# COHOMOLOGY OF SMOOTH SCHUBERT VARIETIES IN PARTIAL FLAG MANIFOLDS

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Abstract. We use the fact that smooth Schubert varieties in partial flag manifolds are iterated fiber bundles over Grassmannians to give a simple presentation for their integral cohomology ring, generalizing Borel's presentation for the cohomology of the partial flag manifold itself. More generally, such a presentation is shown to hold for a larger class of subvarieties of the partial flag manifolds (which we call *subvarieties defined by inclusions*). The Schubert varieties which lie within this larger class are characterized combinatorially by a pattern avoidance condition.

## 1. Introduction

This paper concerns smooth Schubert subvarieties within the partial flag varieties

$$G/P_J := \{0 \subset V_{d_1} \subset V_{d_2} \subset \cdots \subset V_{d_s} \subset \mathbb{C}^n\}.$$

Here $G = GL_n(\mathbb{C})$, and $P_J$ is the subgroup that stabilizes the standard partial flag having $V_{d_i}$ spanned by the first $d_i$ standard basis vectors in $\mathbb{C}^n$. When $J = \emptyset$, $P_J = B$ is the Borel group of upper triangular matrices and $G/B$ is usual (complete) flag variety.

The Schubert varieties $\{X_{wW_J}\}$ in $G/P_J$ are subvarieties indexed by cosets $wW_J$ of the parabolic subgroup $W_J$ of the Weyl group $W = S_n$. The equations which define $X_{wW_J}$ as a subvariety of $G/P_J$ are a conjunction of certain conditions having the general form

$$\dim V_{d_i} \cap \mathbb{C}^e \geq r. \tag{1}$$

Smooth Schubert varieties have several special properties, which make them almost as well-behaved as the partial flag manifolds themselves. We review some of these properties here to provide a context for our results.

Note that condition (1) says

$$\begin{aligned} V_{d_i} &\subset \mathbb{C}^e \text{ when } r = d_i \\ \mathbb{C}^e &\subset V_{d_i} \text{ when } r = e. \end{aligned} \tag{2}$$

Any subvariety $X$ of $G/P_J$ defined by a conjunction of conditions of the special forms in (2) will be called a subvariety of *$G/P_J$ defined by inclusions*. Say that a pair of such inclusion conditions of the form

$$\begin{aligned} V_{d_i} &\subset \mathbb{C}^e \\ \mathbb{C}^{e'} &\subset V_{d_j} \end{aligned}$$

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are *crossing* if $d_i < d_j$ and $e > e'$. If $X$ is defined by inclusions but contains no crossing pair, we say it is *defined by non-crossing inclusions*. By results of Ryan [9] and Wolper [11], this property characterizes the smooth Schubert varieties. Theorem 1.1 below summarizes some of their work, along with a result due independently to Lakshmibai and Sandhya [7] and Haiman [5]. For the purposes of stating it, we introduce two pieces of terminology.

Say that a (complex) algebraic variety $X$ is *fibered by Grassmannians* if it lies within the smallest class of algebraic varieties that contains all Grassmannians $\mathbb{G}r(k, \mathbb{C}^n)$ and has the property that the total space of an algebraic fiber bundle is in the class whenever both its base and fiber are in the class.

Given a coset $wW_J$ in $S_n$ and permutation $u = u_1 \cdots u_k$ in $S_k$ for some $k \leq n$, say that $wW_J$ *avoids the pattern* $u$ if the longest coset representative $w$ for $wW_J$ contains no subsequence $w_{i_1} \cdots w_{i_k}$ whose values are in the same relative order as $u_1 \cdots u_k$ (i.e. the permutation matrix representing $w$ when restricted to columns $\{i_1, \ldots, i_k\}$ and rows $\{w_{i_1}, \ldots, w_{i_k}\}$ gives exactly the permutation matrix representing $u$).

**Theorem 1.1.** *The following are equivalent for a subvariety $X$ of $G/P_J$:*

(i) *$X$ is smooth and $B$-stable.*
(ii) *$X$ is smooth and defined by inclusions.*
(iii) *$X$ is defined by non-crossing inclusions.*
(iv) *$X$ is $B$-stable and fibered by Grassmannians.*
(v) *$X$ is a smooth Schubert variety $X_{wW_J}$.*
(vi) *$X$ is a Schubert variety $X_{wW_J}$ for some $w$ that avoids both patterns* 4231, 3412.

