LINEAR SYZYGIES OF STANLEY-REISNER IDEALS

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ABSTRACT. We give an elementary description of the maps in the linear strand of the minimal free resolution of a square-free monomial ideal, that is, the Stanley-Reisner ideal associated to a simplicial complex Δ. The description is in terms of the homology of the canonical Alexander dual complex Δ^*. As applications we are able to

- prove for monomial ideals and $j = 1$ a conjecture of J. Herzog giving lower bounds on the number of i-syzygies in the linear strand of j^{th}-syzygy modules.
- show that the maps in the linear strand can be written using only ± 1 coefficients if Δ^* is a pseudomanifold,
- exhibit an example where multigraded maps in the linear strand cannot be written using only ± 1 coefficients.
- compute the entire resolution explicitly when Δ^* is the complex of independent sets of a matroid.

1. INTRODUCTION

The goal of this paper is to describe in a simple topological fashion the maps in the linear strand of the minimal free resolution of a Stanley-Reisner ideal. It is an outgrowth of two recent trends in the theory of minimal free resolutions. The first is a series of results [1, 2, 4, 5, 7, 20] giving explicit descriptions of the maps in the minimal free resolutions for various classes of ideals. The second is the realization that when dealing with resolutions of Stanley-Reisner ideals I_Δ associated to a simplicial complex Δ, it can be easier to work with the *canonical Alexander dual* Δ^* (see [3, 8, 14, 22]). Our description of the linear strand for the resolution of I_Δ will be in terms of natural maps on the homology of links of faces in Δ^*.

We note that our description of the linear strand is in some sense not new, as it may be derived with a little work from known results [1, 2, 7, 9, 15]. However, the exact form in which we describe the linear strand seems not to appear elsewhere, and we give an elementary proof of its correctness here. This form is extremely useful for certain applications. In particular, in Section 4 we use it to prove for monomial ideals a special case of a conjecture of J. Herzog [13], asserting that when the linear strand of a homogeneous j^{th}-syzygy module contains p-syzygies for some $p > 0$ then it must contain at least $\binom{p+j}{i+j}$ i-syzygies for each $i < p$. In Section 5 we show that the maps in the linear strand can be written using only ± 1 coefficients whenever Δ^* is a pseudomanifold, and exhibit a small example of a non-pseudomanifold where this fails. Section 6 gives the entire resolution explicitly when Δ^* is the complex of independent sets of a matroid.

Key words and phrases. minimal free resolution, linear strand, Stanley-Reisner ring, square-free monomial ideal.

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2. Minimal free resolutions and notation

We first review minimal free resolutions and their linear strand. Let $A = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{F}$, and $I \subset A$ an ideal which is homogeneous with respect to the standard grading setting $\deg(x_i) = 1$. Regarding I as an A-module, it has a finite minimal free resolution

$$0 \longrightarrow \bigoplus_m A(-m)^{\beta_{p,m}} \xrightarrow{d_p} \cdots \xrightarrow{d_1} \bigoplus_m A(-m)^{\beta_{0,m}} \xrightarrow{d_0} I \to 0 \tag{2.1}$$

in which the notation $A(-m)$ denotes a free A-module with basis element in degree m, with the m's chosen to make the maps homogeneous. It is well-known that the number $\beta_{i,m}$ is the dimension of the m^{th} graded piece $\mathrm{Tor}_i^A(I, \mathbb{F})_m$ of the graded $\mathbb{F}$-vector space $\mathrm{Tor}_i^A(I, \mathbb{F})$. For an ideal I let $t(I)$ be the minimal degree of a homogeneous generator of I. Note that $\mathrm{Tor}_i^A(I, \mathbb{F})_m$ vanishes unless $\deg(m) \geq i + t(I)$.

From grading considerations, the map

$$d_i : \bigoplus_m A(-m)^{\beta_{i,m}} \to \bigoplus_m A(-m)^{\beta_{i-1,m}}$$

has a direct summand

$$d_i^{\mathsf{linear}} : A(-(i + t(I)))^{\beta_{i,i+t(I)}} \to A(-(i-1+t(I)))^{\beta_{i-1,i-1+t(I)}}.$$

We call the collection of maps $\{d_i^{\mathsf{linear}}\}_{i \geq 0}$ the *linear strand* of the resolution. The maps in the linear strand inherit a uniqueness property from the uniqueness of maps in the minimal free resolution. To be more precise, since the maps in the resolution are unique up to a simultaneous A-linear change of bases in the free modules that they map between, the maps in the linear strand are unique up to a simultaneous $\mathbb{F}$-linear change of bases in the lowest graded components of these free modules. We say that I *has linear (or $t(I)$-linear) resolution* if $d_i = d_i^{\mathsf{linear}}$ for all i, or equivalently, if $\mathrm{Tor}_i^A(I, \mathbb{F})_m = 0$ for $m \neq i + t(I)$.

