# CLASSIFICATION OF DING'S SCHUBERT VARIETIES: FINER ROOK EQUIVALENCE

MIKE DEVELIN, JEREMY L. MARTIN, AND VICTOR REINER

Abstract. K. Ding studied a class of Schubert varieties $X_\lambda$ in type A partial flag manifolds, indexed by integer partitions $\lambda$ and in bijection with dominant permutations. He observed that the Schubert cell structure of $X_\lambda$ is indexed by maximal rook placements on the Ferrers board $B_\lambda$, and that the integral cohomology groups $H^*(X_\lambda;\, \mathbb{Z})$, $H^*(X_\mu;\, \mathbb{Z})$ are additively isomorphic exactly when the Ferrers boards $B_\lambda, B_\mu$ satisfy the combinatorial condition of *rook-equivalence*.

We classify the varieties $X_\lambda$ up to isomorphism, distinguishing them by their graded cohomology rings with integer coefficients. The crux of our approach is studying the nilpotence orders of linear forms in the cohomology ring.

## 1. Introduction

The goal of this paper is to classify up to isomorphism a certain class of Schubert varieties within partial flag manifolds of type $A$. Although this is partly motivated as a first step toward the isomorphism classification of all Schubert varieties, we choose here to explain instead our original motivation, stemming from rook theory in combinatorics.

A *board* $B$ is a subset of the squares on an $N \times N$ chessboard, and a *$k$-rook placement* on $B$ is a subset of $k$ squares in $B$, no two in a single row or column. Kaplansky and Riordan [9] considered the problem of when two boards $B, B'$ are *rook-equivalent*, that is, when for each $k \geq 0$, the number $R_k(B)$ of $k$-rook placements is the same as $R_k(B')$.

Foata and Schützenberger [4] solved the problem for the well-behaved subclass of *Ferrers boards* $B_\lambda$; these are the usual Ferrers diagrams associated to partitions[1]

$$\lambda = (0 \leq \lambda_1 \leq \ldots \leq \lambda_n) \tag{1}$$

having all squares left-justified in their row, with $\lambda_1$ squares in the bottom row, $\lambda_2$ in the next, etc. They showed that each rook-equivalence class of Ferrers boards has a unique representative which is a *strict* partition, i.e., satisfying $\lambda_i < \lambda_{i+1}$. Goldman, Joichi and White [8] re-proved this result by showing that $B_\lambda$ and $B_\mu$ are rook-equivalent if and only if the multisets of integers $\{\lambda_i - i\}_{i=1}^n$ and $\{\mu_i - i\}_{i=1}^n$ coincide.

Garsia and Remmel [6] defined *$q$-rook polynomials* $R_k(B_\lambda, q)$ that $q$-count the $k$-rook placements on $B_\lambda$ by a certain "inversion" statistic generalizing inversions of permutations. They also showed that the problem of $q$-rook equivalence is the same as that of rook equivalence. When $\lambda_i \geq i$ for each $i$, this can be deduced from a product formula for $R_n(B_\lambda, q)$ that counts placements of $n$ rooks: up to a factor of $q$ it is

$$\prod_{i=1}^{n} [\lambda_i - i + 1]_q \tag{2}$$

where $[m]_q := \frac{q^m - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{m-1}$.

*Date*: August 24, 2004.

2000 *Mathematics Subject Classification.* Primary 14M15; Secondary 05E05.

*Key words and phrases.* Schubert variety, rook placement, Ferrers board, flag manifold, cohomology ring, nilpotence.

This work was completed while the first author was visiting the Univeristy of Minnesota. First author supported by the American Institute of Mathematics. Second author partially supported by an NSF Postdoctoral Fellowship. Third author partially supported by NSF grant DMS–0245379.

[1] NB: we are writing our partitions with the parts in weakly increasing order, contrary to usual combinatorial conventions, but more convenient in this setting.

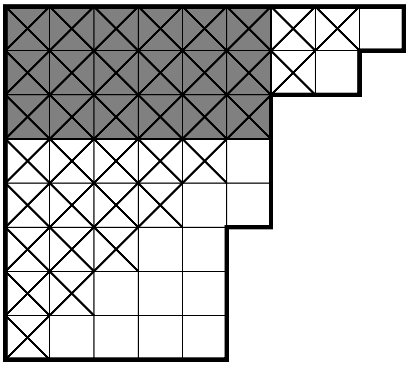

FIGURE 1. A decomposable partition $\lambda$. The unshaded regions are $\lambda^{(1)}$ and $\lambda^{(2)}$.

K. Ding [2, 3] interpreted this product as the Poincaré series for a certain algebraic variety $X_\lambda$ which he called a *partition variety*. Fix a standard complete flag of subspaces

$$0 \subset \mathbb{C}^1 \subset \cdots \mathbb{C}^{N-1} \subset \mathbb{C}^N$$

and define

$$X_\lambda := \{\text{flags } 0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \subset \mathbb{C}^N : \dim_{\mathbb{C}} V_i = i \text{ and } V_i \subset \mathbb{C}^{\lambda_i}\}. \tag{3}$$

The set $X_\lambda$ may be endowed with the structure of a smooth complex projective variety, and (although not stated explicitly in [2]) is in fact a smooth Schubert variety inside the partial flag manifold $X_{N^n}$, where $N^n$ denotes the rectangular board with $n$ rows and $N$ columns. As we shall explain below, the Schubert varieties arising in this way are (in the notation of [5, §10.2]) those of the form $X_w$, where $w$ is a *312-avoiding* permutation. Equivalently, the fundamental cohomology class $[X_w]$ is represented by a Schubert polynomial indexed by a *dominant* or *132-avoiding* permutation. (See [5] for a reference on Schubert varieties, and [10] for a detailed treatment of Schubert polynomials.) Ding observed that the Schubert cell structure inherited by $X_\lambda$ has cells indexed by $n$-rook placements on $B_\lambda$, and with the dimension of the cell governed by Garsia and Remmel's inversion statistic. Since these cells are all even-dimensional, their (co)homology is free abelian, occurring only in even dimension, and the Poincaré series of $X_\lambda$ is given by the $q$-rook polynomial formula (2). From this Ding concluded [3] that two partition varieties $X_\lambda, X_\mu$ have *additively* isomorphic (co)homology groups if and only if $B_\lambda$ and $B_\mu$ are rook-equivalent.

It is natural to ask when two such Ding partition varieties $X_\lambda, X_\mu$ have isomorphic (graded) cohomology *rings*, or even when they are isomorphic *as varieties*. The main result of this paper is that the answers to both questions are the same. We make use of recent results of Gasharov and the third author [7], giving simple explicit cohomology ring presentations[2] for a more general class of Schubert varieties in partial flag manifolds (those defined by a conjunction of inclusion conditions of the forms $\mathbb{C}^j \subset V_i$ and $V_i \subset \mathbb{C}^j$).

To state our main result, we first note one trivial source of isomorphisms among the partition varieties $X_\lambda$. We assume throughout that $\lambda_i \geq i$ for every $i$, for otherwise $X_\lambda = \emptyset$. However, if $\lambda_k = k$ for some $k$, then the condition $V_k \subset \mathbb{C}^k$ with $\dim_{\mathbb{C}} V_k = k$ forces $V_k = \mathbb{C}^k$, so that $X_\lambda$ is isomorphic to $X_{\lambda^{(1)}} \times X_{\lambda^{(2)}}$, where

$$\lambda^{(1)} = (\lambda_1, \ldots, \lambda_{k-1}),$$
$$\lambda^{(2)} = (\lambda_{k+1} - k, \ldots, \lambda_n - k).$$

Here if $k = n$, so that $\lambda_n = n$, there is no partition $\lambda^{(2)}$ and we simply note that $X_\lambda \cong X_{\lambda^{(1)}}$.

Say that $\lambda$ is *decomposable* if this occurs (i.e., if $\lambda_k = k$ for some $k$), and *indecomposable* otherwise. For example, the partition $\lambda = (5, 5, 5, 6, 6, 6, 8, 9)$ shown in Figure 1 is decomposable since $\lambda_6 = 6$. In this case, one has $\lambda^{(1)} = (5, 5, 5, 6, 6)$ and $\lambda^{(2)} = (2, 3)$, as shown in the figure.

[2]It is amusing that these cohomology ring presentations for Schubert varieties are often derived for the purposes of enumerative geometry (Schubert calculus), but are used here for a different classical topological purpose, namely distinguishing non-homeomorphic spaces.

Iterating this, one can decompose $\lambda$ into a multiset of indecomposable partitions $\{\lambda^{(i)}\}_{i=1}^r$, which we will call its *indecomposable components*, such that

$$X_\lambda \cong X_{\lambda^{(1)}} \times \cdots \times X_{\lambda^{(r)}}. \tag{4}$$

Our main result is that the Schubert varieties $X_\lambda$ are determined up to isomorphism by these multisets of indecomposable components. It should be compared with the result of Goldman, Joichi and White [8], which can now be rephrased: the varieties $X_\lambda$ are determined up to additive (co-)homology isomorphism by the multisets of numbers $\{\lambda_i - i\}$.

**Theorem 1.1.** *The following are equivalent for two partitions, $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_{m'})$:*

(i) *The multisets of indecomposable components, $\{\lambda^{(i)}\}_{i=1}^r$ and $\{\mu^{(i)}\}_{i=1}^{r'}$, are identical.*
(ii) *There is an isomorphism $X_\lambda \cong X_\mu$ of varieties.*
(iii) *There is a graded isomorphism of integer cohomology rings $H^*(X_\lambda;\, \mathbb{Z}) \cong H^*(X_\mu;\, \mathbb{Z})$.*

The implications (i) $\implies$ (ii) $\implies$ (iii) are clear; the hard part is to show that (iii) implies (i). It turns out that the key to this implication lies in understanding the *nilpotence orders* of cohomology elements $f \in H^2(X_\lambda;\, \mathbb{Z})$; that is, the least $k$ for which $f^k = 0$.

In Section 2, we review some of Ding's results, and re-prove somewhat more directly the presentation for $H^*(X_\lambda;\, \mathbb{Z})$ from [7]. The three sections that follow are the technical heart of the paper, categorizing elements in $H^2(X^\lambda;\, \mathbb{Z})$ of minimal nilpotence order. We begin in Section 3 by setting up some Gröbner basis machinery that we shall use throughout (for a general reference on Gröbner basis theory, see [1]). Section 4 deals with nilpotents in the cohomology of the complete flag variety (that is, when $\lambda$ is a square Ferrers board) and Section 5 treats the case of arbitrary $X_\lambda$. Using these tools, we prove in Section 6 that an indecomposable partition $\lambda$ may be recovered from the structure of $H^*(X_\lambda;\, \mathbb{Z})$ as a graded $\mathbb{Z}$-algebra. Finally, in Section 7, we show that in the general case, $H^*(X_\lambda;\, \mathbb{Z})$ has an essentially unique decomposition as a tensor product of graded $\mathbb{Z}$-algebras, whose factors correspond to the indecomposable components of the partition $\lambda$.

It is curious that this unique tensor decomposition fails if instead of the integer cohomology ring $H^*(X_\lambda; \mathbb{Z})$ one takes cohomology with coefficients in a ring where 2 is invertible; see Remark 7.6 below.

## 2. Review of $X_\lambda$ and the presentation of $H^*(X_\lambda;\, \mathbb{Z})$

For the sake of completeness, and also to collect facts for future use, we begin by re-proving some of Ding's results from [2], and re-derive somewhat more directly the presentation given in [7] for the cohomology ring of $X_\lambda$. Throughout this paper, all cohomology groups and rings are taken with integer coefficients unless otherwise specified. We begin by identifying the Schubert varieties that arise as Ding's varieties $X_\lambda$. (See [5, §10.6] for more information on Schubert varieties, and [10] for a detailed treatment of Schubert polynomials.)

