Reciprocal domains and Cohen–Macaulay d-complexes in $\mathbb{R}^d$

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Dedicated to Richard P. Stanley on the occasion of his 60th birthday

Abstract

We extend a reciprocity theorem of Stanley about enumeration of integer points in polyhedral cones when one exchanges strict and weak inequalities. The proof highlights the roles played by Cohen–Macaulayness and canonical modules. The extension raises the issue of whether a Cohen–Macaulay complex of dimension d embedded piecewise-linearly in $\mathbb{R}^d$ is necessarily a d-ball. This is observed to be true for $d \leq 3$, but false for $d = 4$.

1 Main results

This note begins by dealing with the relation between enumerators of certain sets of integer points in polyhedral cones, when one exchanges the roles of strict versus weak inequalities (Theorem 1). The interaction of this relation with the Cohen–Macaulay condition then leads us to study piecewise-linear Cohen–Macaulay polyhedral complexes of dimension d in Euclidean space $\mathbb{R}^d$ (Theorem 2).

We start by reviewing a result of Stanley on Ehrhart's notion of reciprocal domains within the boundary of a convex polytope. Good references for much of this material are [3, Chapter 6], [10, Chapter 1], and [8, Part II].

Let $Q \subset \mathbb{Z}^d$ be a saturated affine semigroup, that is, the set of integer points in a convex rational polyhedral cone $C = \mathbb{R}_{\geq 0}Q$. Assume that the cone C is of full dimension d, and pointed at the origin. Denote by $\mathcal{F}$ the facets (subcones of codimension 1) of C. For each facet $F \in \mathcal{F}$, let $\ell_F(x) \geq 0$ be the associated facet inequality, so that the semigroup

$$Q \quad = \quad \{x \in \mathbb{Z}^d \mid \ell_F(x) \geq 0 \text{ for all facets } F \in \mathcal{F}\}$$

is the intersection of the corresponding closed positive half-spaces.

Fix a nonempty proper subset $\mathcal{G}$ of the facets $\mathcal{F}$, and let Δ and Δ', respectively, denote the pure $(d-1)$-dimensional subcomplexes of the boundary complex of C generated by the facets in $\mathcal{G}$ and $\mathcal{F} \smallsetminus \mathcal{G}$, respectively. Ehrhart called the sets $C \smallsetminus \Delta$ and $C \smallsetminus \Delta'$ *reciprocal domains*

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within the boundary complex of C. Examples of reciprocal domains arise when Δ is *linearly separated from* Δ', meaning that some point $p \in \mathbb{R}^d$ satisfies

$$\ell_F(p) > 0 \text{ for } F \in \mathcal{G}$$
$$\text{and } \ell_F(p) < 0 \text{ for } F \in \mathcal{F} \smallsetminus \mathcal{G}.$$

Define the *lattice point enumerator* to be the power series

$$F_{C\smallsetminus\Delta}(x) \quad := \sum_{a\in(C\smallsetminus\Delta)\cap\mathbb{Z}^d} x^a$$

in the variables $x = (x_1, \ldots, x_n)$. This series lies in the completion $\mathbb{Z}[[Q]]$ of the integral semigroup ring $\mathbb{Z}[Q]$ at the maximal ideal $\mathfrak{m} = \langle x^a \mid 0 \neq a \in Q\rangle$ generated by the set of nonunit monomials. General facts about Hilbert series of finitely generated modules over semigroup rings imply that $F_{C\smallsetminus\Delta}(x)$ can be expressed in the complete ring $\mathbb{Z}[[Q]]$ as a rational function whose denominator is a product of terms having the form $1 - x^a$ [8, Chapter 8].

A result of Stanley [9, Proposition 8.3] says that when Δ, Δ' are linearly separated,

$$F_{C\smallsetminus\Delta'}(x^{-1}) \quad = \quad (-1)^d\, F_{C\smallsetminus\Delta}(x)$$

as rational functions in $\mathbb{Q}(x_1, \ldots, x_d)$. Our main result weakens the geometric 'linearly separated' hypothesis on Δ to one that is topological and ring-theoretic.

Let $\Bbbk$ be a field, and denote by $\Bbbk[Q] = \bigoplus_{a\in Q} \Bbbk \cdot x^a$ the $\mathbb{Z}^d$-graded affine semigroup ring corresponding to Q. For each subcomplex Δ of C, this ring contains a radical, $\mathbb{Z}^d$-graded ideal I_Δ consisting of the $\Bbbk$-span of monomials x^a for $a \in C \smallsetminus \Delta$. The *face ring* of Δ is defined to be the quotient $\Bbbk[\Delta] := \Bbbk[Q]/I_\Delta$.