After reviewing combinatorial and topological facts about partial flag manifolds and Schubert varieties in Section 2, in Section 3, we use some very special cases of the fiber bundles from (iv) above to prove a presentation (Theorem 3.1) for the integral cohomology ring of a smooth Schubert variety. This presentation is nearly as simple as Borel's picture for the cohomology of $G/P_J$.

For example, the Schubert variety $X_{352164}$ in the flag variety $G = GL_6(\mathbb{C})/B$ is defined by the following inclusions

$$\{0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5 \subset \mathbb{C}^6 : V_1 \subset \mathbb{C}^3 \subset V_4 \subset \mathbb{C}^5\}$$

and has cohomology ring

$$H^*(X_{352164}, \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, x_3, x_4, x_5, x_6]/I$$

where $I$ is the ideal having generators which are elementary symmetric functions in consecutive variable sets $x_i, x_{i+1}, \ldots, x_j$ of the form $e_r(i,j) := e_r(x_i, x_{i+1}, \ldots, x_j)$, coming from the various inclusion conditions that define $X_{352164}$:

$$\begin{array}{cc}
e_1(1,6), e_2(1,6), e_3(1,6), e_4(1,6), e_5(1,6), e_6(1,6) & \text{from } V_i \subset \mathbb{C}^6 \\
e_3(2,6), e_4(2,6), e_5(2,6) & \text{from } V_1 \subset \mathbb{C}^3 \\
e_2(5,6) & \text{from } V_4 \subset \mathbb{C}^5 \\
e_2(1,4), e_3(1,4), e_4(1,4)) & \text{from } \mathbb{C}^3 \subset V_4.
\end{array}$$

In fact, Theorem 3.1 proves such a presentation more generally for all subvarieties of $G/P_J$ defined by inclusions. In Section 4, the Schubert varieties $X_{wW_J}$ defined by inclusions are characterized combinatorially (Theorem 4.2) as those with $wW_J$ avoiding the four patterns $4231, 35142, 42513, 351624$.

## 2. Preliminaries

We start by defining partial flag manifolds and their Schubert varieties. A good reference for some of this material is [3, Chapter 10].

Let $G = GL_n(\mathbb{C})$ be the general linear group, $B$ the Borel subgroup of upper triangular matrices in $G$, and $W = S_n$ the symmetric group on $n$ letters $[n] := \{1, 2, \ldots, n\}$, which we may also think of as the subgroup of permutation matrices in $G$. The group $W$ has a set of generators $S = \{s_1, \ldots, s_{n-1}\}$, where $s_i$ is the transposition swapping $i, i+1$ and keeping all other numbers fixed, which makes $(W, S)$ a Coxeter system (see [6]).

For any $J = \{s_{i_1}, \ldots, s_{i_p}\} \subseteq S$, let $W_J$ denote the *Young* subgroup of $W$ generated by $J$, let $P_J$ denote the *standard parabolic* subgroup of $G$ generated by $B$ and $J$, and let

$$D(J) = \{0\} \cup \{t \in [n-1] : s_t \notin J\} \cup \{n\} = \{0 < d_1 < d_2 < \cdots < d_s < n\}.$$

$G$ acts transitively on the set of complete flags

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n$$

where $V_i$ is a subspace of dimension $i$ for each $i$. If one fixes the *standard flag* having $V_i = \mathbb{C}^i$, where $\mathbb{C}^i$ is the subspace of $\mathbb{C}^n$ spanned by the first $i$ standard basis vectors $\{e_1, e_2, \ldots, e_i\}$, then $B$ is exactly the subgroup which stabilizes this standard flag. Hence one may identify the coset space $G/B$ with the set of complete flags. More generally, for each $J \subseteq S$ with $D(J)$ defined as above, the stabilizer of the partial flag

$$0 \subset \mathbb{C}^{d_1} \subset \mathbb{C}^{d_2} \subset \cdots \subset \mathbb{C}^{d_s} \subset \mathbb{C}^n$$