Let $\mathfrak{S}_N$ be the symmetric group of permutations of $\{1, \ldots, N\}$, and let $\mathfrak{S}_{\{n+1,n+2,\ldots,N\}}$ be the subgroup of permutations fixing $\{1, \ldots, n\}$ pointwise. Consider the partial flag variety

$$X_{N^n} \;=\; \big\{\text{flags } 0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^N : \;\; \dim V_i = i\big\}.$$

Let $w = w_1 \ldots w_n \in \mathfrak{S}_N$ be a permutation which is a maximum-length representative for its coset in $\mathfrak{S}_N/\mathfrak{S}_{\{n+1,n+2,\ldots,N\}}$. The corresponding Schubert variety $X_w \subset X_{N^n}$ is defined to be

$$X_w \;=\; \big\{\text{flags } 0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{C}^N : \;\; \dim V_i = i,\; \dim V_i \cap \mathbb{C}^j \geq \#\{k \leq i : w_k \leq j\}\big\}.$$

Let $\lambda$ be a partition of the form (1), and let $N = \lambda_n$. It is easy to check that Ding's variety $X_\lambda$ coincides with the Schubert variety $X_w \subset X_{N^n}$, where $w$ is the unique permutation given by the recursive rule

$$w_i \;=\; \max\left(\{1, \ldots, \lambda_i\} \setminus \{w_1, \ldots, w_{i-1}\}\right).$$

Note that if $n = N$, then $w$ corresponds to the maximal rook placement on the Ferrers board $B_\lambda$ given by the following algorithm: let $i$ increase from 1 to $n$, and for each $i$, place a rook in row $i$ and column $w_i$, where $w_i$ is the rightmost square in row $i$ whose column does not already contain a rook. For instance, if $\lambda = (2, 4, 4, 5, 5)$, then $w = 24351 \in \mathfrak{S}_5$. (If $n < N$, then we must first augment $\lambda$ with $N - n$ additional rows of length $\lambda_n$.) It is not hard to verify that the permutations $w$ obtained in this way are exactly those which are *312-avoiding*; that is, there do not exist $i, j, k$ for which $i < j < k$ and $w(i) > w(k) > w(j)$. Equivalently,

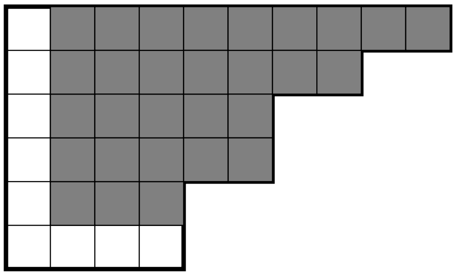

FIGURE 2. A partition $\lambda$ and the subpartition $\nu$ (shaded) such that $H^*(X_\lambda) \,/\, \langle x_1 \rangle = H^*(X_\nu)$.

the cohomology class $[X_w] \in H^*(X_{N^n})$ is represented by a Schubert polynomial which is a single monomial, namely the Schubert polynomial indexed by the dominant (or 132-avoiding) permutation $w_0 w$, where $w_0$ is the unique permutation of maximal length. (We thank Ezra Miller for discussions clarifying these points.)

Because $X_\lambda$ is a Schubert variety, it comes equipped with a Schubert cell decomposition, having cells in only even real dimensions. As observed by Ding, this has important consequences:

**Theorem 2.1** (Ding [2])**.** *The integral cohomology ring $H^*(X_\lambda; \mathbb{Z})$ is free abelian (that is, it has no torsion), is nonzero only in even homological degrees, and has Poincaré series*

$$\mathrm{Poin}(X_\lambda, q) := \sum_{i \geq 0} q^i \operatorname{rank}_{\mathbb{Z}} H^{2i}(X_\lambda;\, \mathbb{Z}) = \prod_{i=1}^{n} [\lambda_i - i + 1]_q.$$

*Proof.* The cohomology is free abelian and concentrated in even degrees because the Schubert cell decomposition for the Schubert variety $X_\lambda$ has cells only in even dimensions.

For the assertion about the Poincaré series, we will induct on $n$. The map

$$\begin{array}{ccc} X_\lambda & \rightarrow & \mathbb{P}(\mathbb{C}^{\lambda_1}) \cong \mathbb{P}_{\mathbb{C}}^{\lambda_1 - 1} \\ \{V_i\}_{i=1}^n & \mapsto & V_1 \end{array}$$

is an (algebraic) fiber bundle, with fiber isomorphic to $X_\nu$, where

$$\nu = (\nu_1, \ldots, \nu_{n-1}) = (\lambda_2 - 1, \ldots, \lambda_n - 1)$$

is the partition obtained by removing the first row and column from $\lambda$ (see Figure 2). The Leray-Serre spectral sequence is particularly simple in this situation, because both base and fiber are simply-connected (again due to the Schubert cell decomposition) and have homology concentrated in even dimension. This causes the spectral sequence to degenerate at the $E^1$-page, yielding

$$\mathrm{Poin}(X_\lambda, q) = \mathrm{Poin}(X_\nu, q) \cdot \mathrm{Poin}(\mathbb{P}_{\mathbb{C}}^{\lambda_1 - 1}, q).$$

The assertion about $\mathrm{Poin}(X_\lambda, q)$ now follows by induction on $n$, using the fact that $[m]_q = \mathrm{Poin}(\mathbb{P}_{\mathbb{C}}^{m-1}, q)$. □

We now set about deriving the presentation for $H^*(X_\lambda)$. To this end, we recall Borel's picture for the cohomology of the complete flag manifold $GL_N(\mathbb{C})/B = X_{N^N}$ and the partial flag manifold $X_{N^n}$; see [5, Chapter 10, §3, §6]. We will use the following notation for symmetric functions in various sets of variables. For integers $1 \leq i \leq j \leq N$ and $m \geq 0$, define the $m^{th}$ *elementary* and *complete homogeneous* symmetric functions, respectively, by

$$\begin{aligned} e_m(i,j) &:= e_m(x_i, x_{i+1}, \ldots, x_j), \\ h_m(i,j) &:= h_m(x_i, x_{i+1}, \ldots, x_j), \end{aligned}$$

where

$$e_m(N) := e_m(1,N) = e_m(x_1,\dots,x_N) = \sum_{1\le i_1<\cdots<i_m\le N} x_{i_1}\cdots x_{i_m},$$
$$h_m(N) := h_m(1,N) = h_m(x_1,\dots,x_N) = \sum_{1\le i_1\le\cdots\le i_m\le N} x_{i_1}\cdots x_{i_m}.$$

According to Borel's picture, $H^*(X_{N^N}) \cong \mathbb{Z}[x_1,\dots,x_N]/J$, where $J = \langle e_1(N),\dots,e_N(N)\rangle$ is the ideal generated by all symmetric functions of positive degree, and where $x_i$ represents the negative of $c_1(\mathcal{L}_i)$, the first Chern class of the line bundle $\mathcal{L}_i$ on $GL_N(\mathbb{C})/B$ whose fiber over the flag $\{V_i\}_{i=0}^N$ is $V_i/V_{i-1}$.

Furthermore, the surjection $X_{N^N} \to X_{N^n}$ which forgets the subspaces of dimension greater than $n$ in a complete flag induces a map $H^*(X_{N^n}) \to H^*(X_{N^N})$ which turns out to be injective, and the image of $H^*(X_{N^n})$ is identified with the invariant subring $H^*(X_{N^N})^{\mathfrak{S}_{\{n+1,n+2,\dots,N\}}}$. This invariant subring may be presented as $S/J'$, where

$$S = \mathbb{Z}[x_1,\dots,x_N]^{\mathfrak{S}_{\{n+1,n+2,\dots,N\}}} = \mathbb{Z}[x_1,\dots,x_n,\ e_1(n+1,N),\dots,e_{N-n}(n+1,N)]$$

and $J' = \langle e_1(N),\dots,e_N(N)\rangle$ is the ideal of $S$ with the same generators as $J$.

The relations in the ideals $J$ and $J'$ induce further relations among various symmetric functions, which we record here for future use.

**Proposition 2.2** (cf. [5, p. 163, eqn. (4)])**.** *For every $m \in \{1,2,\dots,N\}$ and $j \ge 0$, one has*

$$h_j(m) \equiv (-1)^j e_j(m+1,N) \pmod{J}.$$

*Proof.*

$$\prod_{i=1}^m (1+x_it) \prod_{i=m+1}^N (1+x_it) = \prod_{i=1}^N (1+x_it) = \sum_{j=0}^n e_j(N)t^j \equiv 1 \pmod{J}.$$

Hence

$$\sum_{j=0}^\infty (-1)^j h_j(m)t^j = \prod_{i=1}^m (1+x_it)^{-1} \equiv \prod_{i=m+1}^N (1+x_it) = \sum_{j=0}^{N-m} e_j(m+1,N)t^j \pmod{J}.$$

Now comparing coefficients of powers of $t^j$ yields the desired equality. ☐

We now give the general presentation for the integral cohomology of $X_\lambda$ (as pointed out in [7, Remark 3.3]).

**Theorem 2.3.** *Let $\lambda$ be a partition with $1 \le \lambda_1 \le \cdots \le \lambda_n = N$ and $\lambda_i \ge i$ for all $i$. Let*

$$R^\lambda := \mathbb{Z}[x_1,\dots,x_n]/I_\lambda$$

*where $I_\lambda := \langle h_{\lambda_i-i+1}(i) : 1 \le i \le n\rangle$.*

*Then there is a (grade-doubling) ring isomorphism*

$$R^\lambda \to H^*(X_\lambda;\mathbb{Z})$$

*sending $x_i$ to $-c_1(\mathcal{L}_i)$. Here $\mathcal{L}_i$ is the same line bundle as above, but restricted to $X_\lambda$ from the partial flag manifold $X^{N^n}$.*

*Proof.* The obvious inclusion $X_\lambda \hookrightarrow X_{N^n}$ induces a map $H^*(X_{N^n}) \to H^*(X_\lambda)$. This ring map is surjective, because $X_\lambda$ inherits from $X_{N^n}$ a decomposition into Schubert cells, and the dual cocycles to these (even-dimensional) cells additively generate the cohomology in each case.

There are further relations on the Chern classes $x_i$ in $H^*(X_\lambda)$ due to the conditions $V_i \subset \mathbb{C}^{\lambda_i}$. Specifically, the bundle $\mathbb{C}^N/V_i$ on $X_\lambda$ will have the same Chern classes as the direct sum $\mathbb{C}^N/\mathbb{C}^{\lambda_i} \oplus \mathbb{C}^{\lambda_i}/V_i$, in which $\mathbb{C}^N/\mathbb{C}^{\lambda_i}$ is a trivial bundle. Thus when restricted to $X_\lambda$, the bundle $\mathbb{C}^N/V_i$ will have the same Chern classes as the bundle $\mathbb{C}^{\lambda_i}/V_i$ of rank $\lambda_i - i$. Hence its Chern classes $c_m = \pm e_m(i+1,N)$ for $m > \lambda_i - i$ inside $H^*(X_\lambda)$ must vanish. Consequently, we have a surjection of rings

$$\mathbb{Z}[x_1,\dots,x_n,e_1(n+1,N),e_2(n+1,N),\dots,e_{N-n}(n+1,N)]/J_\lambda \ \to\ H^*(X_\lambda) \tag{5}$$

where

$$J_\lambda := J' + \langle e_j(i+1,N) \ : \ 1 \le i \le n \text{ and } j > \lambda_i - i \rangle \,.$$

We now manipulate the quotient ring $\mathbb{Z}[x_1,\ldots,x_N]^{\mathfrak{S}_{\{n+1,n+2,\ldots,N\}}}/J_\lambda$ on the left of (5). We use Proposition 2.2 to draw two conclusions:

(i) Applying Proposition 2.2 with $m = n$ shows that $H^*(X_{N^n})$ and $H^*(X_\lambda)$ are generated as algebras by $x_1,\ldots,x_n$, since their generators of the form $e_i(n+1,N)$ can be expressed modulo $J'$ as (symmetric) polynomials in $x_1,\ldots,x_n$.

(ii) Applying it with $m = i$ for $1 \le i \le n$ shows that $h_{\lambda_i - i+1}(i) = 0$ in $H^*(X_\lambda)$, because for each $j \ge \lambda_i - i$, $h_j(i)$ is congruent modulo $J'$ to $\pm e_j(i+1,N)$.

