
&= \frac{\left(\frac{n+k}{d}-1\right)!}{\left(\frac{n-k}{d}-1\right)!\left(\frac{k}{d}\right)!^2},
\end{aligned}$$

in agreement with (7.2) for $d \geq 3$ and d dividing k. ∎

7.2. **Noncrossing partitions.** Kreweras defined noncrossing partitions in 1972, and they have been intensively studied since; see Simion [32] for a nice survey. To define them, number the vertices of a convex n-gon by $1, 2, \ldots, n$ in a circular order. A partition of $[n]$ is *noncrossing* if its blocks correspond to subsets of these vertices whose convex hulls are pairwise disjoint. Ordering the noncrossing partitions by refinement gives a ranked lattice called $NC(n)$, in which a noncrossing partition having b blocks has rank $n-b$. The cardinality $|NC(n)|$ is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$, and the number of elements at rank k in $NC(n)$ is the *Narayana number* [37, Exer. 6.36]:

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

A natural q-analogue of the Narayana number, introduced by Fürlinger and Hofbauer [14], is given by

$$N(n,k;q) := \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q q^{k(k+1)}. \tag{7.4}$$

This has many combinatorial interpretations; see Brändén [3].

Theorem 7.2. *Let X be the set of noncrossing partitions in $NC(n)$ with rank k. Let the cyclic group C of order n act on X by cyclic rotations of the n-gon. Let $X(q) := N(n,k;q)$.*

Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

Proof. We proceed via direct calculation as in the proof of Theorem 7.1, starting with the left side of condition (i) in the Definition-Proposition from the introduction. A straightforward analysis of numerator and denominator factors in $N(n,k;q)$ similarly shows that for $d \geq 2$ dividing n,

$$[N(n,k;q)]_{q=\omega} = \begin{cases} \frac{n-k}{n}\binom{\frac{n}{d}}{\frac{k}{d}}^2 & \text{if } d|k, \\ \\ \frac{k+1}{n}\binom{\frac{n}{d}}{\frac{k+1}{d}}^2 & \text{if } d|k+1, \\ \\ 0 & \text{otherwise.} \end{cases} \tag{7.5}$$

For the right side of condition (i), we must compute the number of elements of $NC(n)$ having rank k which are invariant under d-fold rotation. Note that when a noncrossing partition is d-fold rotationally symmetric, the cyclic group C_d acts freely on all blocks except possibly for the central block (if present). Thus the number of blocks b is congruent either to 0 or 1 modulo d. Since $d|n$ and $b = n-k$, this means that there will be no such partitions unless $d|k$ or $d|k+1$, in agreement with (7.5).

This leaves the two cases where $d|k$ or $d|k+1$. Here we will utilize a special case of a result of Athanasiadis and the first author [2, Lemma 4.4]: there are

- $\binom{N}{p}^2$ centrally symmetric non-crossing partitions of $[2N]$ that have p antipodal pairs of blocks (where a central block fixed under the antipodal map does *not* count as a pair),

and among these,

- the fraction which have a central block is $\frac{N-p}{N}$, so
- the fraction with no central block is $\frac{p}{N}$.

Case 1: $d|k$, so that $b \equiv 0 \mod d$. Here there is an obvious bijection [27, Proposition 1] to centrally symmetric noncrossing partitions of $[\frac{2n}{d}]$ having $\frac{2b}{d}$ blocks. The latter will have $\frac{b}{d}$ antipodal pairs of blocks and no central block, and so are counted by the following formula with $N = \frac{n}{d}, p = \frac{b}{d}$:

$$\frac{p}{N}\binom{N}{p}^2 = \frac{b/d}{n/d}\binom{\frac{n}{d}}{\frac{b}{d}}^2 = \frac{n-k}{n}\binom{\frac{n}{d}}{\frac{k}{d}}^2.$$

This agrees with (7.5) for $d|k$.

Case 2: $d|k+1$, so that $b \equiv 1 \mod d$. As in Case 1, there is an obvious bijection to centrally symmetric noncrossing partitions of $[\frac{2n}{d}]$ having $\frac{2(b-1)}{d}+1$ blocks. The latter will have $\frac{b-1}{d}$ antipodal pairs of non-central blocks and one central block, and so are counted by the following formula with $N = \frac{n}{d}, p = \frac{b-1}{d}$:

$$\frac{N-p}{N}\binom{N}{p}^2 = \frac{\frac{n}{d}-\frac{b-1}{d}}{\frac{n}{d}}\binom{\frac{n}{d}}{\frac{b-1}{d}}^2 = \frac{k+1}{n}\binom{\frac{n}{d}}{\frac{k+1}{d}}^2.$$

This agrees with (7.5) for $d|k+1$. □

We remark that Theorems 7.1 and 7.2 are somewhat mysterious; it would be desirable to have a more illuminating or unified proof. It is perhaps worth mentioning that

- in both cases, the sets X are in bijection with lattice paths obeying various restrictions, and $X(q)$ is a q-count by a *major index* statistic,
- in both cases, the cardinality $|X|$ has a hook or hook-content formula, and $X(q)$ has a q-hook or q-hook-content formula; see Brändén [3, Corollary 8],

- the cardinalities $|X| = f(n,k)$ and $|X| = N(n,k)$ give, respectively, the entries in the f-vector of the $(n-3)$-dimensional associahedron and the h-vector of the $(n-1)$-dimensional associahedron; see Simion [32].