is $P_J$, hence one can identify the set of partial flags of the form

$$0 \subset V_{d_1} \subset V_{d_2} \subset \cdots \subset V_{d_s} \subset \mathbb{C}^n$$

with $G/P_J$ (the complete flags and $G/B$ are the special case $J = \emptyset$). This homogeneous space $G/P_J$ can be embedded as a closed subvariety in a suitable complex projective space $\mathbb{P}(\mathbb{C}^N)$, and is a smooth variety called the *partial flag manifold* $G/P_J$. In the special case $J = \emptyset$ where $P_J = B$, it is called the *flag manifold* $G/B$. In the special case where $|J| = |S| - 1$ so $D(J) = \{0 < d < n\}$ for some $d$, then $G/P_J$ is called the *Grassmannian* or *Grassman manifold* $\mathrm{Gr}(d, \mathbb{C}^n)$.

The partial flag manifold has a cellular decomposition

$$G/P_J = \coprod_{wW_J \in W/W_J} X^o_{wW_J} \tag{3}$$

where

$$X^o_{wW_J} = BwP_J/P_J$$

are the orbits under the left-translation action of $B$ in $G$, called *Schubert cells* and indexed by cosets $wW_J$. The Schubert cell $X^o_{wW_J}$ has the following description as a collection of flags: if $w = w_1 w_2 \cdots w_n$ is a permutation in one-line notation, that is, $w(i) = w_i$, and $D(J) = \{0 < d_1 < d_2 < \cdots < d_s < n\}$ is defined as above, then

$$\begin{aligned} X^o_{wW_J} = \{0 \subset V_{d_1} \subset \cdots \subset V_{d_s} \subset \mathbb{C}^n :& \\ \dim V_{d_i} \cap \mathbb{C}^e = \#\{k \leq d_i : w_k \leq e\}& \\ \text{for } 1 \leq i \leq s \text{ and } 1 \leq e \leq n\}& \end{aligned} \tag{4}$$

One can check that this is independent of the choice of $w \in wW_J$. The Zariski closure $X_{wW_J} := \overline{X^o_{wW_J}}$ is called a *Schubert variety* in $G/P_J$, and is the subset of partial flags satisfying the same intersection conditions as $X^o_{wW_J}$ except replacing the " $=$ " by " $\geq$ ". We will call these conditions *Schubert conditions* for $X_{wW_J}$.

Using the one-line notation $w = w_1 w_2 \cdots w_n$ for $w$ in $W = S_n$, the usual Coxeter group length function $l : W \to \mathbb{N}$, is in this case the number of *inversions*

$$l(w) = |\{(i,j) : 1 \leq i < j \leq n, w_i > w_j\}|.$$

If $D(J) = \{0, d_1, \ldots, d_s, n\}$, then the freedom in choosing a coset representative for $wW_J$ is exactly the choice of ordering within each block $w_{d_i+1} w_{d_i+2} \cdots w_{d_{i+1}}$. The length function allows one to distinguish two coset representatives $\min wW_J$ and $\max wW_J$ for $wW_J$, which are unique in achieving the minimum and maximum length, respectively. More explicitly, $\min wW_J, \max wW_J$ are the representatives for which each block $w_{d_i+1} w_{d_i+2} \cdots w_{d_{i+1}}$ is in increasing, decreasing order respectively.

The *Bruhat order* is a partial order on the cosets $W/W_J$ defined as follows: $uW_J \leq vW_J$ if $X^o_{uW_J} \subset X_{vW_J}$. The open Schubert cell $X^o_{wW_J}$ is isomorphic to a complex affine space of dimension $l(\min wW_J)$. Thus the cell decomposition (3) has only cells in even (real) dimensions. Note also that since the open Schubert cells are the $B$-orbits of $G/P_J$, any $B$-stable subvariety $X$, such as a Schubert variety itself, will also be a union

$$X = \coprod_{wW_J \in \mathcal{I}} X^o_{wW_J} \tag{5}$$

of open Schubert cells for some subset $\mathcal{I}$ of $W/W_J$. Since $X$ is a closed subvariety, this subset $\mathcal{I}$ must form an order ideal in the Bruhat order on $W/W_J$, and therefore the above decomposition is again a cell decomposition. The fact that the Schubert cells have even real dimensions have several topological consequences, which we collect in the following proposition.