Consequently, there is a surjection of rings

$$\mathbb{Z}[x_1,\ldots,x_n]/\left\langle h_{\lambda_i-i+1}(i) : 1 \le i \le n \right\rangle \ \to \ H^*(X_\lambda). \tag{6}$$

On the other hand, the set

$$\{h_{\lambda_i-i+1}(i) : 1 \le i \le n\}$$

is a Gröbner basis for $I_\lambda$ with respect to the lexicographic term order on $\mathbb{Z}[x_1,\ldots,x_n]$ given by $x_1 < \cdots < x_n$. Indeed, the initial term of $h_{\lambda_i-i+1}(i)$ is $x_i^{\lambda_i-i+1}$, so these generators have pairwise relatively prime, monic initial terms. Consequently, the quotient ring on the left of (6) is a free $\mathbb{Z}$-module of rank $\prod_{i=1}^n(\lambda_i - i+1)$, with $\mathbb{Z}$-basis given by the *standard monomials* (those divisible by none of the initial terms), namely $\{x_1^{a_1}\cdots x_n^{a_n} : a_i \le \lambda_i - i\}$. Since Theorem 2.1 implies that $H^*(X_\lambda)$ is a free $\mathbb{Z}$-module of the same rank, the surjection (6) must be an isomorphism. □

For example, if $\lambda$ is the partition shown in Figure 1, then the Gröbner basis for $I_\lambda$ is

$$h_5(1),\ h_4(2),\ h_3(3),\ h_3(4),\ h_2(5),\ h_1(6),\ h_2(7),\ h_2(8).$$

The previous proof shows that $I_\lambda$ is the *elimination ideal*

$$I_\lambda = \mathbb{Z}[x_1,\ldots,x_n] \cap J_\lambda.$$

This observation has some useful corollaries, which can also be proved by direct combinatorial/algebraic arguments avoiding any use of geometry. The first corollary is the algebraic manifestation of the (surjective) map $R^\lambda \to R^\mu$ induced by the inclusion of Schubert varieties $X_\lambda \to X_\mu$.

**Corollary 2.4.** *Let $\lambda$ and $\mu$ be partitions, both with $n$ nonzero rows, such that $\lambda \supset \mu$.*
*Then $I_\lambda \subset I_\mu$, and consequently, $R^\lambda$ is a quotient of $R^\mu$.*

*Proof.* By definition of $J_\lambda$, one has $J_\lambda \subset J_\mu$ in this situation. □

**Corollary 2.5.** *If $\lambda_i = \lambda_{i+1} = \cdots = \lambda_j$ for some $i < j$ then the ideal $I_\lambda$ is invariant under permutations of the variables $x_i, x_{i+1},\ldots,x_j$.*

*Proof.* It suffices to show that $J_\lambda$ has this same invariance. Note that the generators for $J_\lambda$ of the form

$$e_m(i'+1,N) \quad \text{for } i \le i' < j \text{ and } m > \lambda_{i'} - i'$$

are all redundant, as they lie in the ideal generated by $\{e_m(j+1,N) : m > \lambda_j - j\}$. The latter generators, and all other generators of $J_\lambda$, are symmetric in $x_i, x_{i+1},\ldots,x_j$. □

## 3. Two reduced Gröbner bases

This section examines the Gröbner bases for $I_\lambda$ for two extreme cases of indecomposable partitions. In both cases, one can describe the (unique) reduced Gröbner basis, which will be used in an essential way later in the paper. We assume some familiarity with "Gröbner basics" on the reader's part; a good reference for this topic is [1].

We begin with some notation regarding Gröbner reduction. Since the generators $\{h_{\lambda_i-i+1}(i) : 1 \le i \le n\}$ form a Gröbner basis for $I_\lambda$ with respect to a lexicographic monomial ordering in which $x_1 < \ldots x_n$, we can compute in the quotient $R^\lambda$ by reducing polynomials modulo this Gröbner basis. For a polynomial $f \in \mathbb{Z}[x_1,\ldots,x_n]$, we will denote by $\overline{f}$ this *standard form* of $f$. That is, $\overline{f}$ is the unique $\mathbb{Z}$-linear combination of standard monomials $\{x_1^{a_1}\cdots x_n^{a_n} : a_i \le \lambda_i - i\}$ which is congruent to $f$ modulo $I_\lambda$. Given a standard

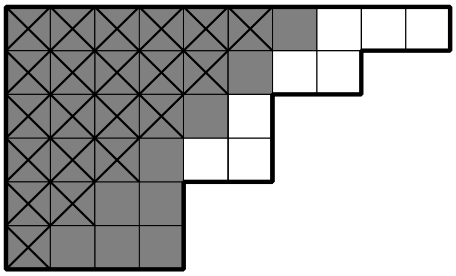

FIGURE 3. An indecomposable partition $\lambda$ and its core subpartition $\mu$ (shaded).

monomial $M$, we denote by $[M]\overline{f}$ the coefficient of $M$ in $\overline{f}$. (This is well-defined, because the standard monomials form a basis for $\mathbb{Z}[x_1, \ldots, x_n]/I_\lambda$ as a free $\mathbb{Z}$-module.)

Let $\lambda = (\lambda_1 \leq \cdots \leq \lambda_n)$ and for some fixed $m \leq n$, let $\mu = (\lambda_1, \ldots, \lambda_m)$. Then the fact that we are using a lexicographic order to perform reductions has the following easy consequence (see also [1, §3.1]), which will be used frequently. It can be viewed as an algebraic consequence of the fibration $X_\lambda \to X_\mu$ that forgets the subspaces of dimension greater than $m$ in a flag, which happens to induce an injective map $H^*(X_\mu) \to H^*(X_\lambda)$.

**Proposition 3.1.** *Let $\lambda$ and $\mu$ be related as above. Suppose that $f$ in $\mathbb{Z}[x_1, \ldots, x_n]$ lies in some subalgebra $\mathbb{Z}[x_1, \ldots, x_m]$, where $m \leq n$.*

*Then the images of $f$ in $R^\lambda$ and $R^\mu$ have the same standard form $\overline{f}$.*

Our first extreme case arises when $\lambda$ is an indecomposable partition with $\lambda_i = p$, and $\mu \subset \lambda$ is the smallest indecomposable partition having $\mu_i = p$, namely $\mu = (2, 3, \ldots, i-1, i, p)$.

**Proposition 3.2.** *Let $\mu = (2, 3, \ldots, i-1, i, p)$. With respect to lexicographic order on $\mathbb{Z}[x_1, \ldots, x_m]$ with $x_1 < \cdots < x_m$, the ideal $I_\mu$ has reduced Gröbner basis*

$$\{x_1 h_1(1),\ x_2 h_1(2),\ \ldots,\ x_{i-1} h_1(i-1),\ x_i^{p-i+1} + x_i^{p-i} h_1(i-1)\}. \tag{7}$$

*Proof.* It is easy to see that the elements of (7) form a reduced Gröbner basis with respect to the lexicographic order for whatever ideal they generate. We observe that this ideal may also be presented as

$$\left\langle h_2(1),\ h_2(2),\ \ldots,\ h_2(i-1),\ x_i^{p-i+1} + x_i^{p-i} h_1(i-1) \right\rangle.$$

We will show that this ideal is exactly $I_\mu$. By Theorem 2.3,

$$I_\mu \ = \ \langle h_2(1), h_2(2), \ldots, h_2(i-1), h_{p-i+1}(i) \rangle\,,$$

so it remains only to show that $h_{p-i+1}(i)$ and $x_i^{p-i+1} + x_i^{p-i} h_1(i-1)$ are congruent modulo the ideal $\langle h_2(1), h_2(2), \ldots, h_2(i-1)\rangle$. Since $h_{p-i+1}(i) = \sum_{j=1}^{p-i+1} x_i^j h_{p-i-j+1}(i-1)$, this congruence is immediate from the fact that

$$h_m(\ell) \in \langle h_2(1), h_2(2), \ldots, h_2(\ell)\rangle \ \text{ for } m \geq 2,$$

which is easily proven by double induction on $m$ and $\ell$ via the identity $h_m(\ell) = x_\ell h_{m-1}(\ell) + h_m(\ell-1)$. $\square$

Our second extreme case arises when $\lambda$ is an indecomposable partition with $n$ rows. Let $k = \lambda_1$, and let $\mu$ be the smallest indecomposable partition with $n$ rows and $\mu_1 = k$. That is,

$$\begin{aligned} &\mu_1 = \mu_2 = \cdots = \mu_{k-1} = k, \\ &\mu_i = i+1 \quad \text{for } k \leq i \leq n. \end{aligned} \tag{8}$$

Then $\mu$ is a subpartition[3] of $\lambda$, which we will call the *core* of $\lambda$. For example, the core of $\lambda = (4, 4, 6, 6, 8, 10)$ is the partition $\mu = (4, 4, 4, 5, 6, 7)$ (see Figure 3).

[3]For the purposes of this paper, the statement "$\mu$ is a subpartition of $\lambda$" means that $\mu_i \leq \lambda_i$ for all rows $\mu_i$ of $\mu$. Equivalently, the Ferrers diagram of $\mu$ is contained inside that of $\lambda$, when both are left- and bottom-justified.

**Proposition 3.3.** *For $k < n$, let $\lambda$ be a partition which is its own core.*

*Then the polynomials*

$$\begin{array}{llll} G_1 = h_k(1), & G_2 = h_{k-1}(2), & \ldots, & G_{k-1} = h_2(k-1), \\ G_k = x_k h_1(k), & G_{k+1} = x_{k+1} h_1(k+1), & \ldots, & G_n = x_n h_1(n) \end{array} \tag{9}$$

*form a reduced Gröbner basis for $I_\lambda$ under the reverse lexicographic term order given by $x_1 < x_2 < \ldots < x_n$.*

*Proof.* The initial terms of the $G_i$'s are (in order) $x_1^k,\ x_2^{k-1},\ \ldots,\ x_{k-1}^2,\ x_k^2,\ \ldots,\ x_n^2$. It is evident that no initial term divides any term of any other $G_i$. Therefore, they are a reduced Gröbner basis for the ideal that they generate.

We claim that for every $r \in \{k, k+1, \ldots, n\}$,

$$\langle G_1, \ldots, G_r \rangle \ = \ \langle h_k(1),\ h_{k-1}(2),\ \ldots,\ h_2(k-1),\ h_2(k),\ \ldots,\ h_2(r) \rangle\,.$$

The claim is trivial for $r = k$. For $r > k$, it follows from induction and the observation that $h_2(r) - h_2(r-1) = x_r h_1(r) = G_r$. In particular, the equality for $r = n$ gives $\langle G_1, \ldots, G_r \rangle = I_\lambda$. □

The form of this reduced Gröbner basis has the following consequence, which we will exploit later.

**Corollary 3.4.** *("Stickiness") Let $\lambda$ be an indecomposable partition which is its own core, and $k := \lambda_1$. Let $M$ be a monomial in $x_1, \ldots, x_n$.*

*Then:*

1. *If $k \leq i \leq n$ and $M$ is divisible by $x_i$, then so is $\overline{M}$.*
2. *If $M$ is not divisible by any of the variables $x_k, \ldots, x_n$, then neither is $\overline{M}$.*

*Proof.* (1) is immediate from the previous discussion. For (2), the only Gröbner basis elements that can be used in the reduction of $M$ are $G_1, \ldots, G_{k-1}$, so the reduction process cannot introduce a monomial divisible by any of $x_k, \ldots, x_n$. □

One useful consequence of "stickiness" is the following.