8. Springer's theorem on regular elements

In this section, we show how Theorem 1.1(b) generalizes to reflection groups.

Recall that an element of $GL_N(\mathbb{C})$ is a *(pseudo-)reflection* if it has finite order and fixes pointwise a (unique) hyperplane in $\mathbb{C}^N$, called its *reflecting hyperplane.* A *finite reflection group* is a subgroup W of $GL_N(\mathbb{C})$ generated by reflections. The theory of these groups is quite rich, including a classification by Shephard and Todd [34] and many results on their polynomials invariants. We introduce some terminology for the sake of stating a beautiful theorem of Springer [35] on regular elements, reformulated[2] below in the fashion of Kraśkiewicz and Weyman [21].

Say that an element c in a finite reflection group W is a *regular element* if it has an eigenvector which does not lie on any of the reflecting hyperplanes for reflections in W. Let A denote the *co-invariant algebra* for W, that is, the quotient of the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ by the ideal generated the W-invariant polynomials of positive degree. Given a regular element c of order n, let $C = \langle c \rangle$ be the cyclic group that it generates, and ω a primitive n^{th} root of unity. We will regard A as a representation of $W \times C$ by having

- W act in the usual way on A (induced on the quotient from substitutions of variables), and
- C act by $c(x_i) = \omega x_i$.

We will also regard the group algebra $\mathbb{C}[W]$ as a representation of $W \times C$ by having W act by left-multiplication and C act by right-multiplication.

Theorem 8.1. *(Springer* [35, Prop. 4.5]*, cf.* [21]*) The coinvariant algebra A and group algebra $\mathbb{C}[W]$ are isomorphic as representations of $W \times C$.*

[2]The result of Kraśkiewicz and Weyman [21] was stated and proved for the classical *Weyl* groups of types A, B, C, D. The authors thank A. Shepler for helping them to understand why Springer's result implies that of [21] for reflection groups in general. They also thank F. Lübeck for checking via computer that the result of [21] holds for the exceptional irreducible finite reflection groups, before they had realized that this follows from Springer's result!

Given a graded $\mathbb{C}$-algebra $R = \bigoplus_{d\geq 0} R_d$, define its *Hilbert series*

$$\mathrm{Hilb}(R,q) := \sum_{d\geq 0} \dim_{\mathbb{C}} R_d \; q^d.$$

The following is the main result of this section, and generalizes Theorem 1.6 from the introduction.

Theorem 8.2. *Let c be a regular element of order n in a finite reflection group W. Let $C = \langle c \rangle$, and let W' be any subgroup of W. Let $A^{W'}$ be the W'-invariant subalgebra of the coinvariant algebra A.*

Then setting $X := W/W'$ and $X(q) = \mathrm{Hilb}(A^{W'}, q)$, one obtains a triple $(X, X(q), C)$ that exhibits the cyclic sieving phenomenon.

To deduce Theorem 1.6 from this theorem, one needs the following fact [19, §IV.4]. For (W, S) a finite Coxeter system (i.e. a finite *Euclidean* reflection group) and any $J \subseteq S$, the minimum length coset representatives W^J for the parabolic subgroup W_J satisfy

$$W^J(q) := \sum_{w\in W^J} q^{\ell(w)} = \mathrm{Hilb}(A^{W_J}, q).$$

Proof of Theorem 8.2. Since Theorem 8.1 tells us that A and $\mathbb{C}[W]$ are isomorphic as representations of $W \times C$, restricting to their W'-invariant subspaces $A^{W'}$ and $\mathbb{C}[W]^{W'}$ gives isomorphic representations of C. By definition of the C-action on $A^{W'}$ and the way we have defined $X(q)$, one has that $A^{W'}$ coincides with the C-representation A_X defined in Proposition 2.1. Meanwhile, $\mathbb{C}[W]^{W'}$ is easily identified with the permutation representation on cosets $\mathbb{C}[W/W'] \cong \mathbb{C}[X]$. Thus the result follows from Proposition 2.1 (iii). $\square$

Theorem 8.2 motivated the search for a generalization of Springer's theory. The general linear group $GL_n(\mathbb{F}_q)$ leads to a q-analogue of the case of the symmetric group, which was studied in [30] and discussed in Section 9 below. This was generalized further to a version of Springer's theory valid over arbitrary fields in [31], which includes a generalization of Theorem 8.2.

As mentioned in the introduction, Theorem 1.6 also gives the first case-free proof for all finite Coxeter groups of the following instance of the $q = -1$ phenomenon. It was originally proven via the classification of irreducible finite Coxeter systems by Eng [13], and later given a case-free proof for Weyl groups in [28].

Theorem 8.3. *Let (W, S) be a finite Coxeter system, and $J \subseteq S$. Let w_0 be the longest element in W. Then*

$$\left[W^J(q)\right]_{q=-1} = |\{\textit{cosets } wW_J : w_0 wW_J = wW_J\}|.$$

In other words, setting $X := W/W_J$, $X(q) := W^J(q)$, and $C = \langle w_0 \rangle$ gives a triple $(X, X(q), C)$ exhibiting the cyclic sieving phenomenon.

The theorem follows immediately from Theorem 1.6 and the following observation.

Lemma 8.4. *For any finite Coxeter system (W, S), the longest element w_0 is a regular element in the sense of Springer.*

Proof. Let V be the natural reflection representation for (W, S), and V^+, V^- the $+1, -1$ eigenspaces for w_0 acting on V. Note that V^+, V^- are orthogonal complements of one another in V, because w_0 acts as an orthogonal involution.