When I is generated by square-free monomials, it is traditional to associate with it a certain simplicial complex Δ, for which $I = I_\Delta$ is the *Stanley-Reisner ideal* of Δ and A/I_Δ is the *Stanley-Reisner ring*. The definition of Δ as a simplicial complex on vertex set $[n] := \{1, 2, \ldots, n\}$ is straightforward: the minimal non-faces of Δ are defined to be the supports of the minimal square-free monomial generators of I. Since I respects the fine $\mathbb{N}^n$-grading by monomials on A, one can find an $\mathbb{N}^n$-graded minimal free resolution for I as an A-module, and define for any monomial $\mathbf{x}^\alpha$

$$\beta_{i,\mathbf{x}^\alpha} := \dim_{\mathbb{F}} \mathrm{Tor}_i^A(I, \mathbb{F})_{\mathbf{x}^\alpha}.$$

Instead of working with Δ (as in [16]), we will instead work with a certain canonical Alexander dual Δ^*,

$$\Delta^* := \{F \subseteq [n] : [n] - F \notin \Delta\}.$$

which in [14] we called the *Eagon complex* associated to I. Alternatively one can describe Δ^* in terms of the square-free monomial ideal as follows: The maximal faces (or *facets*) of Δ^* are the complements of the supports of the minimal square-free monomial generators of $I = I_\Delta$.

3. Description of the linear strand

In this section we describe the linear strand in the minimal free resolution of a Stanley-Reisner ideal I_Δ in the polynomial ring $A = \mathbb{F}[x_1, \ldots, x_n]$. The description will be in terms of the homology of links of faces in the Eagon complex Δ^*.

In [8], the following reformulation of a famous result of Hochster [16] was proved:

Theorem 1.

$$\mathrm{Tor}_i^A(I_\Delta, \mathbb{F})_{\mathbf{x}^\alpha} \cong \tilde{H}_{i-1}(\mathrm{link}_{\Delta^*} F; \mathbb{F})$$

if $\mathbf{x}^\alpha = \mathbf{x}^V := \prod_{i \in V} x_i$ *for some subset* V *of* $[n]$ *whose complement* $F = [n] - V$ *is a face of* Δ^**, and otherwise the above Tor-group vanishes.*

Here $\tilde{H}_\bullet(-;\mathbb{F})$ denotes reduced simplicial homology with coefficients in $\mathbb{F}$. Also the *link*, *star*, and *deletion* of a face F in a simplicial complex K are subcomplexes of K defined by

$$\begin{aligned} \mathrm{link}_K F &:= \{G \in K : G \cup F \in K, G \cap F = \varnothing\} \\ \mathrm{star}_K F &:= \{G \in K : G \cup F \in K\} \\ \mathrm{del}_K F &:= \{G \in K : G \cap F = \varnothing\}. \end{aligned}$$

We begin by noting a consequence of Theorem 1: Only *top*-dimensional homology of links of faces in Δ^* can contribute to the linear strand. To see this, note that the linear i-syzygies are measured by $\mathrm{Tor}_i^A(I_\Delta, \mathbb{F})_{i+t}$, where $t = t(I_\Delta)$ is the minimal degree of a generator of I_Δ. In the notation of Theorem 1, this implies

$$n - |F| = |V| = i + t = i + n - \dim(\Delta^*) - 1$$

or equivalently,

$$\begin{aligned} i - 1 &= \dim(\Delta^*) - |F| \\ &\geq \dim(\mathrm{link}_{\Delta^*} F). \end{aligned}$$

Therefore, the group $\tilde{H}_{i-1}(\mathrm{link}_{\Delta^*} F; \mathbb{F})$ appearing in Theorem 1 will vanish unless $i - 1 = \dim(\mathrm{link}_{\Delta^*} F)$, and in this case it is the top-dimensional homology of this link.

Thus to specify the $\mathbb{N}^n$-graded maps $d_\bullet^{\mathsf{linear}}$ in the linear strand, it suffices to specify for every face F in Δ^* and every vertex v in $[n]$ a map as follows:

$$\partial_{\mathrm{link}_{\Delta^*} F, v} : \tilde{H}_{i-1}(\mathrm{link}_{\Delta^*} F; \mathbb{F}) \to \tilde{H}_{i-2}(\mathrm{link}_{\Delta^*}(F \cup \{v\}); \mathbb{F})$$

where $i - 1 = \dim(\mathrm{link}_{\Delta^*} F)$ as before. Given such a family of maps, one can then postulate the following candidate for the $(\mathbf{x}^V, \mathbf{x}^{V-v})$-graded component of d_i^{linear}, that is, the component of the linear syzygies of multidegree $\mathbf{x}^V$ which are A-linear combinations of the syzygies of multidegree $\mathbf{x}^{V-v}$:

$$(d_i^{\mathsf{linear}})_{(\mathbf{x}^V, \mathbf{x}^{V-v})} : \begin{array}{ccc} A \otimes_{\mathbb{F}} \mathrm{Tor}_i^A(I_\Delta; \mathbb{F})_{\mathbf{x}^V} & \longrightarrow & A \otimes_{\mathbb{F}} \mathrm{Tor}_i^A(I_\Delta; \mathbb{F})_{\mathbf{x}^{V-v}} \\ f \otimes z & \longmapsto & x_v f \otimes \partial_{\mathrm{link}_{\Delta^*} F, v}(z) \end{array} \tag{3.1}$$

where z represents a cycle in $\tilde{H}_{i-1}(\mathrm{link}_{\Delta^*} F; \mathbb{F})$. Since

$$\mathrm{link}_{\Delta^*}(F \cup \{v\}) = \mathrm{link}_{\mathrm{link}_{\Delta^*} F}(v)$$

we can rename the complex $\mathrm{link}_{\Delta^*} F$ by the name K, and it suffices to define in general for any simplicial complex K and any of its vertices v a map:

$$\partial_{K,v} : \tilde{H}_{\dim K}(K; \mathbb{F}) \to \tilde{H}_{\dim K - 1}(\mathrm{link}_K v; \mathbb{F}).$$