A polyhedral subcomplex $\Delta \subseteq C$ is *Cohen–Macaulay over* $\Bbbk$ if $\Bbbk[\Delta]$ is a Cohen–Macaulay ring. This turns out to be a topological condition, as we now explain. Fix a $(d-1)$-dimensional *cross-sectional polytope* $\overline{C}$ of the cone C, and let $\overline{\Delta} := \overline{C} \cap \Delta$, a pure $(d-2)$-dimensional subcomplex of the boundary complex of $\overline{C}$. It is known [12] that Δ is Cohen–Macaulay if and only if the geometric realization $|\overline{\Delta}|$ is *topologically Cohen–Macaulay (over* $\Bbbk$*)*, meaning that its (reduced) homology $\tilde{H}_i(|\overline{\Delta}|; \Bbbk)$ and its local homology groups $\tilde{H}_i(|\overline{\Delta}|, |\overline{\Delta}| \smallsetminus p; \Bbbk)$ vanish for $i < d-2$.

The Cohen–Macaulay condition is known to hold whenever $|\overline{\Delta}|$ is a $(d-2)$-ball, but this sufficient condition is not in general necessary; see Theorem 2 below. Nevertheless, when Δ is linearly separated from Δ', the topological space $|\overline{\Delta}|$ is such a ball, because its facets are shelled as an initial segment of a *(Bruggesser–Mani) line-shelling* [2, Example 4.17] of the boundary complex of the cone C.

Theorem 1 *Let Δ be a dimension $d-1$ subcomplex of a pointed rational polyhedral cone $C \subseteq \mathbb{R}^d$ of dimension d, and let Δ' be the dimension $d-1$ subcomplex of C generated by the facets of C not in Δ. If Δ is Cohen–Macaulay over some field $\Bbbk$, then as rational functions, the lattice point enumerators of the reciprocal domains $C \smallsetminus \Delta$ and $C \smallsetminus \Delta'$ satisfy*

$$F_{C\smallsetminus\Delta'}(x^{-1}) \quad = \quad (-1)^d\, F_{C\smallsetminus\Delta}(x).$$

Theorem 1 raises the issue of whether a d-dimensional Cohen–Macaulay proper subcomplex of the boundary of a $(d+1)$-polytope must always be a d-ball, a question that arises in other contexts within combinatorial topology (such as [1]). Although Theorem 2 below is surely known to some topologists, we have not found its (statement or) proof in the literature. Therefore, we have written down the details of its proof in Section 3.

Theorem 2 *1. Let K be a d-dimensional proper subcomplex of the boundary of a $(d+1)$-polytope. If $d \leq 3$ and K is Cohen–Macaulay over some field $\Bbbk$, then the topological space $|K|$ is homeomorphic to a d-ball.*

2. There exists a proper subcomplex of dimension 4 in the boundary of a 5-polytope that is Cohen–Macaulay over every field but not homeomorphic to a 4-ball.

2 Reciprocal domains via canonical modules

The proof of Theorem 1 relies on the interpretation

$$F_{C \smallsetminus \Delta}(x) \quad = \quad \mathrm{Hilb}(I_\Delta, x)$$

of the lattice point enumerator as the multigraded Hilbert series $\mathrm{Hilb}(M, x)$ of the $\mathbb{Z}^d$-graded module I_Δ. The proof emphasizes the relations between between $\Bbbk[\Delta], I_\Delta$, and $I_{\Delta'}$ by taking homomorphisms into the canonical module. Throughout we will freely use concepts from combinatorial commutative algebra that may be found in [3, Chapter 6], [10, Chapter 1], or [8, Part II].

Hochster [6] showed that the semigroup ring $\Bbbk[Q]$ is Cohen–Macaulay whenever Q is saturated. For a graded Cohen–Macaulay ring R of dimension d, there is the notion of its *canonical module* ω_R. For $R = \Bbbk[Q]$, it is known (see e.g. [10, §I.13], [8, §13.5]) that the canonical module $\omega_{\Bbbk[\Delta]}$ is the ideal in $\Bbbk[Q]$ spanned $\Bbbk$-linearly by the monomials whose exponents lie in the interior of the cone C. Given a Cohen–Macaulay ring R of dimension d, and M a Cohen–Macaulay R-module of dimension e, one can define the *canonical module* of M by

$$\omega_R(M) \quad := \quad \mathrm{Ext}_R^{d-e}(M, \omega_R).$$

Graded local duality implies that $\omega_R(M)$ is again a Cohen–Macaulay R-module of dimension e, and that $\omega_R(\omega_R(M)) \cong M$ as R-modules.