We will show that V^- contains a regular vector, that is, one which lies on no reflecting hyperplanes. Consider a typical reflecting hyperplane H, corresponding to some positive root α. Since $w_0(\alpha)$ is a negative root, we know that α is not fixed by w_0, so $\alpha \notin V^+ = (V^-)^\perp$. Thus $V^- \not\subset \alpha^\perp = H$, and hence $H \cap V^-$ is a codimension one subspace inside V^-. As H ranges over all of the (finitely many) reflecting hyperplanes, these codimension one subspaces cannot exhaust V^-. Hence their complement within V^- consists of regular vectors, all of which are (-1)-eigenvectors for w_0. $\square$

Springer conveniently classified all regular elements in the finite irreducible Coxeter groups [35, §5]. In type A, where $W(A_{N-1}) = \mathfrak{S}_N$, an element c is regular if and only if it acts nearly freely in the sense defined earlier. Thus Theorem 1.6 provides a third proof of Theorem 1.1(b), and also generalizes Proposition 4.4 to nearly free actions.

The Coxeter group $W(B_N)$ of type B is the *hyperoctahedral group*, that is, the subgroup of $GL_N(\mathbb{C})$ consisting of monomial matrices in which the non-zero entries are all ± 1. Springer's classification shows that the only regular elements in $W(B_N)$ are the powers of a Coxeter element. A Coxeter element in type B_N is conjugate to an element which cyclically permutes the N coordinates while introducing a sign change in one particular coordinate; call elements in this conjugacy class *negative N-cycles*. Say that an element conjugate in $W(B_N)$ to a cyclic permutation of the N coordinates with no sign changes is a *positive N-cycle*. For each divisor n of $2N$, there are regular elements in $W(B_N)$ of order n, having form that depends upon the parity of n:

- For n an odd divisor of N, say $N = an$, a regular element of order n consists of a positive cycles each of size n.

- For n an even divisor of $2N$, say $2N = an$, a regular element of order n consists of a negative cycles each of size $\frac{n}{2}$. (When $a = 1$, such elements are the Coxeter elements for $W(B_n)$.)

We will only consider here the *maximal* parabolic subgroups W_J of $W(B_N)$. These have the form

$$W(A_{N-k-1}) \times W(B_k) \text{ for } k = 0, 1, \ldots, N-1.$$

One can identify their minimum length coset representatives W^J with the set of $Q(N, k)$ all vectors in $\{+1, 0, -1\}^N \subset \mathbb{C}^N$ having exactly k zero coordinates (see e.g. [38]). The actions of the appropriate cyclic groups C_n generated by regular elements turn out to be simply the restriction of their actions on $\mathbb{C}^N$. It is easily seen that

$$W^J(q) = \begin{bmatrix} N \\ k \end{bmatrix}_q (-q^{k+1}; q)_{N-k}$$

where we are using the the notation

$$(x; q)_m := (1 - x)(1 - qx)(1 - q^2 x) \cdots (1 - q^{m-1} x).$$

We immediately conclude the following from Theorem 1.6.

Corollary 8.5. *Let n be a divisor of $2N$, and C the cyclic subgroup of $W(B_N)$ generated by a regular element of order n. Let $X := Q(N, k)$, and*

$$X(q) := \begin{bmatrix} N \\ k \end{bmatrix}_q (-q^{k+1}; q)_{N-k}.$$

Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

The previous discussion generalizes easily from $W(B_N)$ to the reflection groups $W = C_m \wr \mathfrak{S}_N$, consisting of the monomial matrices in $GL_N(\mathbb{C})$ whose non-zero entries are m^{th} roots of unity. Although these are not Coxeter groups, they do have distinguished sets of generators S coming from the fact that they are *Shephard groups*, that is the symmetry groups of regular complex polytopes [8].

Let W_J be a "maximal parabolic subgroup", that is, $J = S - \{s\}$ for some $s \in S$. This W_J will be isomorphic to $W(A_{N-k-1}) \times (C_m \wr \mathfrak{S}_k)$ for some $k = 0, 1, \ldots, N-1$. One can check directly that the subring of W_J-invariants A^{W_J} in the coinvariant algebra has Hilbert series

$$\text{Hilb}(A^{W_J}, q) = \begin{bmatrix} N \\ k \end{bmatrix}_q \frac{(q^{(k+1)m}; q^m)_{N-k}}{(q^{k+1}; q)_{N-k}}.$$

The role analogous to that of a *Coxeter element* is played here by an element c which cyclically permutes the N coordinates while introducing a scaling by $\zeta := e^{\frac{2\pi i}{m}}$ in one particular coordinate. It is easy to check that such an element c is regular of order mN, and hence any power of it is also regular. Thus for each divisor n of mN, there will be a cyclic subgroup C_n in $C_m \wr \mathfrak{S}_N$ generated by the regular element $c^{\frac{mN}{n}}$; although one could describe the cycle structure of these powers $c^{\frac{mN}{n}}$, as in type B_N, we omit these descriptions here.

One can check that the cosets W/W_J biject with the set $Q^{(m)}(N,k)$ of vectors in $\mathbb{C}^N$ having $N-k$ coordinates which are m^{th} roots of unity, and k zero coordinates. One can also check that the C_n-action on cosets W/W_J coincides with the restriction of the action on $\mathbb{C}^N$ to the vectors in $Q^{(m)}(N,k)$.