Setting $D := \dim K$, a natural candidate for such a map $\partial_{K,v}$ is the connecting homomorphism in the Mayer-Vietoris exact sequence

$$0 \longrightarrow \tilde{H}_D(\mathrm{star}_K v) \oplus \tilde{H}_D(\mathrm{del}_K v) \longrightarrow \tilde{H}_D(K) \stackrel{\partial_{K,v}}{\longrightarrow} \tilde{H}_{D-1}(\mathrm{link}_K v) \to \cdots \tag{3.2}$$

arising from the decomposition

$$K = \mathrm{star}_K v \cup \mathrm{del}_K v$$
$$\mathrm{link}_K v = \mathrm{star}_K v \cap \mathrm{del}_K v.$$

Note that in the exact sequence (3.2) we have suppressed the field coefficients $\mathbb{F}$ for notational convenience, and we will continue to do so when it causes no confusion.

We point out for future use that if one is given an explicit representing cycle $z \in \tilde{H}_{\dim K}(K)$, then one can obtain $\partial_{K,v} z$ very explicitly in two different ways. On the one hand, from the definition of the maps in any Mayer-Vietoris sequence,

$$\partial_{K,v}(z) = \partial\left(z|_{\mathrm{star}_K v}\right)$$

i.e. one obtains $\partial_{K,v} z$ by applying the simplicial boundary operator ∂ to the chain one gets by restricting z to the faces supported in $\mathrm{star}_K v$. On the other hand, if one defines for each vertex v of K an $\mathbb{F}$-linear map δ_v on oriented simplicial chains by

$$\delta_v[i_1, i_2, \dots, i_r] = \begin{cases} (-1)^j [i_1, i_2, \dots, \hat{i_j}, \dots, i_r] & \text{if } v = i_j \\ 0 & \text{if } v \notin \{i_1, i_2, \dots, i_r\} \end{cases}$$

then we claim that $\partial_{K,v}(z) = \delta_v(z)$. To deduce this description of $\partial_{K,v}(z)$ from the previous one, note that any terms in $\partial\left(z|_{\mathrm{star}_K v}\right)$ supported on faces of K containing v must cancel each other, since the result has to be a cycle supported in $\mathrm{link}_K v$. The remaining terms in the simplicial boundary map are exactly those in δ_v.

We can now prove:

Theorem 2. *The maps d_i^{linear} in the linear strand of the $\mathbb{N}^n$-graded minimal free resolution of a Stanley-Reisner ideal I_Δ are given by equation (3.1), where for a complex K and vertex v, the map $\partial_{K,v}$ is described above.*

Proof. Let d'_i be the map asserted by the theorem to coincide with d_i^{linear}. We first show that $d'_{i-1} d'_i = 0$, so that these maps do form a complex, and then show that this complex must be the linear strand.

The fact that $d'_{i-1} d'_i = 0$ comes from a separate analysis of each $\mathbb{N}$-graded component. The $(\mathbf{x}^V, \mathbf{x}^{V-\{v,w\}})$-component of $d'_{i-1} d'_i$ is determined by a map γ having the form

$$\gamma : \tilde{H}_{\dim K}(K) \longrightarrow \tilde{H}_{\dim K-2}(\mathrm{link}_K\{v,w\})$$

where here $K = \mathrm{link}_{\Delta^*} F$ for $F = [n] - V$. The map γ is the sum of two composite maps

$$\tilde{H}_{\dim K}(K) \stackrel{\partial_{K,v}}{\longrightarrow} \tilde{H}_{\dim K-1}(\mathrm{link}_K\{v\}) \stackrel{\partial_{\mathrm{link}_K v, w}}{\longrightarrow} \tilde{H}_{\dim K-2}(\mathrm{link}_K\{v,w\})$$
$$\tilde{H}_{\dim K}(K) \stackrel{\partial_{K,w}}{\longrightarrow} \tilde{H}_{\dim K-1}(\mathrm{link}_K\{w\}) \stackrel{\partial_{\mathrm{link}_K w, v}}{\longrightarrow} \tilde{H}_{\dim K-2}(\mathrm{link}_K\{v,w\}).$$

From our second description of $\partial_{K,v}$ we have

$$\gamma = \partial_{\mathrm{link}_K v, w} \circ \partial_{K,v} + \partial_{\mathrm{link}_K w, v} \circ \partial_{K,w} = \delta_w \delta_v + \delta_v \delta_w$$

and the righthand side is easily checked to be 0. Hence $d'_{i-1} d'_i = 0$ in each $\mathbb{N}^n$-graded component.

To show that d'_i coincides with d_i^{linear}, we use induction on i. For the case $i = 0$, note that $\tilde{H}_{-1}(\text{link}_{\Delta^*} F)$ vanishes unless F is maximal face of Δ^*, and then it is one-dimensional. When F is a maximal face F of Δ^*, setting $V = [n] - F$ we have that $\mathbf{x}^V$ is a minimal generator of I_Δ, and it is easy to check that the $\mathbf{x}^V$-component of d'_0 is simply the monomial $\mathbf{x}^V$, just as in d_0^{linear}.