Proposition 3 *Let $\Delta \subset C$ be a subcomplex of dimension d, and set $Q = C \cap \mathbb{Z}^d$.*

1. $\Bbbk[\Delta]$ is Cohen–Macaulay if and only if I_Δ is a Cohen–Macaulay $\Bbbk[Q]$-module.

2. I_Δ is Cohen–Macaulay if and only if $I_{\Delta'}$ is Cohen–Macaulay, and in this case there is an isomorphism $I_{\Delta'} \cong \omega_{\Bbbk[Q]}(I_\Delta)$ as $\Bbbk[Q]$-modules.

Proof. For the first assertion we use the fact that a graded module M over $\Bbbk[Q]$ is Cohen–Macaulay if and only if its local cohomology $H^i_{\mathfrak{m}}(M)$ with respect to the maximal ideal $\mathfrak{m} = \langle x^a \mid 0 \neq a \in Q \rangle$ vanishes for i in the range $[0, \dim(M) - 1]$.

The short exact sequence $0 \to I_\Delta \to \Bbbk[Q] \to \Bbbk[\Delta] \to 0$ gives a long exact local cohomology sequence containing the four term sequence

$$H^i_{\mathfrak{m}}(\Bbbk[Q]) \to H^i_{\mathfrak{m}}(\Bbbk[\Delta]) \to H^{i+1}_{\mathfrak{m}}(I_\Delta) \to H^{i+1}_{\mathfrak{m}}(\Bbbk[Q]). \tag{2.1}$$

Cohen–Macaulayness of $\Bbbk[Q]$ implies that the two outermost terms of (2.1) vanish for i in the range $[0, d-2]$, so that the middle map is an isomorphism. As $\Bbbk[\Delta]$ has dimension $d-1$, it is

Cohen–Macaulay if and only if $H^i_{\mathfrak{m}}(\Bbbk[\Delta])$ vanishes for $i \in [0, d-2]$. As I_Δ has dimension d, it is Cohen–Macaulay if and only if $H^{i+1}_{\mathfrak{m}}(I_\Delta)$ vanishes for $i \in [-1, d-2]$. Noting that $H^0_{\mathfrak{m}}(I_\Delta)$ always vanishes due to the fact that I_Δ is torsion-free as a $\Bbbk[Q]$-module, the first assertion follows.

For the second assertion, assuming that I_Δ is Cohen–Macaulay, we prove a string of easy isomorphisms and equalities:

$$\begin{array}{rcl} I_{\Delta'} & = & \big(\omega_{\Bbbk[Q]} : I_\Delta\big) \\ & \cong & \mathrm{Hom}_{\Bbbk[Q]}(I_\Delta, \omega_{\Bbbk[Q]}) \\ & = & \mathrm{Ext}^0_{\Bbbk[Q]}(I_\Delta, \omega_{\Bbbk[Q]}) \\ & = & \omega_{\Bbbk[Q]}(I_\Delta) \end{array} \tag{2.2}$$

in which $(J : I) = \{r \in R : rI \subset J\}$ is the *colon ideal* for two ideals I, J in a ring R.

The last two equalities in (2.2) are essentially definitions. To prove the first equality, we claim that if $x^a \in I'_\Delta$ and $x^b \in I_\Delta$, then $x^a \cdot x^b = x^{a+b} \in \omega_{\Bbbk[Q]}$. Using the linear inequalities from Section 1, this holds because

- $\ell_F(a) \geq 0$ and $\ell_F(b) \geq 0$ for all facets $F \in \mathcal{F}$,
- $\ell_F(a) > 0$ for $F \in \mathcal{G}$,
- $\ell_F(b) > 0$ for $F \in \mathcal{F} - \mathcal{G}$.

Thus $I_{\Delta'} \subset \big(\omega_{\Bbbk[Q]} : I_\Delta\big)$. The reverse inclusion follows by a similar argument.