**Corollary 3.5.** *Let $\lambda$ be an indecomposable partition which is its own core, and $k := \lambda_1$. Let $f = \sum_{i=1}^n a_i x_i$ be an element of the degree-one graded piece $R_1^\lambda$ of $R^\lambda$. Decompose $f$ as $f = g + h$, where*

$$g \ = \ \sum_{i=1}^{k-1} a_i x_i, \qquad\qquad h \ = \ \sum_{i=k}^{n} a_i x_i.$$

*If $f^m = 0$ in $R^\lambda$ for some positive integer $m$, then $g^m = 0$ in $R^\lambda$.*

*Proof.* Note that $f^m = g^m + p$, where $p$ is some polynomial divisible by $a_k x_k + \cdots + a_n x_n$. Passing to the standard forms, we find that $0 = \overline{g^m} + \overline{p}$. By Corollary 3.4, no monomial in $\overline{g^m}$ is divisible by a sticky variable (that is, one of $x_k, \ldots, x_n$), but every monomial in $\overline{p}$ is divisible by a sticky variable. Therefore $\overline{g^m} = 0\ (= \overline{p})$. □

## 4. Nilpotence of linear forms in the cohomology of $G/B$

The main result of this section, Theorem 4.1, concerns the nilpotence orders of degree-1 elements in the graded ring $H^*(G/B)$. This result may be of independent interest, and it would be nice to have a geometric explanation for it.

Recall that $H^*(G/B) = R^{n^n} \cong \mathbb{Z}[x_1, \ldots, x_n]/J$, where

$$J = \langle e_i(n) : 1 \leq i \leq n \rangle = I_{n^n} = \langle h_{n-i+1}(i) : 1 \leq i \leq n \rangle\,. \tag{10}$$

We digress to discuss graded $\mathbb{Z}$-algebras and nilpotence. A *standard graded $\mathbb{Z}$-algebra* is a ring $R$ with a $\mathbb{Z}$-module direct sum decomposition $R = \bigoplus_{d \geq 0} R_d$ in which each $R_d$ is a free $\mathbb{Z}$-module, $R_d \cdot R_e \subset R_{d+e}$ and $R$ is generated over the subalgebra $R_0 = \mathbb{Z}$ by $R_1$. Let $R$ be a ring and $f \in R$ a nilpotent element (that is, some power of $f$ is zero). The *nilpotence order* of $f$ is defined as the smallest integer $k$ such that $f^k = 0$; we will sometimes say that $f$ is *$k$-nilpotent.* (So $f$ has nilpotence order 1 if and only if $f = 0$.)

By Theorem 2.3, $R^\lambda = \mathbb{Z}[x_1, \ldots, x_n]/I_\lambda$ is a standard graded $\mathbb{Z}$-algebra, with $R_1^\lambda \cong H^2(X_\lambda; \mathbb{Z})$. Furthermore, every element of $R_1^\lambda$ is nilpotent, since $R^\lambda$ has finite rank as a $\mathbb{Z}$-module. The nilpotence order of these linear forms will be our main tool in distinguishing the rings $R^\lambda$. In this section, we study the case that $\lambda = n^n$; we treat the general case in Section 5.

Note that the images of the variables $x_i$ in $R^{n^n}$ satisfy $x_i^n = 0$. Indeed, by Corollary 2.5, it is sufficient to prove that $x_1^n = 0$, which follows from (10) since $I_{n^n}$ contains the element $h_{n-1+1}(1) = h_n(1) = x_1^n$. In fact, more is true:

**Theorem 4.1.** *Let $f \in H^2(G/B) \cong (R^{n^n})_1$.*

*Then $f$ has nilpotence order greater than or equal to $n$, with equality if and only if $f$ is congruent, modulo $J$, to a scalar multiple of one of the variables $x_1, \dots, x_n$.*

We first show that $n$ is the minimal nilpotence order achieved by any linear form.

**Proposition 4.2.** *Let $f \in R_1^{n^n}$ be a linear form. If $f^{n-1} = 0$, then $f = 0$.*

*Proof.* Let $\hat{f}$ be a preimage of $f$ under the quotient map $\mathbb{Z}[x_1, \dots, x_n] \twoheadrightarrow R^{n^n}$. Then $f^{n-1} = 0$ means $\hat{f}^{n-1} \in J$. By degree considerations, this means that $\hat{f}^{n-1}$ belongs to the ideal

$$I := \langle e_i(n) : 1 \leq i \leq n-1 \rangle \ \subset \ \mathbb{Z}[x_1, \dots, x_n]. \tag{11}$$

Thus it suffices to show that $I$ is a radical ideal, since then $\hat{f} \in I$ and $f = 0$ in $R^{n^n}$. We will show something slightly stronger: that the ideal $I' := \langle e_i(n) : 1 \leq i \leq n-1 \rangle \subset \mathbb{C}[x_1, \dots, x_n]$ is radical. Indeed, any nonzero nilpotent in $\mathbb{Z}[x_1, \dots, x_n]/I$ would give rise to a nonzero nilpotent in $\mathbb{C}[x_1, \dots, x_n]/I'$.

Let $\zeta$ be a primitive $n^{th}$ root of unity. We claim that $I'$ is the vanishing ideal $I(V)$ for the variety $V \subset \mathbb{C}^n$, defined as the union of all lines whose slope vector is any permutation of $(1, \zeta, \dots, \zeta^{n-1})$. Note that there are exactly $(n-1)!$ such lines, because two such slope vectors that differ by multiplication by a root of unity give rise to the same line. Equating coefficients of powers of $t$ in the equation

$$t^n - 1 = \prod_{i=1}^{n} (t - \zeta^i) = \sum_{i=0}^{n} e_i(1, \zeta, \dots, \zeta^{n-1}) t^i$$

shows that $I' \subset I(V)$. For the reverse inclusion, note that $e_1(n), \dots, e_n(n)$ is a regular sequence in $\mathbb{C}[x_1, \dots, x_n]$, and therefore cuts out scheme-theoretically a complete intersection of Krull dimension 1, that is, a set of curves with various multiplicities. By Bézout's Theorem, the sum of the degrees of those curves, counted with multiplicities, must be

$$\deg(e_1(n)) \cdot \deg(e_2(n)) \cdots \deg(e_n(n)) \ = \ 1 \cdot 2 \cdots (n-1) \ = \ (n-1)!$$

But this complete intersection contains at least $(n-1)!$ lines in $V$, each of degree 1. Therefore it contains no other curves, and each line occurs with multiplicity 1; that is, $I' = I(V)$. □

To complete the proof of Theorem 4.1, we must show that the scalar multiples of the variables $x_i$ are the only $n$-nilpotent linear forms in $R^{n^n}$. In what follows, we regard a linear form $f = \sum_{i=1}^n a_i x_i$ as a $\mathbb{C}$-linear functional, mapping $v = (v_1, \dots, v_n) \in \mathbb{C}^n$ to $\sum_{i=1}^n a_i v_i$.

**Lemma 4.3.** *Let $f = \sum_{i=1}^n a_i x_i$, with $a_i \in \mathbb{C}$, and let $\alpha \in \mathbb{C}^*$ be a nonzero constant. Suppose that $f(v)^n = \alpha$ for all $v \in \mathbb{C}^n$ whose coordinates are permutations of the distinct $n^{th}$ roots of unity.*

*Then $f \in \mathbb{C}x_i + \mathbb{C}e_1(n)$ for some $i$.*

*Proof.* Let $\zeta$ be a primitive $n^{th}$ root of unity. Let the symmetric group $\mathfrak{S}_n$ act on $\mathbb{C}^n$ by permuting coordinates, and for a permutation $\sigma \in \mathfrak{S}_n$, abbreviate $f(\sigma(1, \zeta, \dots, \zeta^{n-1}))$ by $f(\sigma)$. Replacing $f$ with $f/\alpha$, we may assume that $f(\sigma)^n = 1$ for all $\sigma \in \mathfrak{S}_n$. That $f$ has the desired form is equivalent to the statement that at least $n-1$ of the coefficients $a_1, \dots, a_n$ are equal. This is trivial if $n = 1$ or $n = 2$, and can be checked by direct calculation if $n = 3$. Therefore, suppose $n \geq 4$. By transitivity, it suffices to show that if two coefficients $a_i$ are different, then the other $n-2$ are mutually equal.

Suppose that $a_1 \neq a_2$. Choose $i \neq j \in [n]$ so as to maximize $|\zeta^i - \zeta^j|$, and let $\sigma \in \mathfrak{S}_n$ such that $\sigma(1) = i$ and $\sigma(2) = j$. Then $f(\sigma)$ and $f((12) \circ \sigma)$ are both $n^{th}$ roots of unity, and

$$f(\sigma) - f((12) \circ \sigma) = (a_1 - a_2)(\zeta^i - \zeta^j). \tag{12}$$

Taking the magnitude of both sides, the choice of $i$ and $j$ implies that $|a_1 - a_2| \leq 1$. On the other hand, if we choose $i' \neq j' \in [n]$ to minimize $|\zeta^{i'} - \zeta^{j'}|$, the same argument implies that $|a_1 - a_2| \geq 1$. We conclude that $|a_1 - a_2| = 1$.

Note that $\zeta^i$ and $\zeta^j$ are the only $n^{th}$ roots of unity whose difference is $\zeta^i - \zeta^j$. (This may be seen most easily by plotting the $n^{th}$ roots of unity in the complex plane, and observing that no two of the line segments joining two maximally distant roots are parallel.) Therefore, the equation (12) implies that the values $f(\sigma)$ and $f((12) \circ \sigma)$ do not depend on $\sigma(3), \ldots, \sigma(n)$. Hence $a_3 = \ldots = a_n$ as desired. □

**Proposition 4.4.** *Let $f \in R_1^{n^n}$ be a linear form such that $f^n = 0$.*
*Then $f \in \mathbb{Z}x_i$ for some $i$.*

*Proof.* Let $\hat{f}$ be a preimage of $f$ under the quotient map $\mathbb{Z}[x_1, \ldots, x_n] \twoheadrightarrow R^{n^n}$; that is, $\hat{f}^n \in J$. By degree considerations, there is a constant $\alpha \in \mathbb{Z}$ such that $\hat{f}^n \equiv \alpha e_n(n)$ modulo $I$. As in the proof of Proposition 4.2, the ideal $I$ vanishes on all vectors $v$ whose coordinates are a permutation of the distinct $n^{th}$ roots of unity. Therefore $\hat{f}^n(v) = \alpha e_n(n)(v) = (-1)^{n-1}\alpha$ for all such vectors $v$. By Lemma 4.3, there is some $i$ such that $\hat{f} \in \mathbb{C}x_i + \mathbb{C}e_1(n)$. As $\hat{f} \in \mathbb{Z}[x_1, \ldots, x_n]$, this implies $\hat{f} \in \mathbb{Z}x_i + \mathbb{Z}e_1(n)$. Consequently $f \in \mathbb{Z}x_i$ in $R^{n^n}$. This completes the proof of the proposition and of Theorem 4.1. □

## 5. Nilpotence of linear forms in the cohomology of $X_\lambda$

Throughout this section, $\lambda$ will be an indecomposable partition. We continue our study of nilpotence orders of linear forms in the graded $\mathbb{Z}$-algebra $R^\lambda = H^*(X^\lambda)$. The main result is the following classification of linear forms of minimal nilpotence order, generalizing Theorem 4.1.