Corollary 8.6. *Let n be a divisor of mN and $C = C_n$ the cyclic subgroup of $C_m \wr \mathfrak{S}_N$ generated by $c^{\frac{mN}{n}}$. Let $X := Q^{(m)}(N,k)$, and*

$$X(q) := \begin{bmatrix} N \\ k \end{bmatrix}_q \frac{(q^{(k+1)m}; q^m)_{N-k}}{(q^{k+1}; q)_{N-k}}.$$

Then $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.

As an example of Corollary 8.6 take

$$N = 4, \ m = 3, \ k = 0, \ n = 12,$$

so $|Q^{(3)}(4,0)| = 3^4 = 81$. Altering the notation slightly, one can identify the alphabet $\{1, \zeta, \zeta^2\}$ where $\zeta = e^{\frac{2\pi i}{3}}$ with the alphabet $\{0,1,2\} = \mathbb{Z}/3\mathbb{Z}$. Then $Q^{(3)}(4,0)$ consists of words of length 4 in the alphabet $\mathbb{Z}/3\mathbb{Z}$, with the generator c of C_{12} acting as follows:

$$c(w_1, w_2, w_3, w_4) = (w_4 + 1, w_1, w_2, w_3) \mod 3.$$

It turns out that there are 6 free orbits of C_{12} on $Q^{(3)}(4,0)$, one orbit with stabilizer-order 2, namely $\{2200, 1220, 1122, 0112, 0011, 2001\}$, and one orbit with stabilizer-order 4, namely $\{2102, 0210, 1021\}$. This is reflected in the coefficients of the sieved polynomial

$$\begin{aligned}\frac{(q^3;q^3)_4}{(q;q)_4} &\equiv 8 + 6q + 7q^2 + 6q^3 + 8q^4 + 6q^5 \\ &\quad + 7q^6 + 6q^7 + 8q^8 + 6q^9 + 7q^{10} + 6q^{11} \mod q^{12} - 1.\end{aligned}$$

9. Counting cyclic orbits over finite fields

In this section we propose q-analogues of some of the preceding formulae, involving orbit-counting of cyclic groups acting on (flags of) subspaces in finite vector spaces. We hope that the reader will forgive our re-use of the variable q in these new polynomials; since they have finite field interpretations when q is a prime power, we adhere to convention.

We first introduce some terminology for the finite field q-analogues of cyclic group actions. Let q be a prime power and let $\mathbb{F}_q$ denote the field with q elements. Consider the tower of extensions

$$\mathbb{F}_q \subset \mathbb{F}_{q^n} \subset \mathbb{F}_{q^{an}}$$

for positive integers a, n. The field extension $\mathbb{F}_{q^{an}}$ is in particular an $\mathbb{F}_q$-vector space, in which the multiplicative group $\mathbb{F}_{q^n}^\times$ of the subfield $\mathbb{F}_{q^n}$ acts $\mathbb{F}_q$-linearly. Consequently, $\mathbb{F}_{q^n}^\times$ acts on the set of all $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^{an}}$. In this action, the subgroup $\mathbb{F}_q^\times$ acts trivially, so it descends to an action of the quotient group $\mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$.

Note that $\mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$ is a cyclic group, since it is a quotient of the multiplicative group $\mathbb{F}_{q^n}^\times$ (which is well-known to be cyclic). When $a = 1$, the cyclic permutation in $\mathfrak{S}_{\mathbb{F}_{q^n}^\times}$ corresponding to multiplication by a cyclic generator of $\mathbb{F}_{q^n}^\times$ is called a *Singer cycle* (see e.g. [11]). Also note that the cardinality of $\mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$ is $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$, suggesting that $\mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$ is a natural q-analogue of the cyclic group C_n.

This analogy is tightened further by the following observation: just as every subgroup of C_n is of the form C_d for some $d|n$, the only subgroups of $\mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$ which can occur as the stabilizer of an $\mathbb{F}_q$-subspace V in $\mathbb{F}_{q^{an}}$ are those of the form $\mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times$ for some $d|n$. This is because the set $\{\alpha \in \mathbb{F}_{q^n} : \alpha V \subset V\}$ is easily seen to form a subfield of $\mathbb{F}_{q^n}$, and every such subfield is of the form $\mathbb{F}_{q^d}$ for $d|n$. Note that the size of such a $\mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$-orbit in which the stabilizer is $\mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times$ will be

$$\frac{[n]_q}{[d]_q} = [e]_{q^d} \qquad \text{where } n = de.$$

Bearing this in mind, the following finite field analogue of Proposition 4.1 is also a straightforward exercise in Möbius inversion. Define q-analogues of the Euler-phi function and Ramanujan sum by

$$\phi_q(m) := \sum_{d|m} \mu\left(\frac{m}{d}\right)[d]_q$$

$$c_d(\ell; q) := \sum_{s|d,\ell} \mu\left(\frac{d}{s}\right)[s]_q.$$

Proposition 9.1. *Let $C = \mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$ act on a finite set X of subspaces or flags of subspaces in $\mathbb{F}_{q^n}$, and let*

$$\beta_q(d) := |\{x \in X : x \text{ is fixed by at least the subgroup } \mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times \text{ of } C\}|.$$