The isomorphism in the second line of (2.2) follows from a general fact: for any two $\mathbb{Z}^d$-graded ideals I, J in $\Bbbk[Q]$, one has

$$\mathrm{Hom}_{\Bbbk[Q]}(I, J) \quad \cong \quad (J : I).$$

To prove this, assume $\phi : I \to J$ is a $\Bbbk[Q]$-module homomorphism that is $\mathbb{Z}^d$-homogeneous of degree c. Since each $\mathbb{Z}^d$-graded component of I or J is a $\Bbbk$-vector space of dimension at most 1, for each monomial x^a in I there exists a scalar $\lambda_a \in \Bbbk$ such that $\phi(x^a) = \lambda_a x^{a+c}$. We claim that these scalars λ_a are all equal to a single scalar λ. Indeed, given x^a, x^b in I, the fact that ϕ is a $\Bbbk[Q]$-module homomorphism forces both $\lambda_a = \lambda_{a+b}$ and $\lambda_b = \lambda_{a+b}$. Thus for some $\lambda \in \Bbbk$, one has $\phi = \lambda \cdot \phi_c$, where $\phi_c(x^a) = x^{a+c}$. Furthermore, if $\lambda \neq 0$ then $x^c \in (J : I)$. We conclude that the map

$$\begin{array}{rcl} (J : I) & \to & \mathrm{Hom}_{\Bbbk[Q]}(I, J) \\ x^c & \mapsto & \phi_c \end{array}$$

is an isomorphism of $\Bbbk[Q]$-modules. $\square$

Proof of Theorem 1. The key fact (see [10, §I.12], for instance) is that for any Cohen–Macaulay $\Bbbk[Q]$-module M of dimension d,

$$\mathrm{Hilb}(\omega_{\Bbbk[Q]}(M); x^{-1}) \quad = \quad (-1)^d \mathrm{Hilb}(M; x).$$

Therefore when I_Δ is Cohen–Macaulay, Proposition 3 gives

$$\begin{array}{rcl} F(C \smallsetminus \Delta'; x^{-1}) & = & \mathrm{Hilb}(I_{\Delta'}; x^{-1}) \\ & = & (-1)^d \mathrm{Hilb}(I_\Delta; x) \\ & = & (-1)^d F(C \smallsetminus \Delta; x). \end{array}$$ $\square$

3 Cohen–Macaulay d-complexes in $\mathbb{R}^d$

The goal of this section is to prove Theorem 2. In this section, K will be a finite polyhedral complex embedded piecewise linearly in $\mathbb{R}^d$. That is, K is a finite collection of convex polytopes in $\mathbb{R}^d$ containing the faces of any polytope in K, and for which any two polytopes in K intersect in a common (possibly empty) face of each.

It will be convenient to pass between PL-embeddings of such polyhedral complexes into $\mathbb{R}^d$, and PL-embeddings into the boundary of a $(d+1)$-polytope. In one direction, this passage is easy, as we now show.

Proposition 4 *Let K be a finite polyhedral d-dimensional complex PL-embedded as a proper subset of the boundary of a $(d+1)$-polytope P (but not necessarily as a subcomplex of the boundary). Then K has a PL-embedding into $\mathbb{R}^d$.*

Proof. We first reduce to the case where K avoids at least one facet of P entirely. Since K is a compact proper subset of the boundary of P, there exists at least one facet F of P whose interior is not contained in K. Let σ be a d-dimensional simplex PL-embedded in the complement $F \setminus K$, and let P' be a $(d+1)$-simplex obtained by taking the pyramid over σ whose apex is any interior point of P. Then projecting K from any interior point of P' onto the boundary of P' gives a PL-embedding of K into this boundary, avoiding the facet σ of P' entirely.

Once K avoids a facet F of P entirely, it is PL-homeomorphic to a subcomplex of a *Schlegel diagram* for P in $\mathbb{R}^d$ [13, Definition 5.5] with F as the bounding facet. $\square$

For the other direction, we use a construction of J. Shewchuk.

Theorem 5 (Shewchuk) *Let K be a polyhedral complex PL-embedded in $\mathbb{R}^d$. Then K is PL-homemorphic to a subcomplex of the boundary of a $(d+1)$-polytope.*

Proof. Consider an arrangement $\mathcal{A} = \{H_i\}$ of finitely many affine hyperplanes in $\mathbb{R}^d$ with the property that every polytope P in K is an intersection of closed halfspaces bounded by some subset of the hyperplanes in $\mathcal{A}$; since K contains only finitely many polytopes, such arrangements exist.