**Theorem 5.1.** *Let $\lambda = (0 < \lambda_1 \leq \cdots \leq \lambda_n)$ be an indecomposable partition, and let $k := \lambda_1$. Then $k$ is the minimal nilpotence order of any linear form in $R^\lambda$. Moreover, if $\lambda$ has exactly $m$ parts equal to $k$ (that is, $k = \lambda_1 = \cdots = \lambda_m < \lambda_{m+1}$), then the elements of $R_1^\lambda$ of nilpotence order exactly $k$ are classified as follows:*

Case I. *Either $\lambda_{k-1} > k$, or $n < k - 1$.*
*Then the $k$-nilpotents in $R_1^\lambda$ are the multiples of $x_1, \ldots, x_m$.*

Case II. *$\lambda_{k-1} = k$ (that is, $m = k - 1$).*
Subcase IIa. *Either $\lambda_k > k + 1$, or $k$ is odd.*
*Then the $k$-nilpotents are $x_1, \ldots, x_{k-1}$, and $x_1 + \cdots + x_{k-1}$.*

Subcase IIb. *Both $\lambda_k = k + 1$ and $k$ is even.*
*Then the $k$-nilpotents are $x_1, \ldots, x_{k-1}$, $x_1 + \cdots + x_{k-1}$, and $x_1 + \cdots + x_{k-1} + 2x_k$.*

By way of motivation for the rather technical matter of this section, we explain how the classification of nilpotents will be used in the next two sections to recover a partition from its cohomology ring. Theorem 5.1 implies immediately that $\lambda_1$ is an isomorphism invariant of $R^\lambda$. Moreover, by the presentation of Theorem 2.3, the quotient ring $R^\lambda \,/\, \langle x_1 \rangle$ may be identified with the ring $R^\nu$, where $\nu = (\lambda_2 - 1, \lambda_3 - 1 \ldots, \lambda_n - 1)$ is the partition obtained by removing the first row and column from $\lambda$ (see Figure 2). However, it is really necessary to describe $R^\nu$ as a quotient $R^\lambda \,/\, \langle f \rangle$, where $f$ is some linear form identified intrinsically from the structure of $R^\lambda$ as a standard graded $\mathbb{Z}$-algebra, that is, in a way that *does not depend on the presentation.* The classification of nilpotents in Theorem 5.1 is the tool that allows this. It turns out that we will require almost all, but not quite all of the last assertion in the theorem, so we only prove the parts that will be used. (The arguments we omit are very similar to those that we include.)

In the first part of this section, culminating in Proposition 5.4, we prove the first assertion of Theorem 5.1, namely that $k = \lambda_1$ is the minimal nilpotence order of any linear form in $R^\lambda$. We begin with a weaker statement: that no linear form in the first $k - 1$ variables has nilpotence order less than $k$.

**Lemma 5.2.** *Let $\lambda$ be indecomposable with $k := \lambda_1$. Let $f = \sum_{i=1}^{k-1} a_i x_i \in R_1^\lambda$; that is, $f$ is supported only on the first $k - 1$ variables. Then, in $R^\lambda$,*

(a) *$f^{k-1} = 0$ if and only if $f = 0$, and*

(b) *if* $f^k = 0$, *then* $f$ *is a scalar multiple of one of the following:* $x_1,\ \ldots,\ x_{k-1},\ x_1 + \cdots + x_{k-1}$.

*Proof.* By Proposition 3.1 and the hypothesis that $f$ is supported only on the first $k-1$ variables, we may assume without loss of generality that $n \le k-1$. By Corollary 2.4, we may decrease the part sizes of $\lambda$ (if necessary), so as to assume that $\lambda = k^n$. But then using Proposition 3.1 again, we can re-introduce parts $\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_k$ all of size $k$, and work in the ring $R^{k^k} \cong GL_k(\mathbb{C})/B$, where assertion (a) follows from Theorem 4.1.

In fact, assertion (b) also follows from Theorem 4.1. The degree-1 graded piece of $I_{k^k}$ is generated by $e_1(k)$, so the elements of $R_1^{k^k}$ listed above are the only ones that are congruent modulo $I_{k^k}$ to a scalar multiple of a variable $x_i$ (here we use the fact that $x_1 + \cdots + x_{k-1} = e_1(k) - x_k$). □

An immediate consequence of Lemma 5.2 is that every linear form of nilpotence order $\lambda_1 - 1$ must be supported on at least one of the variables $x_k, \ldots, x_n$. This is where the concept of "stickiness" introduced in Corollary 3.4 first comes into play.

**Proposition 5.3.** *Let* $\lambda$ *be indecomposable with* $k := \lambda_1$, *and let* $f \in R_1^\lambda$. *Then* $f^{k-1} = 0$ *if and only if* $f = 0$ *in* $R^\lambda$.

*Proof.* Assume $f \neq 0 \in R_1^\lambda$, but $f^{k-1} = 0$ in $R^\lambda$. By Lemma 5.2(a), we may assume $n \ge k$. By Proposition 3.1, we may assume without loss of generality that $\lambda$ is its own core.

Writing $f = g+h$, where $g = a_1x_1 + \cdots + a_{k-1}x_{k-1}$ and $h = a_kx_k + \cdots + a_nx_n$, it follows from Corollary 3.5 that $g^{k-1} = 0$. Hence $g = 0$ by Lemma 5.2. That is, $f = h$. If $f$ is not supported on $x_n$ (that is, $a_n = 0$), then we may replace $\lambda$ with the partition obtained by removing the $n$th (largest) row. Repeating this as many times as necessary, we may assume without loss of generality that $a_n \neq 0$.

Now let $M$ be any monomial in the variables $x_1, \ldots, x_{k-1}$. Note that

$$[x_nM]\,\overline{f^{k-1}} \;=\; [x_nM]\,\overline{(a_nx_n)^{k-1}} \tag{13}$$

because the variables $x_k, \ldots, x_{n-1}$ are sticky (Corollary 3.4). Reducing $(a_nx_n)^{k-1}$ using the Gröbner basis element $G_n$ of (9), we find that

$$\begin{aligned} (a_nx_n)^{k-1} &= -a_nx_n^{k-2}(x_1 + \cdots + x_{n-1}) \\ &= a_n^2x_n^{k-3}(x_1 + \cdots + x_{n-1})^2 \\ &= \cdots \\ &= \alpha x_n(x_1 + \cdots + x_{n-1})^{k-2}, \end{aligned} \tag{14}$$

where $\alpha = (-1)^{k-2}a_n^{k-2} \neq 0$. Combining this with (13) yields

$$\begin{aligned} [x_nM]\,\overline{f^{k-1}} &= \alpha\,[x_nM]\,\overline{x_n(x_1 + \cdots + x_{n-1})^{k-2}} \\ &= \alpha\,[x_nM]\,\overline{x_n(x_1 + \cdots + x_{k-1})^{k-2}} \qquad (15\text{a}) \\ &= \alpha\,[M]\,\overline{(x_1 + \cdots + x_{k-1})^{k-2}} \qquad (15\text{b}) \end{aligned}$$

where (15a) follows from stickiness, and (15b) from the fact that only $G_1, \ldots, G_{k-1}$ are used in reducing (15a).

The polynomial $x_1 + \cdots + x_{k-1}$ is nonzero in $R^\lambda$ since $\lambda$ is indecomposable. Thus Lemma 5.2 implies that $(x_1 + \cdots + x_{k-1})^{k-2} \neq 0$ as well, and so there exists some monomial $M$ in the variables $x_1, \ldots, x_{k-1}$ for which $[M]\overline{(x_1 + \cdots + x_{k-1})^{k-2}} \neq 0$. Note that $x_nM$ is also a standard monomial for $I_\lambda$. Therefore $[x_nM]\overline{f^{k-1}} \neq 0$, a contradiction. □

**Proposition 5.4.** *When* $\lambda$ *is indecomposable, the number* $k = \lambda_1$ *is an isomorphism invariant of* $R^\lambda$ *as a graded ring: namely, it is the minimum nilpotence order achieved by any linear form.*

*Proof.* Proposition 5.3 states that no nonzero linear form can have nilpotence order strictly less than $k = \lambda_1$. On the other hand, $x_1$ has nilpotence order at most $k$, because $x_1^k = h_k(1) \in I_\lambda$. □

In the second part of this section, we show that the various linear forms mentioned in Theorem 5.1 are the only possible $k$-nilpotents in $R^\lambda$. We begin by determining the nilpotence order of each variable.

**Proposition 5.5.** *When* $\lambda$ *is indecomposable, the variable* $x_i$ *is* $\lambda_i$*-nilpotent in* $R^\lambda$.

*Proof.* Let $p = \lambda_i$. First, we show that $x_i^p = 0$ in $R^\lambda$. Let $\kappa$ be the partition given by

$$\kappa \ := \ (\underbrace{p, \ldots, p}_{i \text{ times}}, \ \lambda_{i+1}, \ \lambda_{i+2}, \ \ldots, \ \lambda_n).$$

Then $\lambda$ is a subpartition of $\kappa$, so $R^\lambda$ is a quotient of $R^\kappa$ by Lemma 2.4. It suffices to show that $x_i^p = 0$ in $R^\kappa$, which follows from Corollary 2.5 since $x_1^p \in I_\kappa$.

It remains to show that $x_i^{p-1} \neq 0$ in $R^\lambda$. By Proposition 3.1 and Corollary 2.4, it suffices to show that $x_i^{p-1} \neq 0$ in $R^\mu$, where $\mu$ is the subpartition of $\lambda$ given by

$$\mu \ := \ (2, 3, \ldots, i-1, i, p).$$

Note that $\mu$ is indecomposable, and that $R^\mu$ has a reduced Gröbner basis given by (7). A Gröbner reduction similar to (14), using the Gröbner basis element $x_i^{p-i+1} + x_i^{p-i} h_1(i-1)$ yields the equation

$$x_i^{p-1} \ \equiv \ (-1)^{i-1} x_i^{p-i} h_1(i-1)^{i-1} \pmod{I_\mu}.$$

Since further reductions modulo $I_\mu$ can only involve the other generators $h_2(1), h_2(2), \ldots, h_2(i-1)$, we may conclude that $x_i^{p-1} \neq 0$ in $R^\mu$, provided that $h_1(i-1)^{i-1} \neq 0$ in $R^{(2,3,\cdots,i-1,i)}$. Using the fact that $h_1(i-1) = e_1(i-1)$, this follows from the following more general assertion: for any $m \geq 1$ and $i \geq 1$,

$$e_1(i-1)^m \ = \ e_m(i-1) \ \neq \ 0 \qquad \text{in } R^{(2,3,\cdots,i-1,i)}. \tag{16}$$

This is trivially true for $i \leq 2$. For $i > 2$, we prove it by induction on $i$:

$$\begin{aligned}
e_1(i-1)^m &= (x_{i-1} + e_1(i-2))^m \\
&= \sum_{j=0}^{m} \binom{m}{j} x_{i-1}^j e_1(i-2)^{m-j} \\
&= e_1(i-2)^m + \sum_{j=1}^{m} \binom{m}{j} x_{i-1}^j e_1(i-2)^{m-j} \\
&\equiv e_1(i-2)^m + \sum_{j=1}^{m} \binom{m}{j} (-1)^{j-1} x_{i-1} e_1(i-2)^{m-1} \pmod{I^{(2,3,\cdots,i-1,i)}}.
\end{aligned}$$

This last expression follows from using $x_{i-1} h_1(i-1) = x_{i-1}^2 + x_1 h_1(i-2) = x_{i-1}^2 + x_1 h_1(i-2)$ to perform repeated Gröbner reduction on each summand. By induction, $e_1(i-2)^m = e_m(i-2)$, so we obtain

$$\begin{aligned}
e_1(i-1)^m \ &= \ e_m(i-2) + x_{i-1} e_{m-1}(i-2) \sum_{j=1}^{m} \binom{m}{j} (-1)^{j-1} \\
&= \ e_m(i-2) + x_{i-1} e_{m-1}(i-2) \\
&= \ e_m(i-1),
\end{aligned}$$

establishing (16) as desired. □

**Proposition 5.6.** *Let $f = \sum_{i=1}^n a_i x_i \in R^\lambda$. Suppose that $f^k = 0$.*

*Then $f$ is a scalar multiple of one of the following:*

$$\begin{aligned}
&x_1, \ x_2, \ \ldots, \ x_{k-1}, \\
&x_1 + \cdots + x_{k-1}, \\
&x_1 + \cdots + x_{k-1} + 2x_k.
\end{aligned} \tag{17}$$

*The last case can occur only if $k$ is even.*

*Proof.* By Corollary 2.4, we may replace $\lambda$ with its core. Let $g = \sum_{i=1}^{k-1} a_i x_i$ be the part of $f$ in the non-sticky variables. Then $g^k = 0$ by Corollary 3.5. By Lemma 5.2(b), $g$ is either zero or of the form $\alpha x_i$ for some $i \in \{1, 2, \ldots, k-1\}$, or $\alpha(x_1 + \cdots + x_{k-1})$, where $\alpha$ is a nonzero scalar. Without loss of generality, we may assume that $\alpha = 1$.