Then the number of C-orbits

(i) *in total is*

$$\frac{1}{[n]_q}\sum_{d|n}\phi_q(d)\beta_q(d).$$

(ii) *whose stabilizer-order divides* $[\ell]_q$ *is*

$$\frac{1}{[n]_q}\sum_{d|n}c_d(\ell;q)\beta_q(d).$$

(iii) *of size* $[e]_{q^d}$, *where* $n = de$, *is*

$$\frac{1}{[e]_{q^d}}\sum_{s:\, d|s|n}\mu\left(\frac{s}{d}\right)\beta_q(s).$$

We now consider the case where X is the set of all $\mathbf{k}$-*flags* of $\mathbb{F}_q$-subspaces in $\mathbb{F}_{q^{an}}$, for some composition $\mathbf{k} = (k_1, \ldots, k_m)$ of an:

$$0 \subset V^{k_1} \subset V^{k_1+k_2} \subset \cdots \subset V^{k_1+\cdots+k_{m-1}} \subset V^{k_1+\cdots+k_{m-1}+k_m} = \mathbb{F}_{q^{an}}$$

where $\dim_{\mathbb{F}_q} V^k = k$. Note that such a $\mathbf{k}$-flag is stabilized by the subgroup $\mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times$ if and only if each $\mathbb{F}_q$-subspace in the flag is actually an $\mathbb{F}_{q^d}$-subspace. One concludes that in this situation

$$\beta_q(d) = \begin{bmatrix} \frac{an}{d} \\ \frac{k_1}{d}, \ldots, \frac{k_m}{d} \end{bmatrix}_{q^d}. \tag{9.1}$$

From this and Proposition 9.1, we immediately deduce the following generalization of a recent result of Drudge [11], who proved the special case where $a = 1, m = 2$.

Proposition 9.2. *(cf.* [11, Theorem 2.1]*) When* $\mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$ *acts on the set of all* $\mathbf{k}$*-flags of* $\mathbb{F}_q$*-subspaces in* $\mathbb{F}_{q^{an}}$*, the total number of* $\mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$*-orbits is*

$$O^{a,n,\mathbf{k}}(q) := \frac{1}{[n]_q}\sum_{d|\gcd(n,\mathbf{k})}\phi_q(d)\begin{bmatrix} \frac{an}{d} \\ \frac{k_1}{d}, \ldots, \frac{k_m}{d} \end{bmatrix}_{q^d} \tag{9.2}$$

and the number of orbits of size $[e]_{q^d}$ *for* $n = de$ *is*

$$O_d^{a,n,\mathbf{k}}(q) := \frac{1}{[e]_{q^d}}\sum_{s:\, d|s|n}\mu\left(\frac{s}{d}\right)\begin{bmatrix} \frac{an}{s} \\ \frac{k_1}{s}, \ldots, \frac{k_m}{s} \end{bmatrix}_{q^s}. \tag{9.3}$$

The relevant cyclic sieving phenomenon that accompanies Proposition 9.2 requires the definition of an appropriate generating function[3] $X(t)$ for the set X of all $\mathbf{k}$-flags in $\mathbb{F}_{q^{an}}$.

[3]The variable t is used in $X(t)$ here rather than our previous variable q because the variable q is needed in its traditional role as the order of the field $\mathbb{F}_q$.

Definition 9.3. Given a composition $\mathbf{k} = (k_1, \ldots, k_m)$ of N, let

$$\sigma_s := k_1 + k_2 + \cdots + k_s \quad \text{for } 1 \leq s \leq m,$$

and define the (q,t)*-multinomial coefficient*

$$\begin{bmatrix} N \\ k_1, \ldots, k_m \end{bmatrix}_{q,t} := \frac{\prod_{i=1}^{N}(1 - t^{q^N - q^{N-i}})}{\prod_{s=1}^{m} \prod_{i=1}^{k_s}(1 - t^{q^{\sigma_s} - q^{\sigma_s - i}})}.$$

Although not obvious at the moment, we will see that the (q,t)-multinomial coefficient is a polynomial in t with nonnegative integer coefficients, whose degree is a polynomial in q depending upon $\mathbf{k}$. Also note that

$$\lim_{t \to 1} \begin{bmatrix} N \\ k_1, \ldots, k_m \end{bmatrix}_{q,t} = \begin{bmatrix} N \\ k_1, \ldots, k_m \end{bmatrix}_{q}.$$

The (q,t)-multinomial with $N = an$ is the appropriate choice to obtain the cyclic sieving phenomenon; see Theorem 9.4 below. The motivation for this choice was guided by the proof of Theorem 1.6, and results from invariant theory, as we now explain; we refer the reader to [30] for more details.

Let $G = GL_N(\mathbb{F}_q)$, which acts transitively on $\mathbf{k}$-flags of $\mathbb{F}_q$-subspaces in $\mathbb{F}_{q^N}$. If P denotes the parabolic subgroup which fixes some particular $\mathbf{k}$-flag, then $X := G/P$ is identified with the set of $\mathbf{k}$-flags. Letting $S = \mathbb{F}_{q^n}[x_1, \ldots, x_N]$, a theorem of Hewett [18] describes the P-invariant polynomials S^P as a polynomial subalgebra of S with homogeneous generators of degree $q^{\sigma_s} - q^{\sigma_s - i}$ for $1 \leq s \leq m, 1 \leq i \leq k_s$. His result generalizes a well-known theorem of Dickson asserting that the G-invariant polynomials S^G form a polynomial subalgebra with homogeneous generators of degree $q^N - q^{N-i}$ for $1 \leq i \leq N$. A little commutative algebra of Cohen-Macaulay rings then implies that S^P is a free module over the polynomial subalgebra S^G. Consequently, the quotient ring $S^P/(S^G_+)$, where (S^G_+) denotes the ideal in S^P generated by the G-invariants S^G_+ of positive degree, has Hilbert series

$$\mathrm{Hilb}(S^P/(S^G_+), t) = \frac{\mathrm{Hilb}(S^P, t)}{\mathrm{Hilb}(S^G, t)} = \begin{bmatrix} N \\ k_1, \ldots, k_m \end{bmatrix}_{q,t}.$$