Let K' be the subdivision of K induced by its intersection with the hyperplanes of $\mathcal{A}$, so that K' is a finite subcomplex of the polyhedral subdivision $\hat{K}$ of $\mathbb{R}^d$ induced by $\mathcal{A}$. Then $\hat{K}$ is a *regular* (or *coherent* [5, Definition 7.2.3]) subdivision; its cells are exactly the domains of linearity for the piecewise-linear convex function

$$\begin{array}{rcl} f : \mathbb{R}^d & \to & \mathbb{R} \\ x & \mapsto & \sum_i d(x, H_i) \end{array}$$

in which $d(x, H)$ denotes the (piecewise-linear, convex) function defined as the distance from x to the affine hyperplane H. Since K is finite, f achieves a maximum value, say M, on K. Then for any $\epsilon > 0$, the $(d+1)$-dimensional convex polytope

$$\{(x, x_{d+1}) \in \mathbb{R}^{d+1} \mid f(x) \leq x_{d+1} \leq M + \epsilon\}$$

contains the graph

$$\{(x, f(x)) \mid x \in K\}$$

of the restricted function $f|_K$ as a polyhedral subcomplex of its lower hull. Furthermore, the projection $\mathbb{R}^{d+1} \to \mathbb{R}^d$ gives an isomorphism of this subcomplex onto K'. $\square$

Theorem 2.2 will follow immediately from the following construction of B. Mazur, which is famous in the topology community.

Proposition 6 *$\mathbb{R}^4$ contains a PL-embedded finite simplicial complex K that triangulates a contractible 4-manifold, but whose boundary is not simply-connected. In particular, K is Cohen–Macaulay over every field $\Bbbk$ but not homeomorphic to a 4-ball.*

Proof. Mazur [7, Corollary 1] constructs a finite simplicial complex K that is contractible, has non-simply-connected boundary ∂K, and enjoys the further property that its "double" $2K$ (obtained by identifying two disjoint copies of K along their boundaries) is *PL*-isomorphic to the boundary of a 5-cube. Thus K is *PL*-embedded as a proper subset of the boundary of a 5-polytope, and hence has a *PL*-embedding in $\mathbb{R}^4$ by Proposition 4. $\square$

Proof of Theorem 2.2. This is a consequence of Theorem 5 and Proposition 6. $\square$

We next turn to Theorem 2.1. Fix a field $\Bbbk$. For each nonnegative integer d, consider two related assertions A_d and A'_d concerning finite d-dimensional polyhedral complexes, where we write 'CM' for 'Cohen–Macaulay over $\Bbbk$'.

A_d: Every *PL*-embedded CM d-complex in $\mathbb{R}^d$ is homeomorphic to a d-ball.

A'_d: Every *PL*-embedded CM d-complex in $\mathbb{R}^d$ is a d-manifold with boundary.

Assertion A_d is false for $d = 4$, as shown by Proposition 6. We wish to show it is true for $d \leq 3$, as this would in particular prove the assertion of Theorem 2.1.

Throughout the remainder of this section, all homology and cohomology groups are reduced, and taken with coefficients in $\Bbbk$. We will also use implicitly without further mention the fact that any Cohen–Macaulay d-complex K embedded in $\mathbb{R}^d$ must necessarily be $\Bbbk$-acyclic: the Cohen–Macaulay hypothesis gives $H_i(K) = 0$ for $i < d$, and Alexander duality within the one-point compactification of $\mathbb{R}^d$ implies $H_d(K) = 0$.

In the proof of the next lemma, we use the notion of *links* (sometimes also called *vertex figures*) of faces (polytopes) F in a polyhedral complex K that is *PL*-embedded in $\mathbb{R}^d$. For each face F, we (noncanonically) construct a polyhedral complex $\mathrm{link}_K(F)$ that models the link. First, write $\mathbb{R}^d/F$ for the quotient of $\mathbb{R}^d$ by the unique linear subspace parallel to the affine span of F. Then choose a small simplex σ containing the point $F/F \in \mathbb{R}^d/F$ in its interior. Each face G of K containing F has an image G/F in $\mathbb{R}^d/F$ whose intersection with each face of σ is a polytope. These polytopes constitute the faces of a polyhedral complex *PL*-embedded in the boundary of σ, and we take $\mathrm{link}_K(F)$ to be this complex.