For the inductive step, assume that d'_{i-1} coincides with d_{i-1}^{linear}. Since $d'_{i-1} d'_i = 0$, it follows that the A-linear combinations of the $(i-1)$-syzygies defined by d'_i are all genuine i-linear syzygies. Hence they must form an $\mathbb{F}$-subspace in the space of all linear i-syzygies. By construction d'_i produces exactly the same number of $\mathbf{x}^V$-graded linear i-syzygies as predicted by Theorem 1 for the space of *all* $\mathbf{x}^V$-graded linear i-syzygies, although some of these syzygies might, a priori, be linearly dependent over $\mathbb{F}$. Therefore, it suffices to show for each set V that all of the $\mathbf{x}^V$-graded linear i-syzygies produced by d'_i are linearly independent over $\mathbb{F}$. Setting $K = \text{link}_{\Delta^*} F$ for $F = [n] - V$ as before, this amounts to showing injectivity of the map

$$\tilde{H}_{\dim K}(K) \stackrel{\oplus_v \partial_{K,v}}{\longrightarrow} \bigoplus_v \tilde{H}_{\dim K - 1}(\text{link}_K v).$$

To prove injectivity, we use the following lemma.

Lemma 3. *Let z be a cycle in $\tilde{H}_{\dim K}(K)(= \tilde{Z}_{\dim K}(K))$. Then $\partial_{K,v}(z) = 0$ if and only if z does not involve the vertex v.*

Proof. Consider the Mayer-Vietoris exact sequence (3.2). Since $\text{star}_K v$ is a cone, it is contractible and $\tilde{H}_{\dim K}(\text{star}_K v)$ vanishes. Hence exactness at $\tilde{H}_{\dim K}(K)$ shows that if $\partial_{K,v}(z) = 0$, then z is represented by a cycle in $\text{del}_K v$. But since there are no $(\dim K)$-boundaries to create ambiguity, this means that z does not involve v. The converse is obvious. ☐

Injectivity of the map $\oplus_v \partial_{K,v}$ is now clear: Any cycle z in $\tilde{H}_{\dim K}(K)$ which is sent to 0 must involve no vertices of K by Lemma 3.

This concludes the proof of Theorem 2. ☐

4. A conjecture by Herzog in the case of monomial ideals

In this section we prove for monomial ideals a conjecture by J. Herzog [13] about linear syzygies. Recall that a j^{th}-*syzygy module* is a module that occurs as the kernel in the $(j-1)^{st}$-homological degree in the resolution of a finitely generated module N over the polynomial ring A. For example any ideal I in the polynomial ring A is a first syzygy module, since I is the kernel of the natural surjection $A \to N = A/I$.

Conjecture 4 (Herzog [13])**.** *Let M be a j^{th}-syzygy module for some graded A-module N, where $A = \mathbb{F}[x_1, \ldots, x_n]$ is given the standard grading. If M has non-zero p-syzygies in its linear strand for some $p \geq 0$, then it will have at least $\binom{p+j}{i+j}$ i-syzygies in its linear strand for each i.*

A homogeneous ideal I in A is a graded first syzygy module. So when M is such an ideal I we can reformulate the conjecture as follows: if $\text{Tor}_p^A(I, \mathbb{F})_{p+t(I)}$ does not vanish for some $p \geq 0$, then

$$\dim_{\mathbb{F}} \text{Tor}_i^A(I, \mathbb{F})_{i+t(I)} \geq \binom{p+1}{i+1}$$

for each i.

Herzog's conjecture is motivated by a result of M. Green [12] (see also [10]) that contains the case $i = 0$, $j = 1$, and his own results that show the conjecture in full generality for $j = 0$ [13].

By the technique of polarization [11], the $j = 1$ case of Conjecture 4 follows for all monomial ideals $I(= M)$ once it is proven for square-free monomial ideals. For square-free monomial ideals, we will in fact show something slightly stronger:

Theorem 5. *Let I be a square-free monomial ideal in A. If $\mathrm{Tor}_p^A(I, \mathbb{F})_{\mathbf{x}^V} \neq 0$ for some set V with $|V| = p + t(I)$, then for each i there exist at least $\binom{p+1}{i+1}$ subsets $V' \subseteq V$ having $|V'| = i + t(I)$ and $\mathrm{Tor}_i^A(I, \mathbb{F})_{\mathbf{x}^{V'}} \neq 0$.*

Proof. The result will be deduced from the following lemma, whose statement involves only simplicial topology, but whose proof relies heavily on the maps $\partial_{K,v}$ which appear in the linear strand:

Lemma 6. *Let K be a $(q-1)$-dimensional simplicial complex and $\mathbb{F}$ any field. If $\tilde{H}_{q-1}(K; \mathbb{F}) \neq 0$, then there exist at least $q+1$ vertices v of K with the property that $\tilde{H}_{q-2}(\mathrm{link}_K v; \mathbb{F}) \neq 0$.*

Proof. Let z be a non-trivial cycle in $\tilde{H}_{q-1}(K; \mathbb{F})$. Because it is a non-trivial $(q-1)$-cycle, z must involve at least $q + 1$ vertices. Hence by Lemma 3, at least $q + 1$ of the maps $\partial_{K,v}$ must have $\partial_{K,v}(z) \neq 0$. Therefore at least $q + 1$ vertices must have $\tilde{H}_{q-2}(\mathrm{link}_K v; \mathbb{F}) \neq 0$. ☐