In [30], it is directly verified that if $c \in \mathbb{F}_{q^n}^{\times}$ generates the subfield $\mathbb{F}_q(c) = \mathbb{F}_{q^d}$, and ω is a complex root of unity with the same multiplicative order as c, then

$$\begin{bmatrix} N \\ k_1, \ldots, k_m \end{bmatrix}_{q,t=\omega} = \begin{bmatrix} \frac{N}{d} \\ \frac{k_1}{d}, \ldots, \frac{k_m}{d} \end{bmatrix}_{q^d}. \tag{9.4}$$

This can be used to prove the following.

Theorem 9.4. *Let $C = \mathbb{F}_{q^n}^\times$ act on the set X of $\mathbf{k}$-flags of $\mathbb{F}_q$-subspaces in $\mathbb{F}_{q^{an}}$. Let $X(t) = \begin{bmatrix} an \\ k_1, \ldots, k_m \end{bmatrix}_{q,t}$.*

Then $(X, X(t), C)$ exhibits the cyclic sieving phenomenon.

Proof. Let $c \in C = \mathbb{F}_{q^n}^\times$ generate the extension $\mathbb{F}_q(c) = \mathbb{F}_{q^d} \subset \mathbb{F}_{q^n}$. Then the $\mathbf{k}$-flags stabilized by c are exactly those counted by $\beta_q(d)$ in (9.1) above. Since the root of unity $\omega(c)$ has the same multiplicative order as c, comparing $\beta_q(d)$ with (9.4) directly verifies condition (i) of the Definition-Proposition from the introduction. ☐

10. Polynomiality, nonnegativity, and a conjecture

In this section we examine the rational functions $O_d^{a,n,\mathbf{k}}(q)$ from Proposition 9.2 that count orbits of $\mathbf{k}$-flags in $\mathbb{F}_{q^{an}}$. We observe that they are actually polynomials in q with integer coefficients, and conjecture that these coefficients are nonnegative. We then discuss the easy special case where $\gcd(n, \mathbf{k}) = 1$, which is related to results of Haiman and Andrews.

Our main tool in this section is Proposition 10.1 below. Parts (i),(ii) give obvious criteria for a rational function in $\mathbb{Q}(q)$ to be a polynomial in $\mathbb{Q}[q]$ or $\mathbb{Z}[q]$. Part (iii) abstracts an argument implicit in Andrews [1], giving a simple but useful criterion for a polynomial in $\mathbb{Z}[q]$ to lie in $\mathbb{N}[q]$, that is, to have nonnegative coefficients.

Proposition 10.1. *Let $f(q) \in \mathbb{Q}(q)$.*

(i) *If $f(N) \in \mathbb{Z}$ for infinitely many $N \in \mathbb{Z}$, then $f(q) \in \mathbb{Q}[q]$.*

(ii) *If furthermore $f(q) = \frac{a(q)}{b(q)}$ where $a(q), b(q) \in \mathbb{Z}[q]$ and $b(q)$ is monic, then $f(q) \in \mathbb{Z}[q]$.*

(iii) *If furthermore $f(q) = \frac{a(q)}{[n]_q}$ for some positive integer n, and $a(q) \in \mathbb{N}[q]$ has symmetric, unimodal coefficient sequence, then $f(q) \in \mathbb{N}[q]$, and $f(q)$ has symmetric coefficient sequence.*

Proof. For (i), express $f(q) = \frac{a(q)}{b(q)}$ with $a(q), b(q) \in \mathbb{Q}[q]$. Divide $b(q)$ into $a(q)$ with quotient $c(q)$ and remainder $r(q)$. If $D \in \mathbb{Z}$ is a common denominator for the coefficients of $c(q)$, then

$$Df(q) = Dc(q) + D\frac{r(q)}{b(q)}.$$

where $Dc(q) \in \mathbb{Z}[q]$. Since $Dc(N) \in \mathbb{Z}$ for *any* integer N, we conclude that $D\frac{r(N)}{b(N)} \in \mathbb{Z}$ for infinitely many integers N. As r has lower degree than b, one will have $\left|D\frac{r(N)}{b(N)}\right| < 1$ for $|N|$ large, and hence $D\frac{r(N)}{b(N)} = 0$ for infinitely many N. Hence $r(q) = 0$, so $f(q) = c(q) \in \mathbb{Q}[q]$.

For (ii), note that when $a(q), b(q) \in \mathbb{Z}[q]$ and $b(q)$ is monic, the quotient $c(q)$ also lies in $\mathbb{Z}[q]$.