If $f = g$ then we are done. Otherwise, we must show that $f$ is a scalar multiple of $x_1 + \cdots + x_{k-1} + 2x_k$, and $k$ is even. By Proposition 3.1, we may assume without loss of generality that $f$ involves the variable $x_n$ with non-zero coefficient; that is,

$$f = g + h + ax_n,$$

where $a := a_n \neq 0$ and $h$ is a linear form in the variables $x_k, \ldots, x_{n-1}$. We consider in turn each of the three possibilities: namely, $g = 0$, $g = x_i$, or $g = x_1 + \cdots + x_{k-1}$.

Case 1: $g = 0$.
We will rule out this case by deriving a contradiction from the assumption that $f^k = 0$ in $R^\lambda$. Taking the further quotient of $R^\lambda$ by the variables $x_k, \ldots, x_{n-1}$, one obtains a ring isomorphic to $R^\mu$, where

$$\mu = (\underbrace{k, \ldots, k}_{k-1 \text{ times}}, k+1)$$

is an indecomposable partition, with $k$ parts, equal to its own core. If $f^k = 0$ in $R^\lambda$, then $(ax_k)^k = a^k x_k^k = 0$ in $R^\mu$. So $x_k^k = 0$ in $R^\mu$ (because $a \neq 0$). But this contradicts Corollary 5.5, since $\mu_k = k + 1$.

Case 2: $g = x_i$, where $i \in \{1, 2, \ldots, k-1\}$.
Assume that $k \geq 3$ (the case $k = 2$ falls under Case 3 below). As in Case 1, we wish to reach a contradiction. Consider the quotient ring

$$S \ := \ R^\lambda / \langle x_k, \ \ldots, \ x_{n-1}, \ x_1 + x_2 + \cdots + x_{k-1} + x_n \rangle,$$

which is isomorphic to $R^{k^k}$. Let $\tilde{f} = x_i - a(x_1 + \cdots + x_{k-1})$ be the image of $f$ in $S$; then $\tilde{f}^k = 0$. By Theorem 4.1, $\tilde{f}$ must be a scalar multiple of some variable. This is possible only if $k = 3$ and $a = 1$; that is, $f$ is a scalar multiple of either $x_1 + x_3$ or $x_2 + x_3$. All that remains is to check that neither $(x_1 + x_3)^3$ nor $(x_2 + x_3)^3$ belongs to the ideal $I_{3^3} = \langle h_3(1), h_2(2), h_2(3) \rangle$; this is a routine calculation. Thus $f^k \neq 0$ in all cases, a contradiction. Case 2 is therefore ruled out.

Case 3: $g = x_1 + \ldots + x_{k-1}$.
Let $M$ be any standard monomial for $I_\lambda$ of degree $k-1$ in the non-sticky variables $x_1, \ldots, x_{k-1}$; then $x_n M$ is also standard. Using stickiness of the variables $x_k, \ldots, x_{n-1}$ and the fact that $G_n = x_n(x_1 + \cdots + x_n) \in I_\lambda$, we have for every such monomial

$$\begin{aligned}
[x_n M] f^k \ &= \ [x_n M](g + ax_n)^k \ = \ [x_n M] \sum_{i=0}^{k} \binom{k}{i} a^i x_n^i g^{k-i} \ = \ [x_n M] \sum_{i=1}^{k} \binom{k}{i} a^i x_n^i g^{k-i} \\
&= \ [x_n M] \sum_{i=1}^{k} \binom{k}{i} a^i g^{k-i} (-1)^{i-1} x_n (x_1 + \cdots + x_{n-1})^{i-1} \\
&= \ [x_n M] \sum_{i=1}^{k} \binom{k}{i} a^i g^{k-i} (-1)^{i-1} x_n (x_1 + \cdots + x_{k-1})^{i-1} \\
&= \ [M] \sum_{i=1}^{k} \binom{k}{i} a^i g^{k-1} (-1)^{i-1} \ = \ \left( \sum_{i=1}^{k} \binom{k}{i} a^i (-1)^{i-1} \right) [M] g^{k-1} \\
&= \ \left(1 - (1-a)^k\right) [M] g^{k-1}.
\end{aligned}$$

This last expression must be zero since $f^k = 0$ in $R^\lambda$. On the other hand, $g^{k-1} \neq 0$ in $R^\lambda$, so there is at least one such monomial $M$ in $x_1, \ldots, x_{k-1}$ for which $[M] g^{k-1} \neq 0$. It follows that $1 - (1-a)^k = 0$. Since $a \neq 0$, the only possibility is that $k$ is even and $a = 2$. If $n = k$, then we are done; we need to rule out the case $n > k$.

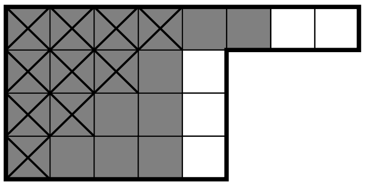

FIGURE 4. The subpartition $\mu$ of Proposition 5.7 (shaded).

Suppose that $n > k$. Replacing $x_n$ with $x_k$ in the above calculation, we find that the coefficient $a_k$ is either 0 or 2. Bearing in mind that $g + x_k = x_1 + \cdots + x_{k-1} + x_k = h_1(k)$, we pass to the quotient ring

$$\begin{aligned} T &:= R^\lambda \,/\, \langle x_{k+1},\, x_{k+2},\, \ldots,\, x_{n-1},\, g + x_k \rangle \\ &\cong \mathbb{Z}[x_1, \ldots, x_k, x_n] \,/\, \langle h_k(1),\, h_{k-1}(2),\, \ldots,\, h_2(k-1),\, x_k(g+x_k),\, x_n(g+x_k+x_n),\, g+x_k \rangle \\ &\cong \mathbb{Z}[x_1, \ldots, x_k, x_n] \,/\, \left\langle h_k(1),\, h_{k-1}(2),\, \ldots,\, h_2(k-1),\, g+x_k,\, x_n^2 \right\rangle \\ &\cong R^{k^k}[x_n] \,/\, \left\langle x_n^2 \right\rangle. \end{aligned}$$

Note that since $f$ equals either $g + 2x_n$ or $g + 2x_k + 2x_n$, and $x_k = -g$ in $T$, the image $p$ of $f$ in $T$ is of the form $p = \pm g + 2x_n$. Since $x_n^2$ and $g^k$ are both zero in $T$, we have

$$p^k \;=\; \sum_{j=0}^{k} \binom{k}{j} (2x_n)^j (\pm g)^{k-j} \;=\; \pm 2k x_n g^{k-1}.$$

But $g^{k-1} \neq 0$ in $R^{k^k}$ by Theorem 4.1, so $x_n g^{k-1} \neq 0$ in $T$. Hence $p^k \neq 0$ in $T$, which implies that $f^k \neq 0$ in $R^\lambda$, as desired. □

We now know that every $k$-nilpotent linear form in $R^\lambda$ is, up to scalar multiplication, one of the linear forms (17). However, if $\lambda$ is not its own core, then we must consider the possibility that one or more of these linear forms actually has nilpotence order strictly greater than $k$. We examine each candidate in turn; Proposition 5.5 immediately takes care of the possible nilpotents $x_1, \ldots, x_{k-1}$.

**Proposition 5.7.** *Let $\lambda$ be indecomposable with $n \geq k-1$ parts and $k = \lambda_1$. Let $g = x_1 + \cdots + x_{k-1} \in R^\lambda$. Then $g^k = 0$ if and only if $\lambda_1 = \cdots = \lambda_{k-1} = k$.*

*Proof.* By Proposition 3.1, we may assume that $n = k-1$. Suppose that $\lambda_1 = \cdots = \lambda_{k-1} = k$. Then

$$\begin{aligned} R^\lambda = R^{k^{k-1}} &= \mathbb{Z}[x_1, \ldots, x_{k-1}] \quad / \; \langle h_k(1),\, h_{k-1}(2),\, \ldots,\, h_2(k-1) \rangle \\ &\cong \mathbb{Z}[x_1, \ldots, x_{k-1}, x_k] \,/\, \langle h_k(1),\, h_{k-1}(2),\, \ldots,\, h_2(k-1), h_1(k) \rangle \\ &= R^{k^k}, \end{aligned}$$

and $g = -x_k$ in $R^{k^k}$, so $g^k = 0$ follows from Theorem 4.1.

Conversely, suppose that $\lambda_{k-1} > k$. We will show that $g^k \neq 0$. Let $\mu$ be the subpartition of $\lambda$ given by

$$\mu = (\underbrace{k-1, \ldots, k-1}_{k-2 \text{ times}}, k+1)$$

(see Figure 4). By Corollary 2.4, it will suffice to show that $g^k \neq 0$ in $R^\mu$. We may rewrite the presentation of $R^\mu$ as

$$\begin{aligned} R^\mu &= \mathbb{Z}[x_1, \ldots, x_{k-1}] \,/\, \langle h_{k-1}(1),\, h_{k-2}(2),\, \ldots,\, h_2(k-2),\, h_3(k-1) \rangle \\ &= \mathbb{Z}[x_1, \ldots, x_{k-1}] \,/\, \left\langle h_{k-1}(1),\, h_{k-2}(2),\, \ldots,\, h_2(k-2),\, x_{k-1}^3 + x_{k-1}^2 h_1(k-2) \right\rangle, \end{aligned}$$

using the fact that

$$\begin{aligned} h_3(k-1) &= x_{k-1}^3 + x_{k-1}^2 h_1(k-2) + x_{k-1}h_2(k-2) + h_3(k-2) \\ &= x_{k-1}^3 + x_{k-1}^2 h_1(k-2) + x_{k-1}h_2(k-2) + h_3(k-3) + x_{k-2}h_2(k-2), \end{aligned}$$

Therefore $x_{k-1}^j \equiv (-1)^j x_{k-1}^2 h^{j-2}$ for all $j \geq 3$. Letting $h = h_1(k-2) = x_1 + \cdots + x_{k-2}$, so that $g = h + x_{k-1}$, we have in $R^\mu$

$$\begin{aligned} g^k &= h^k + \binom{k}{1} h^{k-1} x_{k-1} + \binom{k}{2} h^{k-2} x_{k-1}^2 + \sum_{j=3}^{k} \binom{k}{j} h^{k-j} (-1)^j x_{k-1}^2 h^{j-2} \\ &= h^k + kh^{k-1} x_{k-1} + h^{k-2} x_{k-1}^2 \sum_{j=2}^{k} (-1)^j \binom{k}{j} \\ &= h^k + kh^{k-1} x_{k-1} + (k-1) h^{k-2} x_{k-1}^2. \end{aligned}$$

No further Gröbner reduction is possible, so $g^k$ is zero if and only if $h^k$, $kh^{k-1}$, and $(k-1)h^{k-2}$ are all zero. But $k > 1$, and $h^{k-2} \neq 0$ by Proposition 5.3. We conclude that $g^k \neq 0$ in $R^\mu$ as desired. ☐

For the remaining assertions of Theorem 5.1, we are left only to consider the potentially $k$-nilpotent linear form $g = x_1 + \ldots + x_{k-1} + 2x_k$. Rather than determining exactly when $g$ is $k$-nilpotent as in the theorem (which can be done by an argument similar to Proposition 5.7), we content ourselves with checking directly the case $k = 2$, since this is all we need for the present study. Here $g = x_1 + 2x_2$, and by Proposition 3.1) we may work in the ring

$$R^{(2,\lambda_2)} = \mathbb{Z}[x_1, x_2] / \left\langle x_1^2, h_{\lambda_2 - 1}(2) \right\rangle.$$

Then it is easily seen that $g^2 = x_1^2 + 4x_1x_2 + 4x^2$ is zero in this ring if and only if $\lambda_2 = 3$.

## 6. The indecomposable case

We now use the results of the previous section to prove that an indecomposable partition is determined uniquely by the cohomology ring of the corresponding Schubert variety.