To deduce Theorem 5, we use induction on $p-i$. It is trivially true for $p-i = 0$. In the inductive step, assume that there are at least $\binom{p+1}{i+1}$ subsets $V'' \subseteq V$ having $|V''| = i + t(I)$ and $\mathrm{Tor}_i^A(I; \mathbb{F})_{\mathbf{x}^{V''}} \neq 0$. For each such V'', we can apply Lemma 6 and Theorem 1 to $K = \mathrm{link}_{\Delta^*} F''$ where $F'' = [n] - V''$. In this way, we obtain a collection $\mathcal{C}_{V''}$ of at least $i + 1$ subsets $V' \subset V''$ with $|V'| = i - 1 + t(I)$ and $\mathrm{Tor}_{i-1}^A(I; \mathbb{F})_{\mathbf{x}^{V''}} \neq 0$. It remains then to show that the cardinality of the union $\bigcup_{V''} \mathcal{C}_{V''}$ is at least $\binom{p+1}{i}$. If we fix attention on a particular subset $V' \subset V$ having $|V'| = i - 1 + t(I)$ elements, then it can occur in at most $p + 1 - i$ different collections $\mathcal{C}_{V''}$ since $|V - V'| = p + 1 - 1$. Therefore

$$\begin{aligned}
\left| \bigcup_{V''} \mathcal{C}_{V''} \right| &\geq \frac{1}{p+1-i} \sum_{V''} |\mathcal{C}_{V''}| \\
&\geq \frac{1}{p+1-i} \binom{p+1}{i+1} (i+1) \\
&= \binom{p+1}{i}.
\end{aligned}$$

This concludes the proof of Theorem 5. ☐

5. Pseudomanifolds and a counterexample

In this section we give some consequences of Theorem 2. It is easy to see for any monomial ideal I, the matrix entries in the maps in the multigraded minimal free resolution will always be single terms, that is a monomial times some coefficient.

We show that the coefficients occurring in the linear strand can always be chosen to be ± 1 whenever Δ^* is a pseudomanifold, and give a non-pseudomanifold example where this property fails.

Say that a d-dimensional simplicial complex K is a *pseudomanifold without boundary*, or just a *pseudomanifold*, if every $(d-1)$-face lies in exactly two d-faces. Examples of such complexes are triangulations of manifolds or singular spaces whose singularities have real codimension at least two, such as complex varieties. We emphasize that our definition differs somewhat from the definition of pseudomanifolds sometimes given in the literature (e.g. [18, p. 261]), where it is further assumed K is pure, and that any two d-faces are *gallery-connected*, i.e. connected by a path of d-faces in which every pair of d-faces forming a step in the path share a common $(d-1)$-face. Note however, that dropping the assumption of purity gives us only a spurious extra generality with regard to results on the linear strand: The linear strand in the minimal free resolution of I_Δ depends only on the minimal generators of I_Δ of lowest degree, and hence depends only on the pure subcomplex of Δ^* generated by its faces of maximum dimension.

Recall that $\tilde{H}_d(K;\mathbb{F})$ for a d-dimensional pseudomanifold K takes on a particularly simple form and has a canonical basis (up to negating basis vectors) of *orientation cycles* ([18, p. 394]). To be precise, let $\Gamma_1, \ldots, \Gamma_m$ be the *gallery-equivalence* classes of d-faces of K, where gallery-equivalence is the equivalence relation generated by the relation $F \sim F'$ if the two d-faces F, F' share a common $(d-1)$-face $F \cap F'$. For each class Γ_i it is easy to see that either there exists a unique way (up to an overall sign change) to choose signs so that

$$z_{\Gamma_i} = \sum_{F \in \Gamma_i} \pm [F]$$

forms an cycle (called an orientation cycle), or else Γ_i contributes no orientation cycle to the basis of $\tilde{H}_d(K;\mathbb{F})$. Note that the latter cannot happen if $\mathbb{F}$ has characteristic 2. Also note that any cycle z in $\tilde{H}_d(K;\mathbb{F})$ is completely determined by its coefficients $c_{[F_i]}$ on any system of representatives of d-faces $F_i \in \Gamma_i$ of the classes Γ_i: If z_{Γ_i} is normalized so that the oriented simplex $[F_i]$ has coefficient $+1$, then

$$z = \sum_i c_{[F_i]} z_{\Gamma_i}.$$

As a consequence of this discussion, we have the following:

Proposition 7. *If Δ^* is a pseudomanifold, then the maps in the linear strand of the minimal free resolution of I_{Δ^*} can be written using only ± 1 coefficients.*

Proof. Note that if Δ^* is a d-dimensional pseudomanifold and F is a face of Δ with

$$\dim(\mathrm{link}_{\Delta^*} F) = d - |F|$$

(i.e. F is a face whose link in Δ^* has top homology that might contribute to the linear strand), then $\mathrm{link}_{\Delta^*} F$ is also a pseudomanifold. Therefore by Theorem 2, it suffices to show that for any pseudomanifold K and vertex v, the map $\partial_{K,v}$ is a ± 1-matrix when written with respect to the basis of orientation cycles for $\tilde{H}_d(K;\mathbb{F})$ and $\tilde{H}_{d-1}(\mathrm{link}_K v;\mathbb{F})$.