For (iii), since the polynomial $f(q) = \frac{a(q)}{[n]_q}$ is a quotient of two polynomials with symmetric coefficients, it will also have symmetric coefficients. One has

$$\deg(f) = \deg(a) - n + 1 \leq \deg(a),$$

and hence by the symmetry of $f(q)$, it suffices to show that the coefficient of q^k in $f(q)$ is nonnegative for $0 \leq k \leq \deg(a)/2$. Rewrite

$$f(q) = (1-q)\,a(q)\,\cdot\,\frac{1}{1-q^n}.$$

Since $a(q)$ is symmetric and unimodal, the coefficient of q^k in $(1-q)\,a(q)$ is nonnegative for $0 \leq k \leq \deg(a)/2$. Since $\frac{1}{1-q^n} \in \mathbb{N}[[q]]$, the coefficient of q^k in $f(q)$ is also nonnegative for $0 \leq k \leq \deg(a)/2$, as desired. $\square$

Corollary 10.2. *For any composition* $\mathbf{k}$ *of* an*,*

$$O^{a,n,\mathbf{k}}(q),\;\; O_d^{a,n,\mathbf{k}}(q) \;\in \mathbb{Z}[q].$$

Proof. Apply Proposition 10.1 (i),(ii) to Theorem 9.2. $\square$

Conjecture 10.3. *For any composition* $\mathbf{k}$ *of* an*,*

$$O^{a,n,\mathbf{k}}(q),\;\; O_d^{a,n,\mathbf{k}}(q) \;\in \mathbb{N}[q].$$

Note that since

$$O^{a,n,\mathbf{k}}(q) = \sum_{d|n} O_d^{a,n,\mathbf{k}}(q), \qquad \text{and}$$

$$O_d^{a,n,\mathbf{k}}(q) = O_e^{a,e,\frac{\mathbf{k}}{d}}(q^d) \qquad \text{if } n = de,$$

this conjecture immediately reduces to the case $d = 1$, where $O_1^{a,n,\mathbf{k}}(q)$ counts *free* orbits.

Things simplify greatly, and Conjecture 10.3 is easy, when the action of $\mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$ is free. As in the discussion of Section 9, this will happen if and only if $\gcd(n, \mathbf{k}) = 1$: an element $c \in \mathbb{F}_{q^n}^\times$ that generates the subfield $\mathbb{F}_q(c) = \mathbb{F}_{q^d}$ for $d|n$ will stabilize a $\mathbf{k}$-flag if and only if it is actually a $\frac{\mathbf{k}}{d}$-flag of $\mathbb{F}_{q^d}$-subspaces, requiring $d|k_1, \ldots, k_m, n$. So if $\gcd(n, \mathbf{k}) = 1$ this forces c to lie in $\mathbb{F}_q^\times$.

Thus when $\gcd(n, \mathbf{k}) = 1$, the $\mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$-action is free, and every $\mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$-orbit of $\mathbf{k}$-flags has cardinality $[n]_q$. The number of $\mathbb{F}_{q^n}^\times/\mathbb{F}_q^\times$-orbits of $\mathbf{k}$-flags of $\mathbb{F}_q$-subspaces in $\mathbb{F}_{q^{an}}$ is then

$$\frac{1}{[n]_q}\begin{bmatrix} an \\ k_1, \cdots, k_m \end{bmatrix}_q. \tag{10.1}$$

Since q-multinomial coefficients are known (see e.g. [38]) to have non-negative, symmetric, and unimodal coefficient sequence (in q), Proposition 10.1(iii) immediately implies the following.

Corollary 10.4. *When* $\gcd(n, \mathbf{k}) = 1$,

$$O^{a,n,\mathbf{k}}(q) = O_1^{a,n,\mathbf{k}}(q) = \frac{1}{[n]_q}\begin{bmatrix} an \\ k_1, \cdots, k_m \end{bmatrix}_q \in \mathbb{N}[q]$$

and has symmetric coefficient sequence.

As $O_d^{a,n,\mathbf{k}}(q)$ is non-zero only for $d = 1$ when $\gcd(n, \mathbf{k}) = 1$, this verifies Conjecture 10.3 in this case.

Question 10.5. What is a combinatorial interpretation for the non-negative integer coefficients in (10.1) when $\gcd(n, \mathbf{k}) = 1$?

For $m = 2$ (i.e. the case of k-subspaces rather than $\mathbf{k}$-flags), and $a = 1$, Corollary 10.4 becomes the assertion

$$\frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \in \mathbb{N}[q] \ \text{ if } \gcd(n, k) = 1. \tag{10.2}$$

This was proven by Andrews [1], where he implicitly introduced Proposition 10.1(iii). It also turns out to be a special case of the following result of Haiman, whose original proof is somewhat tricky. We deduce it here by Andrews' method.

Theorem 10.6. *(*[17, Prop. 2.5.1, 2.5.2, 2.5.3]*) When* $\gcd(n, |\lambda|) = 1$,

$$\frac{1}{[n]_q} s_\lambda(1, q, \cdots, q^{n-1}) \in \mathbb{N}[q].$$

Proof. Nonnegativity, symmetry and unimodality of the coefficients of $s_\lambda(1, q, \cdots, q^{n-1})$ are well-known [24, Ex. I.8.4]. Polynomiality of the quotient is not hard: it is [17, Prop. 2.5.1] or can be established using n-cores as in the proof of Theorem 4.3, or as in [24, Ex. I.3.17(a)]). Now apply Proposition 10.1(iii). ☐

One can apply the same technique to generalize Corollary 10.4 to generating functions of the form $\frac{W^J(q)}{[n]_q}$. Recall that $W^J(q)$ is the length generating function for minimal length coset representatives of a parabolic subgroup W_J in a finite Coxeter group W. Stanley [38] showed that when W is a *Weyl group*, $W^J(q)$ always has symmetric, unimodal coefficient sequence. Thus Proposition 10.1(iii) applies to give the following.