**Theorem 6.1.** *Every indecomposable partition $\lambda$ may be recovered from the structure of the ring $R^\lambda$ as a graded $\mathbb{Z}$-algebra. In particular, if $\lambda$ and $\mu$ are different indecomposable partitions, then $R^\lambda$ and $R^\mu$ are not isomorphic.*

*Proof.* We induct on $n$, the number of parts of $\lambda$. Since $\lambda$ is indecomposable, $n$ is the rank of $R_1^\lambda$ as a free $\mathbb{Z}$-module. By Theorem 5.1, the smallest part $k := \lambda_1$ is the minimal nilpotence order of any member of $R_1^\lambda$. Moreover, as mentioned at the beginning of Section 5, $R^\lambda / \langle x_1 \rangle \cong R^\nu$, where $\nu$ is obtained from $\lambda$ by deleting the first row and column (see Figure 2). By induction, it suffices to show that we can describe $R^\nu$ up to isomorphism in a way that is independent of the presentation.

We proceed by examining the same two cases as in Theorem 5.1; however, we subdivide Case II slightly differently into subcases.

Case I. $\lambda_{k-1} > k$ or $n < k-1$.

Let $m$ be the greatest index such that $\lambda_m = k$. Then Theorem 5.1 tells us that the $k$-nilpotent linear forms in $(R^\lambda)_1$ are (up to $\mathbb{Z}$-multiples) $x_1, \ldots, x_m$. Consequently, up to sign, these are exactly the *primitive* $k$-nilpotents, that is, those $k$-nilpotents $f$ which can only be expressed as a scalar multiple $\alpha g$ for another $k$-nilpotent $g$ and $\alpha \in \mathbb{Z}$ if $\alpha = \pm 1$.

By Corollary 2.5, one has $R^\lambda / \langle x_i \rangle \cong R^\lambda / \langle x_1 \rangle$ ($\cong R^\nu$) for every $i \in \{1, 2, \ldots, m\}$, and hence $R^\nu$ may be identified intrinsically as the quotient of $R^\lambda$ by an arbitrary primitive $k$-nilpotent linear form.

Case II. $\lambda_{k-1} = k$.

Then the primitive $k$-nilpotents are (up to sign) $x_1, \ldots, x_{k-1}, x_1 + \cdots + x_{k_1}$, and if $k$ is even, possibly also $x_1 + \cdots + x_{k-1} + 2x_k$.

Subcase IIA. $k > 2$.

If $k$ is odd, then the "extraneous" primitive $k$-nilpotent $x_1 + \cdots + x_{k-1} + 2x_k$ is absent. If $k$ is even, then $x_1 + \cdots + x_{k-1} + 2x_k$ is distinguished intrinsically as the unique primitive $k$-nilpotent which is $\mathbb{Z}$-linearly independent of all the others.

Thus, in all cases when $k > 2$, we can intrinsically identify the primitive $k$-nilpotents $x_1, \ldots, x_{k-1}$, $x_1 + \cdots + x_{k-1}$, up to sign. By Corollary 2.5, the first $k-1$ forms on this list all have $R^\lambda / \langle x_i \rangle \cong R^\lambda / \langle x_1 \rangle \cong R^\nu$. Hence $R^\nu$ can be identified intrinsically by "majority rule": it is the $\mathbb{Z}$-algebra that occurs (up to isomorphism) as the quotient of $R^\lambda$ by at least $k-1$ of the $k$ different primitive $k$-nilpotent linear forms (other than the one, namely $x_1 + \cdots + x_{k-1} + 2x_k$, that is linearly independent from the rest, as above). Note that the fact that $k-1$ out of $k$ is a well-defined "majority" uses the assumption that $k > 2$.

Subcase IIB. $k = 2$.

If $\lambda_2 > 3$, then $x_1$ is the unique primitive $k$-nilpotent up to sign, so it is distinguished intrinsically, as is $R^\nu \cong R/\langle x_1 \rangle$.

If $\lambda_2 = 3$, then there are two primitive $k$-nilpotents up to sign, namely $x_1$ and $x_1 + 2x_2$. We claim that the graded $\mathbb{Z}$-algebra map $\omega : R^\lambda \to R^\lambda$ defined by

$$\omega(x_1) = x_1 + 2x_2, \qquad \omega(x_2) = -x_2, \qquad \omega(x_i) = x_i \quad \text{for } 3 \le i \le n$$

is an automorphism of $R^\lambda$ interchanging $x_1$ with $x_1 + 2x_2$. Indeed, it is a routine calculation to check that $\omega$ lifts to an automorphism of $\mathbb{Z}[x_1, \ldots, x_n]$, and that $\omega(I_\lambda) = I_\lambda$. In particular, $R^\nu \cong R^\lambda / \langle x_1 \rangle \cong R^\lambda / \langle x_1 + 2x_2 \rangle$ may again be described up to isomorphism as the quotient of $R^\lambda$ by an arbitrary primitive $k$-nilpotent linear form. □

## 7. The decomposable case

We now consider the case that $\lambda$ is decomposable, with indecomposable components $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)}$. In this case, $X_\lambda \cong X_{\lambda^{(1)}} \times \cdots \times X_{\lambda^{(r)}}$. Since each $X^{\lambda^{(i)}}$ has no torsion in its (co-)homology by Theorem 2.1, the Künneth formula [11, §61] implies a tensor decomposition for the associated cohomology rings:

$$H^*(X_\lambda; \mathbb{Z}) \;\cong\; \bigotimes_{i=1}^{r} H^*(X_{\lambda^{(i)}}; \mathbb{Z}). \tag{18}$$

Together with the uniqueness result for indecomposable partitions (Theorem 6.1), it would seem that we are done. However, there is one remaining technical point: to verify that the partitions $\lambda^{(i)}$ can be read off intrinsically from the structure of $H^*(X_\lambda)$ as a graded $\mathbb{Z}$-algebra, we must check that the tensor decomposition (18) is unique.

To do this, we make further use of the facts about nilpotence established in Section 5. But first we must make precise the notion of tensor decomposition, and point out how it interacts with order of nilpotence.

For $R$ a standard graded $\mathbb{Z}$-algebra, a *tensor decomposition* is an isomorphism of graded $\mathbb{Z}$-algebras $R \cong T^{(1)} \otimes \cdots \otimes T^{(r)}$ in which each $T^{(i)}$ is a standard graded $\mathbb{Z}$-algebra. Note that any such decomposition is completely determined by the associated direct sum decomposition of free $\mathbb{Z}$-modules $R_1 \cong \bigoplus_{i=1}^r T_1^{(i)}$, since $T^{(i)}$ is then the subalgebra of $R$ generated by the direct summand $T_1^{(i)}$ of $R_1$. Say that a tensor decomposition of $R$ is *nontrivial* if $T^{(i)} \neq \mathbb{Z}$ for all $i$. Say $R$ is *tensor-indecomposable* if it is not $\mathbb{Z}$ itself, and has no nontrivial tensor decomposition.

**Lemma 7.1.** *Suppose that $R = T^{(1)} \otimes \cdots \otimes T^{(r)}$. Let $x \in R_1$; that is,*

$$x \;=\; x_1 \otimes 1 \otimes \cdots \otimes 1 \;+\; 1 \otimes x_2 \otimes 1 \cdots \otimes 1 \;+\; \ldots \;+\; 1 \otimes \cdots \otimes 1 \otimes x_r$$

*where $x_i \in T_1^{(i)}$. Let $k_i$ be the nilpotence order of $x_i$. (Recall that $k_i = 1$ if and only if $x_i = 0$.)*

*Then the nilpotence order of $x$ is*

$$c \;=\; k_1 + k_2 + \cdots + k_r - r + 1.$$

*Proof.* By the pigeonhole principle, each term of the multinomial expansion of $x^c$ is divisible by $x_i^{k_i}$ for some $i$; therefore, $x^c = 0$ in $R$. For the same reason, all but one term of the multinomial expansion of $x^{c-1}$

vanishes; the exception is

$$\binom{c}{k_1-1,\ \ldots,\ k_n-1} x_1^{k_1-1}\otimes x_2^{k_2-1}\otimes\cdots\otimes x_n^{k_n-1},$$

which is nonzero, since it is nonzero in each tensor factor. □

This calculation has immediate useful consequences.

**Corollary 7.2.** *Let $R$ be a standard graded $\mathbb{Z}$-algebra with a nontrivial tensor decomposition $R=\bigotimes_{i=1}^r T^{(i)}$. Then any linear form $f\in R_1$ that achieves the minimal nilpotence among all elements in $R_1$ must lie in $T^{(i)}$ for some $i$.*

Combining Lemma 7.1 with Proposition 5.5 yields the following.

**Corollary 7.3.** *Let $\lambda$ be a partition with indecomposable components $\{\lambda^{(j)}\}_{j=1}^r$. If $\lambda_i$ corresponds to $\lambda_k^{(j)}$ in this decomposition, then $x_i$ is $\lambda_k^{(j)}$-nilpotent in $R^\lambda$.*

For example, if $\lambda$ is the decomposable partition shown in Figure 1, then $\lambda_1,\ldots,\lambda_5$ correspond to the rows of $\lambda^{(1)}$, and $\lambda_7,\lambda_8$ to the rows of $\lambda^{(2)}$. Thus the variables $x_1,\ldots,x_5$ have nilpotence orders $5,5,5,6,6$, respectively, in $R^\lambda$ (and in $R^{\lambda^{(1)}}$), and $x_7,x_8$ have nilpotence orders 2 and 3, respectively. (Note that these seven variables are a $\mathbb{Z}$-basis for $R_1^\lambda$; $x_6\equiv-(x_1+\cdots+x_5)$ does not correspond to a variable in the presentation for $R^{\lambda^{(1)}}$.)

**Proposition 7.4.** *Let $\lambda$ be an indecomposable partition. Then the ring $R^\lambda$ is tensor-indecomposable.*

*Proof.* Let $n$ denote the number of parts in $\lambda$, and $k=\lambda_1$ its smallest part. We proceed by induction on $n$.

If $n=1$, then clearly $R^\lambda=\mathbb{Z}[x_1]/\langle x_1^k\rangle$ is indecomposable. Otherwise, suppose that $R^\lambda=T^{(1)}\otimes T^{(2)}$ is a nontrivial tensor decomposition; we will obtain a contradiction.

By Proposition 5.4, $x_1$ is a nilpotent of minimal order, and hence by Corollary 7.2, without loss of generality, $x_1\in T^{(1)}$. Then $R^\lambda/\langle x_1\rangle=T^{(1)}/\langle x_1\rangle\otimes T^{(2)}$. On the other hand, $R^\lambda/\langle x_1\rangle\cong R^\nu$, where $\nu$ is the partition obtained from $\lambda$ by removing the first row and column. Since $\lambda$ is indecomposable, so is $\nu$. By the inductive hypothesis, the decomposition $T^{(1)}/\langle x_1\rangle\otimes T^{(2)}$ must be trivial; that is, $T^{(1)}/\langle x_1\rangle\cong\mathbb{Z}$, and $T^{(1)}$ must be generated by $x_1$ as a $\mathbb{Z}$-algebra, i.e., $T^{(1)}=\mathbb{Z}[x_1]/\langle x_1^k\rangle$. Therefore, exactly one member of the set

$$L=\{x_2+\alpha x_1:\alpha\in\mathbb{Z}\}$$

belongs to $T_1^{(2)}$. Let $\ell$ be the nilpotence order of that one form; then all other elements of $L$ have nilpotence order $k+\ell-1>\ell$ by Lemma 7.1. Let $m=\lambda_2$; note that $m\geq 3$ since $\lambda$ is indecomposable. By Proposition 3.1 we can work in the algebra $R^{(\lambda_1,\lambda_2)}=R^{(k,m)}$, namely the quotient of $\mathbb{Z}[x_1,x_2]$ by the ideal

$$\left\langle G_1=x_1^k,\qquad G_2=x_2^{m-1}+x_2^{m-2}x_1+\ldots+x_2^{m-k}x_1^{k-1}\right\rangle.$$

Let $\alpha\in\mathbb{Z}$ be arbitrary. We will show that no linear form $x_2+\alpha x_1$ has nilpotence order strictly less than $m$. Indeed,

$$\begin{aligned}
(x_2+\alpha x_1)^{m-1} &= \sum_{j=0}^{m-1}\binom{m-1}{j}x_2^j\alpha^{m-j-1}x_1^{m-j-1}\\
&= \left(\sum_{j=0}^{m-2}\binom{m-1}{j}x_2^j\alpha^{m-j-1}x_1^{m-j-1}\right)+G_2-\sum_{j=0}^{m-2}x_2^jx_1^{m-j-1}\\
&= G_2+\sum_{j=0}^{m-2}\left(\binom{m-1}{j}\alpha^{m-j-1}-1\right)x_2^jx_1^{m-j-1}\\
&\equiv \sum_{j=m-k}^{m-2}\left(\binom{m-1}{j}\alpha^{m-j-1}-1\right)x_2^jx_1^{m-j-1}\qquad \bmod \langle G_1,G_2\rangle.
\end{aligned}$$

This last expression is exactly the standard form of $(x_2+\alpha x_1)^{m-1}$. For $j=m-2$, the summand is $((m-1)\alpha-1)x_2^{m-2}x_1$; since $m>3$ and $\alpha$ is an integer, the coefficient is nonzero. Therefore $(x_2+\alpha x_1)^{m-1}\neq 0$.