Lemma 7 *If assertion A_δ holds for every $\delta < d$ then assertion A'_d holds.*

Proof. Assume that K satisfies the hypotheses of A'_d. To show that K is a d-manifold with boundary, it suffices to show that $\mathrm{link}_K(F)$ is either a δ-sphere or a δ-ball for every e-dimensional face F of K, where $\delta = d-e-1$. We use the fact that the link of any face in any Cohen–Macaulay d-complex is Cohen–Macaulay. This holds in our case because Cohen–Macaulayness is a topological property (see Section 1), so we can barycentrically subdivide and use the corresponding fact for simplicial complexes (which follows from Reisner's criterion for Cohen–Macaulayness via links [10, §II.4]).

By construction, $\mathrm{link}_K(F)$ is a δ-dimensional polyhedral complex PL-embedded in the boundary of the small $(\delta+1)$-simplex σ around F/F. If the barycenter of F is an interior point of the manifold K, then $\mathrm{link}_K(F)$ is a polyhedral subdivision of the entire (topologically δ-spherical) boundary of σ; otherwise it is embedded as a proper subset. In the latter case, Proposition 4 and assertion A_δ apply to show that $\mathrm{link}_K(F)$ is a δ-ball, as desired. $\square$

Theorem 8 *Assertion A_d holds for $d \leq 3$.*

Proof. Assertions A_0, A_1 are trivial. Together they imply assertion A_2' via Lemma 7. From this, deducing the stronger assertion A_2 is a straightforward exercise using

- the fact that the boundary ∂K is a disjoint union of 1-spheres (possibly nested) embedded in $\mathbb{R}^2$,
- the Jordan Curve Theorem, and
- $H_0(K) = H_1(K) = 0$.

To prove A_3, we may assume A_3' by Lemma 7, and hence assume that K is a Cohen–Macaulay 3-manifold with boundary, embedded in $\mathbb{R}^3$. Thus $H_1(K) = 0$, and hence Lemma 9 below forces $H_1(\partial K) = 0$. Since ∂K is orientable, this implies that ∂K is a disjoint union of (possibly nested) 2-spheres. It is then another straightforward exercise using the Jordan–Brouwer Separation Theorem, along with the fact that $H_0(K) = H_2(K) = 0$, to deduce that ∂K must consist of a single 2-sphere, with K its interior. The Alexander–Schoenflies Theorem then implies that K is a 3-ball. $\square$

Proof of Theorem 2.1. Immediate from Theorem 8 and Proposition 4. $\square$

The authors thank T.-J. Li for pointing out the following lemma and proof (cf. [11, proof of Theorem 6.40]), which was used in the proof of Theorem 8.

Lemma 9 *For any compact 3-manifold K with boundary ∂K,*

$$\dim_{\Bbbk} H_1(K;\Bbbk) \quad \geq \quad \frac{1}{2}\dim_{\Bbbk} H_1(\partial K;\Bbbk).$$

Proof. Consider the following diagram, in which the two squares commute:

$$\begin{tikzcd}
\mathrm{Hom}(H_1(K),\Bbbk) \arrow[r, "\mathrm{Hom}(i_*,\Bbbk)"] & \mathrm{Hom}(H_1(\partial K),\Bbbk) & \\
H^1(K) \arrow[u] \arrow[r, "i^*"] \arrow[d] & H^1(\partial K) \arrow[u] \arrow[d] & \\
H_2(K,\partial K) \arrow[r, "j_*"] & H_1(\partial K) \arrow[r, "i_*"] & H_1(K)
\end{tikzcd}$$

The vertical maps are all isomorphisms. The two vertical maps in the top square come from the universal coefficient theorem relating cohomology and homology with coefficients in $\Bbbk$. The two vertical maps in the bottom square are duality isomorphisms, the left coming from Poincaré–Lefschetz duality for $(K, \partial K)$ and the right from Poincaré duality for ∂K.

The inclusion $\partial K \overset{i}{\hookrightarrow} K$ induces three of the horizontal maps. The last row is exact at its middle term, forming part of the long exact sequence for the pair $(K, \partial K)$, in which j_* is a connecting homorphism. Thus

$$\mathrm{nullity}(i_*) \quad = \quad \mathrm{rank}(j_*) \quad = \quad \mathrm{rank}(i^*) \quad = \quad \mathrm{rank}(\mathrm{Hom}(i_*, \Bbbk)) \quad = \quad \mathrm{rank}(i_*).$$

On the other hand,

$$\begin{aligned} \dim_{\Bbbk} H_1(\partial K) &= \mathrm{rank}(i_*) + \mathrm{nullity}(i_*) \\ &= 2\,\mathrm{rank}(i_*) \\ &\leq 2 \dim_{\Bbbk} H_1(K), \end{aligned}$$

which completes the proof. □

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