Corollary 10.7. *Let (W, S) be a finite Weyl group, $J \subset S$, and n a positive integer.*

If $\frac{W^J(q)}{[n]_q} \in \mathbb{Z}[q]$, then it is actually in $\mathbb{N}[q]$.

In particular, for the classical Weyl groups of types B, D and their maximal parabolic subgroups W_J, one can check the conditions for polynomiality of $\frac{W^J(q)}{[n]_q}$ case-by-case. One deduces the following.

Type B_n: If $0 \le k \le n-1$ and $\gcd(n,k)$ is a power of two, then

$$\frac{1}{[2n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q (-q^{k+1}; q)_{n-k} \ \in \mathbb{N}[q].$$

Type D_n: If $2 \le k \le n-2$ and both $\gcd(n-1,k)$ and $\gcd(n-1,k-1)$ are powers of two, then

$$\frac{1}{[2(n-1)]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q (-q^{k}; q)_{n-k} \ \in \mathbb{N}[q].$$

We close with some remarks.

Remark 10.8. An explicit statistic for the C_n-orbits on k-subsets of $[n]$ with $\gcd(n,k) = 1$ can be given to explain the nonnegativity of coefficients in (10.2) for $k = 2, 3$ or 4. We explain here such a statistic for $k = 4$ and comment on $k = 2, 3$ below.

Denote by (a, b, c, d) on a circle the spacing between the elements in the 4-subset of $[n]$ when considered circularly modulo n. Then one may always rotate the circle to assume that $a \le c$ and $b < d$ (note that $n - 4 = a + b + c + d$ must be odd, since $\gcd(n, 4) = 1$), and this representation is unique. The statistic for this 4-tuple is $a + 2b + 3c$, that is, one can check that

$$\sum_{\substack{0 \le a \le c \\ 0 \le b < d \\ a+b+c+d=n-4}} q^{a+2b+3c} = \frac{1}{[n]_q} \begin{bmatrix} n \\ 4 \end{bmatrix}_q .$$

For $k = 3$, similarly denote by (a, b, c) the spacing between elements of the 3-subset of $[n]$ (so $a+b+c = n-3$). Choose C_n-orbit representatives that have (a, b, c) first in lexicographic order in their orbit. Then the statistic $a+2b$ can be checked to work. For $k=2$, the problem is trivial, since for n odd,

$$\frac{1}{[n]_q}\begin{bmatrix} n \\ 2 \end{bmatrix}_q = 1+q^2+q^4+\cdots+q^{n-3}.$$

Remark 10.9. Let a_ℓ be the coefficient of q^ℓ in $\begin{bmatrix} n \\ k \end{bmatrix}_q \mod q^n-1$. Combining Theorem 1.1(b) with Propositions 4.1(ii), 4.2(ii) tells us that

$$a_\ell = \frac{1}{n}\sum_{d|n} c_d(\ell)\binom{\frac{n}{d}}{\frac{k}{d}}. \tag{10.3}$$

Note that this may be rephrased

$$a_\ell = \left\langle h_{(n-k,k)}, L_n^{(\ell)} \right\rangle \tag{10.4}$$

where $h_{(n-k,k)}$ is the product $h_{n-k}h_k$ of complete homogeneous symmetric functions, and

$$L_n^{(\ell)} := \frac{1}{n}\sum_{d|n} c_d(\ell) p_d^{n/d}$$

is the Frobenius characteristic of a representation of $\mathfrak{S}_n$ induced from a character on an n-cycle [21], [22], [37, Exercise 7.88], generalizing the well-known *Lie character* when $\ell = 1$.

Equation (10.4) also follows from results of Désarménien on q-Kostka polynomials, as we now explain. He proved [10, Theorem 2.2] that the coefficient of q^ℓ in

$$q^{n(\lambda)-n(\lambda')}K_{\lambda,1^n}(q) \mod q^n-1$$

equals

$$\frac{1}{n}\sum_{d|n}\chi^\lambda(d^{n/d})c_d(\ell) = \left\langle s_\lambda, L_n^{(\ell)} \right\rangle.$$

Combining Désarménien's result with the (three) facts

$$\begin{bmatrix} n \\ j \end{bmatrix}_q - \begin{bmatrix} n \\ j-1 \end{bmatrix}_q = q^{n((n-j,j))-n((n-j,j)')}K_{(n-j,j),1^n}(q),$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{j=0}^{k}\left(\begin{bmatrix} n \\ j \end{bmatrix}_q - \begin{bmatrix} n \\ j-1 \end{bmatrix}_q\right),$$

$$h_{(n-k,k)} = \sum_{j=0}^{k} s_{(n-j,j)},$$

gives an alternate proof of (10.3).

The specialization $a = 1, m = 2, \ell = 0$ in (9.2) is a q-version of the right side of (10.3). Perhaps an appropriate q-generalization of $L_n^{(\ell)}$ can be used to resolve Conjecture 10.3?

Acknowledgments

The authors thank Jay Goldman, Jon McCammond, John Stembridge, and an anonymous referee for useful comments and suggestions.

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