On the other hand, $x_2^m = 0$ in $R^\lambda$ by Proposition 5.5. Therefore $x_2$ must be the unique element of $L$ with minimal nilpotence order $m = \ell$, and every other element of $L$ must have nilpotence order $k + m - 1$. But there are no standard monomials in $x_1, x_2$ of degree greater than $(k-1)+(m-2) = k+m-3$, which implies that every element of $L$ has nilpotence order $k + m - 2$ or less. This contradiction completes the proof. $\square$

We now establish the key fact of the decomposable case, that these decompositions are actually unique.

**Lemma 7.5.** *The ring $R^\lambda$ has a unique tensor decomposition into tensor-indecomposables. Specifically, if $\lambda$ has indecomposable components $\lambda^{(1)}$, $\lambda^{(2)}$, $\ldots$, $\lambda^{(r)}$, then*

$$R^\lambda = R^{\lambda^{(1)}} \otimes \cdots \otimes R^{\lambda^{(r)}}$$

*is the unique tensor decomposition of $R^\lambda$, up to permuting the factors.*

*Proof.* The existence is immediate, since each $R^{\lambda^{(i)}}$ is tensor-indecomposable by Lemma 7.4. For uniqueness, we proceed by induction on the number of rows of $\lambda$. If $\lambda$ has only one row, the statement is trivial.

Suppose that $R^\lambda = \otimes_{i=1}^s T^{(i)}$ is a tensor decomposition with each $T^{(i)}$ tensor-indecomposable, so that

$$\bigotimes_{i=1}^{s} T^{(i)} \quad = \quad R^\lambda \quad = \quad \bigotimes_{j=1}^{r} R^{\lambda^{(j)}}, \tag{19a}$$

$$\bigoplus_{i=1}^{s} T_1^{(i)} \quad = \quad R_1^\lambda \quad = \quad \bigoplus_{j=1}^{r} R_1^{\lambda^{(j)}}. \tag{19b}$$

Let $k$ be the minimal nilpotence order of any element of $R_1^\lambda$. Then $k = \min\{\lambda_1^{(j)} : 1 \le j \le r\}$ by Corollary 7.3. Without loss of generality, we may re-index so that $k = \lambda_1^{(1)}$; then $x_1$ is a linear form of nilpotence order $k$. By Corollary 7.2, $x_1$ must belong to one of the $T^{(i)}$, say $T^{(1)}$. Let $\nu, \nu^{(1)}$ be the partitions obtained by removing the left column and bottom row of $\lambda, \lambda^{(1)}$, respectively. Then

$$T^{(1)}/\langle x_1 \rangle \otimes \bigotimes_{i=2}^{s} T^{(i)} \quad = \quad R^\lambda/\langle x_1 \rangle \quad = \quad R^{\nu^{(1)}} \otimes \bigotimes_{j=2}^{r} R^{\lambda^{(j)}}, \tag{20a}$$

$$T_1^{(1)}/\mathbb{Z}x_1 \oplus \bigoplus_{i=2}^{s} T_1^{(i)} \quad = \quad R_1^\lambda/\mathbb{Z}x_1 \quad = \quad R_1^{\nu^{(1)}} \oplus \bigoplus_{j=2}^{r} R_1^{\lambda^{(j)}}. \tag{20b}$$

By induction, the rightmost expression in (20a) is the unique tensor decomposition of $R^\nu$ into tensor-indecomposables (possibly with a superfluous factor $R^{\nu^{(1)}} = \mathbb{Z}$ if $\lambda^{(1)}$ has only one part). Thus the rightmost expression in (20b) is unique—clearly not as a direct sum decomposition of $R_1^\lambda/\mathbb{Z}x_1$ as a $\mathbb{Z}$-module, but as a direct sum decomposition which induces a tensor decomposition of $R^\lambda/\langle x_1 \rangle$.

Now assume that $\lambda^{(1)}$ has $m$ rows, so that $x_1, x_2, \ldots, x_m$ generate $R^{\lambda^{(1)}}$ as a $\mathbb{Z}$-subalgebra of $R^\lambda$. For each $\ell \in \{2, \ldots, m\}$, consider the image $\bar{x}_\ell$ of $x_\ell$ in $R_1^\nu = R_1^\lambda/\mathbb{Z}x_1$. Since each $\bar{x}_\ell$ belongs to the direct summand $R_1^{\nu^{(1)}}$ on the left side of the unique decomposition (20b), it must belong either to $T_1^{(1)}/\mathbb{Z}x_1$, or to $T_1^{(i)}$ for some $i \ge 2$. On the other hand, Corollary 7.3 tells us that $x_\ell$ is $\lambda_\ell^{(1)}$-nilpotent in $R^\lambda$, but $\bar{x}_\ell$ is $\nu_{\ell-1}^{(1)}$-nilpotent in $R^\nu$. That is, the nilpotence order of $x_\ell$ drops by 1 in the quotient by $x_1$ (because $\nu_{\ell-1}^{(1)} = \lambda_\ell^{(1)} - 1$). If $\bar{x}_\ell \in T_1^{(i)}$ for some $i \ge 2$, then this last observation contradicts Lemma 7.1. Therefore $\bar{x}_\ell \in T_1^{(1)}/\mathbb{Z}x_1$, from which we conclude that $T_1^{(1)}/\mathbb{Z}x_1 \supseteq R_1^{\nu^{(1)}}$.

Consequently, the uniqueness property of the decomposition (20b) implies that

$$T_1^{(1)}/\mathbb{Z}x_1 \;=\; R_1^{\nu^{(1)}} \oplus \bigoplus_{u \in U} R_1^{\lambda^{(j_u)}}$$

for some subset $U \subset \{2, 3, \ldots, r\}$. Since $x_1$ lies in both $T^{(1)}$ and $R^{\lambda^{(1)}}$, we conclude that

$$T_1^{(1)} \;=\; R_1^{\lambda^{(1)}} \oplus \bigoplus_{u \in U} R_1^{\lambda^{(j_u)}}$$

and, since $T^{(1)}$ is a standard graded $\mathbb{Z}$-algebra,

$$T^{(1)} \;=\; R^{\lambda^{(1)}} \otimes \bigotimes_{u \in U} R^{\lambda^{(j_u)}}.$$

But $T^{(1)}$ was assumed to be indecomposable, so this forces $U = \emptyset$. Hence $T_1^{(1)} = R_1^{\lambda^{(1)}}$ and $T_1^{(1)}/\mathbb{Z}x_1 = R_1^{\nu^{(1)}}$. By the uniqueness property of (20b), we must have $r = s$, and after re-indexing, $T_1^{(i)} = R_1^{\lambda^{(i)}}$ for $i = 2, 3, \ldots, r$. Thus the two tensor decompositions in (19a) are identical. $\square$

The nontrivial implication (iii) $\Longrightarrow$ (i) in the main result, Theorem 1.1, is now immediate from Lemma 7.5 and Theorem 6.1.

*Remark* 7.6. As we shall now demonstrate, it was essential to study the cohomology of $X_\lambda$ with integer coefficients. If $A$ is a coefficient ring in which 2 is invertible, then Proposition 7.4, Lemma 7.5 and Theorem 1.1 would all fail to hold if "graded $\mathbb{Z}$-algebra" was replaced with "graded $A$-algebras". That is, Ding's Schubert varieties are *not* classified up to isomorphism by their cohomology with $A$-coefficients. For example, consider the indecomposable partition $\lambda = (2, 3)$. By completing the square, one has

$$\begin{aligned} R^{(2,3)} \otimes_{\mathbb{Z}} A \;&\cong\; A[x_1, x_2] \,/ \left\langle x_1^2,\, x_2^2 + x_1x_2 + \frac{1}{4}x_1^2 \right\rangle \\ &=\; A[x_1, x_2] \,/ \left\langle x_1^2,\, \left(x_2 + \frac{1}{2}x_1\right)^2 \right\rangle \\ &\cong\; A[x_1] \,/ \left\langle x_1^2 \right\rangle \,\otimes\, A[y] \,/ \left\langle y^2 \right\rangle. \end{aligned}$$

Thus indecomposable partitions do not lead to tensor-indecomposable graded $A$-algebras. This also leads to "extra" isomorphisms among the cohomology rings $H^*(X_\lambda; A) \cong R^\lambda \otimes_{\mathbb{Z}} A$. For example, the partition $\mu = (2, 2, 4)$ has indecomposable components $\mu^{(1)} = \mu^{(2)} = (2)$. Since $R^{(2)} \cong \mathbb{Z}[x]/\left\langle x^2 \right\rangle$, one has

$$R^\mu \otimes_{\mathbb{Z}} A \;\cong\; A[x] \,/ \left\langle x^2 \right\rangle \otimes_A A[x] \,/ \left\langle x^2 \right\rangle \;\cong\; R^\lambda \otimes_{\mathbb{Z}} A$$

even though $\lambda = (2, 3)$ and $\mu = (2, 2, 4)$ do not have the same indecomposable partition components.

## References


[1] D. Cox, J. Little, and D. O'Shea, Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra. Second edition. *Undergraduate Texts in Mathematics*, Springer-Verlag, New York, 1997.
[2] K. Ding, Rook placements and cellular decompositions of partition varieties. *Discrete Math.* **170** (1997), 107-151.
[3] K. Ding, Rook placements and classification of partition varieties $B\backslash M_\lambda$, *Commun. Contemp. Math.* **3** (2001), 495–500.
[4] D. Foata and M.-P. Schützenberger, On the rook polynomials of Ferrers relations. Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), pp. 413–436. North-Holland, Amsterdam, 1970.
[5] W. Fulton, Young tableaux. *London Mathematical Society Student Texts* **35**. Cambridge University Press, Cambridge, 1997.
[6] A.M. Garsia and J.B. Remmel, $Q$-counting rook configurations and a formula of Frobenius. *J. Combin. Theory Ser. A* **41** (1986), 246–275.
[7] V. Gasharov and V. Reiner, Cohomology of smooth Schubert varieties in partial flag manifolds. *J. London Math. Soc.* **66** (2002), 550–562.
[8] J.R. Goldman, J.T. Joichi, and D.E. White, Rook theory. I. Rook equivalence of Ferrers boards. *Proc. Amer. Math. Soc.* **52** (1975), 485–492.
[9] I. Kaplansky and J. Riordan, The problem of the rooks and its applications. *Duke Math. J.* **13** (1946), 259–268.
[10] I.G. Macdonald, Notes on Schubert polynomials. Publications du LACIM, Université du Québec à Montréal, 1991.
[11] J.R. Munkres, Elements of algebraic topology. Addison-Wesley Publishing Company, Menlo Park, CA, 1984.



Mike Develin, American Institute of Mathematics, 360 Portage Ave., Palo Alto, CA 94306-2244, USA
*E-mail address*: `develin@post.harvard.edu`

Jeremy L. Martin, School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
*E-mail address*: `martin@math.umn.edu`

Victor Reiner, School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
*E-mail address*: `reiner@math.umn.edu`