To see this, let Γ_i (resp. Γ'_j) be a gallery-equivalence class of d-faces (resp. $(d-1)$-faces) for K (resp. $\mathrm{link}_k v$), which gives rise to an orientation cycle z_{Γ_i} (resp. $z_{\Gamma'_j}$) over $\mathbb{F}$. Then it is easy to check from the description $\partial_{K,v} = \delta_v$ that the coefficient of $z_{\Gamma'_j}$ in $\partial_{K,v}(z_{\Gamma_i})$ will be 0 unless any chosen $(d-1)$-face F' in Γ'_j has the property that $F = F' \cup \{v\}$ is in Γ_i. In the latter case, the coefficient will be ± 1, depending upon the sign of $[F]$ in z_{Γ_i} and of $[F']$ in $z_{\Gamma'_j}$. $\square$

The proof of Proposition 7 shows that the maps in the linear strand when Δ^* is a pseudomanifold are essentially "incidence matrices" for the orientation cycles of all of the links of faces in Δ^*. An interesting special case of this occurs when Δ^* is a *homology sphere over* $\mathbb{F}$, that is when every face F of Δ^* has

$$\tilde{H}_i(\mathrm{link}_{\Delta^*} F; \mathbb{F}) = \begin{cases} \mathbb{F} & \text{if } i = \dim \Delta^* - |F| \\ 0 & \text{else .} \end{cases}$$

This condition may also be phrased in terms of local homology groups $\tilde{H}(|\Delta^*|, |\Delta^*| - x; \mathbb{F})$, and hence is topologically invariant. In [8] it was pointed out that for homology spheres Δ^*, the Betti numbers in the (linear) resolution of I_Δ coincide with the *f-vector* listing the number of faces of various dimensions in Δ^*. In [21, page 3] Sturmfels further remarked that this linear resolution is essentially the *coboundary complex* for the simplicial complex Δ^*. Although this can be deduced from Theorem 2, it is easy enough to prove directly so we omit the proof.

The previous results raise the question of whether there exists a monomial ideal I for which the maps *cannot* be written using only ± 1 coefficients. The following example illustrates that this can happen, even when the whole resolution is linear, and even when it is linear over an arbitrary field $\mathbb{F}$.

Example:
Let I_Δ be the ideal in $A = \mathbb{F}[x_1, x_2, x_3, x_4, x_5, x_6, x_v]$

$$\begin{array}{lllll} (x_3x_4x_5x_6, & x_2x_4x_5x_6, & x_1x_4x_5x_6, & & \\ x_3x_4x_5x_v, & x_2x_4x_5x_v, & x_1x_3x_5x_v, & x_1x_2x_4x_v, & x_1x_4x_6x_v, \\ x_1x_5x_6x_v, & x_3x_4x_6x_v, & x_2x_5x_6x_v, & x_2x_3x_6x_v, & x_1x_2x_3x_v) \end{array}$$

where $\mathbb{F}$ does not have characteristic 2 or 3. Then Δ^* is the simplicial complex on vertex set $\{1, 2, 3, 4, 5, 6, v\}$ with maximal faces

$$\{12v, 13v, 23v, 126, 136, 246, 356, 235, 234, 125, 134, 145, 456\},$$

and can be described in the following fashion: The induced subcomplex of Δ^* on the vertices $\{1, 2, 3, 4, 5, 6\}$ is a minimal triangulation of the real projective plane having the property that 123 is *not* a 2-face, and Δ^* is obtained from this subcomplex by adding the three triangles $v12, v13, v23$. One can think of this addition as first adding in the missing 2-face 123 and then subdividing this 2-face with a vertex v into three smaller triangles.

It is easy to check that the order in which the maximal faces of Δ^* are listed above is a *shelling order*, and hence Δ^* is Cohen-Macaulay over any field $\mathbb{F}$, so that I_Δ always has a 4-linear resolution whose Betti numbers are independent of $\mathbb{F}$ (see [8]). The complex Δ^* has the interesting property that even though it is shellable and homotopy equivalent to a 2-sphere, the orientation cycle in $\tilde{H}_2(\Delta^*, \mathbb{F})$ cannot be written as a linear combination of 2-faces with ± 1 coefficients. This property was shown to us by G. M. Ziegler in a similar example, where the triangle 123 is not subdivided into three smaller triangles. Unfortunately, this original example had a resolution with only ± 1 coefficients in the maps, requiring the subdivision by vertex v.

A computation with the computer algebra package Macaulay shows that the 4-linear resolution of I_Δ has the form

$$0 \longrightarrow A(-7)^1 \longrightarrow A(-6)^{10} \longrightarrow A(-5)^{21} \longrightarrow A(-4)^{13} \longrightarrow I_\Delta \longrightarrow 0.$$

This resolution has all ± 1 coefficients in the maps except for the last map

$$A(-7)^1 \longrightarrow A(-6)^{10}$$

which contains some coefficients of ± 2. Call this Macaulay resolution $\mathcal{M}_\bullet$. We now assume that we have such a multigraded minimal free resolution $\mathcal{F}_\bullet$ which uses only coefficients ± 1, in its maps, and will reach a contradiction.

Obviously, the map $A(-4)^{13} \longrightarrow I_\Delta$ in $\mathcal{F}_\bullet$ must simply list the minimal generators of I_Δ, possibly with ± 1 coefficients in front, so without loss of generality, we may alter the basis for $A(-4)^{13}$ in $\mathcal{F}_\bullet$ by $\pm$ signs so that the coefficients are all $+1$, as in $\mathcal{M}_\bullet$. Since the edges $e = v1, v2, v3, 16, 26, 36, 46, 56$ have links in Δ^* which are 0-spheres, they each give rise to a unique 1-syzygy having multidegree complementary to e by Theorem 1. In $\mathcal{M}_\bullet$ these 1-syzygies have only ± 1 coefficients, and so $\mathcal{F}_\bullet$ must choose a ± 1 multiple of these same syzygies, hence we may assume that the columns of the map $A(-5)^{21} \longrightarrow A(-4)^{13}$ in these multidegrees are the same as in $\mathcal{M}_\bullet$. A similar remark applies to the vertices $v, 6$ whose links in Δ^* are 1-spheres, once we note for reasons of multidegree that the unique 2-syzygies to which they correspond will only involve the 1-syzygies which correspond to the edges e listed previously, and hence are the same in $\mathcal{F}_\bullet$ as in $\mathcal{M}_\bullet$. Let us name the basis elements of $A(-6)^{10}$ corresponding to these two 2-syzygies by e_v, e_6, respectively, in both $\mathcal{M}_\bullet$ and $\mathcal{F}_\bullet$.

Lastly, we compare the unique 3-syzygy in $\mathcal{M}_\bullet$ and $\mathcal{F}_\bullet$. In $\mathcal{M}_\bullet$, this 3-syzygy has coefficient x_6 on e_6 and $2x_v$ on e_v. Since $\mathbb{F}$ does not have characteristic 2 or 3, there is no way for $\mathcal{F}_\bullet$ to rescale this unique 3-syzygy to have both a $\pm x_6$ on e_6 and $\pm x_v$ on e_v. Contradiction. $\square$

6. The resolution for matroidal ideals

In this section we use Theorem 2 to explicitly describe the minimal free resolution when I_Δ is *matroidal*, i.e. if m, m' are two minimal monomial generators of I and x_i divides m, then there exists a j such that x_j divides m' and $\frac{x_i}{x_j} m'$ is in I_Δ. In this case, the supports of the square-free monomial generators of I form the bases of a *matroid* M on ground set $[n]$ (see [19] for definitions and facts about matroids). It was observed in [8] that in this case Δ^* is the complex of independent sets $IN(M^*)$ or *matroid complex* associated to the *dual matroid* M^*. Since matroid complexes are pure and shellable (see [6]) this implies that I_Δ has a linear resolution for any field $\mathbb{F}$. To make this resolution explicit, we recall some terminology from matroid theory from [6].

Given a base B of a matroid M and an element e not in B, there is a unique circuit $ci_M(B, e)$ contained in $B \cup \{e\}$ called the *basic circuit* for e and B. Dually for any b in B there is a unique *bond* $bo_M(B, b)$ contained in $([n] - B) \cup \{b\}$ called the *basic bond* for b and B. An element e in $[n] - B$ (resp. b in B) is called *externally (resp. internally) active* with respect to B if it is the smallest element of its basic circuit (resp. bond) in the usual order on $[n]$. The *external (resp. internal) activity* of a base B is the number of elements of $[n]$ which are externally (resp. internally) active with respect to B.

Given a subset $V \subseteq [n]$ such that its complement $F = [n] - V$ is a face of $\Delta^* = IN(M^*)$, it is easy to check that

$$\mathrm{link}_{\Delta^*} F = IN(M^*/F)$$

where M^*/F denotes the *quotient matroid* of M^* by F. Alternatively, we have

$$M^*/F \cong (M|_V)^*$$

where $M|_V$ is the restriction of M to the ground set V.

In order to apply Theorem 2, we need a basis for the top homology of $\mathrm{link}_{\Delta^*} F = IN(M^*/F)$. There are two (conjecturally equivalent) choices for us to use. Given any matroid M,

- Björner [6, Prop. 7.8.4] constructs a basis $\{\sigma_B\}$ for the top homology of $IN(M)$ which is indexed by bases B of M having internal activity 0, or
- from Theorem 14 and Remark 15 of [17], one obtains a basis $\{\tau_C\}$ for the top homology of of $IN(M)$ which is indexed by bases C of M^* having external activity 0, sometimes called *nbc-bases* for M^*.

We choose to use the latter basis for convenience. Fortunately, bases of internal activity 0 for M are the same as bases of external activity 0 for M^* by complementation within the ground set $[n]$, since the internal activity of b with respect to B in M equals the external activity of b with respect to $[n] - B$ in M^*. Conjecturally, $\tau_C = \sigma_{[n]-C}$, but we will not need this.

For the purpose of stating the theorem, let us fix the following notation. Let I_Δ be matroidal for a matroid M, so that $\Delta^* = IN(M^*)$. Given $V \subseteq [n]$ and $v \in V$ and $F = [n] - V$ as usual, choose as bases for the top homology of

$$\mathrm{link}_{\Delta^*} F = IN(M^*/F)$$
$$\mathrm{link}_{\Delta^*} F + v = IN(M^*/(F + v))$$

the sets $\{\tau_C\}, \{\tau_{C'}\}$ where C, C' run through the bases of external activity 0 for $M|_V, M|_{V-v}$. With respect to the above bases, let $d_{C,C'}$ denote the (C, C')-entry of the $(\mathbf{x}^V, \mathbf{x}^{V-v})$-graded component of the map in the minimal free resolution of I_Δ.

Theorem 8. *The matrix entry $d_{C,C'}$ is 0 unless either*

- *$C = C'$, or*
- *$C = C' - \{w\} \cup \{v\}$ for some w in $ci_{M|_{V-v}}(C', v)$.*

In either of these cases, $d_{C,C'} = (-1)^j x_v$ where v is the j^{th} smallest element of C,

Proof. The basis elements τ_C constructed in the proof of [17, Theorem 14] have the following properties:

- The coefficient of τ_C on any particular oriented simplex is 0 or ± 1,
- The coefficient of τ_C on the oriented simplex $[[n] - C]$ is $+1$,
- The coefficient of τ_D on $[[n] - C]$ is 0 for any other basis D of external activity 0.

As a consequence, from Theorem 2 and the description $\partial_{K,v} = \delta_v$, computing $d_{C,C'}$ comes down to checking whether $[([n] - v) - C' + v] = [[n] - C']$ appears with non-zero coefficient in τ_C. It remains to show that this happens exactly under the circumstances described in the theorem (and under those circumstances the appropriate ± 1 coefficient then follows easily).

To simplify the analysis, we let $y_C = f(\tau_C)$ where f maps chains in $IN((M|_V)^*)$ to chains in $IN(M|_V)$ by sending the oriented simplex $[B]$ to the oriented simplex $[V-B]$. We then need to show that C' occurs with non-zero coefficient in y_C only in the circumstances of the theorem. From the proof of [17, Theorem 14], the chain y_C has a fairly simple description: If one writes the elements $c_1 < c_2 < \cdots < c_r$ of C in increasing order, then

$$y_C = \sum_{(e_1,\dots,e_r)} [e_1, \dots, e_r]$$

where $(e_1, \dots, e_r)$ runs over all sequences of r elements in $[n]$ such that e_i is in the *flat* $\overline{C_i}$ spanned by $C_i := \{c_r, c_{r-1}, \dots, c_{i+1}, c_i\}$ but not in the flat $\overline{C_{i+1}}$. Renaming the restricted matroid $M|_V$ by N, we are left with proving the following claim:

> **Claim**: If N is a matroid, v an element of its ground set, and C, C' bases of external activity 0 for $N, N-v$ respectively, then $[C']$ occurs with non-zero coefficient in y_C if and only if $C = C'$ or $C = C' - w + v$ for some w in $ci_N(C', v)$.

Before proving this claim, we recall two important properties of bases of external activity 0 from the proof of [17, Theorem 14]:

(a) If C is a base of external activity 0, then any non-empty subset $C_0 \subseteq C$ is a base of external activity 0 for the flat $\overline{C_0}$.
(b) If $C = \{c_1 < \dots < c_r\}$ and $C_i = \{c_r, c_{r-1}, \dots, c_i\}$ as above, then c_i is the smallest element of $\overline{C_i} - \overline{C_{i+1}}$.

To prove the claim, there are two cases to check depending on v's external activity with respect to C.
Case 1: The element v is not externally active for C'. In this case, C' still has external activity 0 when considered as a base for N rather than $N - v$. It then follows from the bulleted facts at the beginning of the proof that C' occurs with non-zero coefficient in y_C if and only if $C = C'$.
Case 2: The element v is externally active for C'. We wish to show in this case that $C = C' - \{w\} \cup \{v\}$ for some w in $ci_{(M|_{V-v})^*}(C', v)$. Recalling the notation $C_i = \{c_r, c_{r-1}, \dots, c_i\}$, let s be the unique index such that v lies in $\overline{C_s} - \overline{C_{s+1}}$. Since C' occurs with non-zero coefficient in y_C, by the definition of y_C we can also number the elements $\{c'_1, \dots, c'_r\} = C'$ in such a way that $C'_i := \{c'_r, c'_{r-1}, \dots, c'_i\}$ has C'_i, C_i span the same flat of N for all i. We will now show that $w = c'_s$ has property asserted in the claim.

Since v is not in $\overline{C_{s+1}} (= \overline{C'_{s+1}}\ )$, it follows from assertion (a) that C_{s+1}, C'_{s+1} are both bases of external activity 0 for this flat. But by construction, $[C'_{s+1}]$ appears as a term in $y_{C_{s+1}}$, so we must have $C'_{s+1} = C_{s+1}$. Now v, c_s, c'_s all lie in $\overline{C_s} - \overline{C_{s+1}} (= \overline{C'_s} - \overline{C'_{s+1}})$, so by assertion (b), c_s must be the smallest element of $\overline{C_s} - \overline{C_{s+1}}$. On the other hand, the fact that C' has external activity 0 for $N - v$ but not for N implies that C'_s has external activity 0 for $\overline{C_s} - v$ but not for $\overline{C_s}$. Therefore, v must be externally active for C'_s in $\overline{C'_s}$, which means that it is the smallest element of $\overline{C_s} - \overline{C_{s+1}}$ and so we must have $v = c_s$. Consequently, $C_s = C'_s - c'_s + v = C'_s - w + v$. The fact that $c'_i = c_i$ for $i > s$ follows again from assertion (b), since $\overline{C_i} - \overline{C_{i+1}} = \overline{C'_i} - \overline{C'_{i+1}}$ for $i > s$. Therefore $C = C' - w + v$ as desired. $\square$

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