**Proposition 2.1.** *Let $Y$ be a finite CW-complex with cells $\{\sigma_j\}_{j \in J}$ occurring only in even dimensions, and exactly one 0-cell. Let $X$ be any subcomplex, with cells $\{\sigma_i\}_{i \in I}$ for some subset $I \subset J$. (In particular, this applies when $Y$ is any $B$-stable subvariety of $G/P_J$, and $X$ is any $B$-stable subvariety of $Y$).*

*Then*

(i) *$X$ is simply-connected.*
(ii) *The integral homology $H_*(X;\mathbb{Z})$ and cohomology $H^*(X;\mathbb{Z})$ are free abelian groups which are non-vanishing only in even dimensions. They form dual lattices under the Kronecker pairing, having dual $\mathbb{Z}$-bases given by the cellular homology classes $\{[\sigma_i]\}_{i \in I}$ and their (Kronecker) dual cohomology classes $\{[\sigma_i]\}^*_{i \in I}$, respectively.*
(iii) *Consequently,*

$$\begin{aligned}\mathrm{Poin}(X,q) &:= \sum_{k \geq 0} \mathrm{rank}_{\mathbb{Z}} H_k(X;\mathbb{Z})\, q^i \\ &= \sum_{i \in I} q^{\dim \sigma_i}.\end{aligned}$$

(iv) *The maps $i_*, i^*$ induced on homology and cohomology*

$$H_*(X;\mathbb{Z}) \xrightarrow{i_*} H_*(Y;\mathbb{Z})$$
$$H^*(Y;\mathbb{Z}) \xrightarrow{i^*} H^*(X;\mathbb{Z})$$

*by the inclusion $X \overset{i}{\hookrightarrow} Y$ are injective and surjective, respectively, with*

$$i_*([\sigma_i]) = [\sigma_i] \text{ for all } i \text{ in } I$$
$$i^*([\sigma_j]^*) = \begin{cases} [\sigma_j]^* & \text{if } j \in I \\ 0 & \text{if } j \in J - I \end{cases}$$

(v) *(cf.* [1, Cor. 4.4]*) Consequently, the kernel of the surjection $i^*$ is generated by $\{[\sigma_j]^*\}_{j \in J-I}$, both additively and as an ideal in $H^*(X;\mathbb{Z})$.* □

*Proof.* Since $X$ has only one 0-cell, it is connected. Since it has no 1-cells, it is simply connected. For the rest, note that the absence of odd-dimensional cells implies that all boundary and coboundary maps are zero in the cellular chain and cochain complexes computing $H_*(X;\mathbb{Z}), H^*(X;\mathbb{Z})$. □

We have seen the geometric significance of the coset representative $\min wW_J$, whose length gives the dimension of $X_{wW_J}$. The other distinguished coset representative $\max wW_J$ has the following geometric interpretation.

**Proposition 2.2.** *Let $w = \max wW_J$. Then the Schubert conditions on flags in $X_w$ inside $G/B$ are all implied by the conditions on $\dim V_d \cap \mathbb{C}^e$ with $d \in D(J) = \{0, d_1, \ldots, d_s, n\}$.*

*Consequently, the canonical surjection*

$$\begin{array}{ccc} G/B & \xrightarrow{p} & G/P_J \\ 0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n & \longmapsto & 0 \subset V_{d_1} \subset \cdots \subset V_{d_s} \subset \mathbb{C}^n \end{array}$$

*is an algebraic fiber bundle with fiber isomorphic to a product of flag manifolds, and restricts to a fiber bundle map $X_{\max wW_J} \xrightarrow{p} X_{wW_J}$ with the same fiber.*

*Proof.* See Proposition 4.1. □

Such fiber bundle maps (and others that will appear later) have strong combinatorial and geometric consequences, via the following proposition.

**Proposition 2.3.** *Let $F, E, B$ be complex algebraic varieties, and $E \xrightarrow{p} B$ an algebraic fiber bundle map with fiber isomorphic to $F$.*

*If $F, B$ are simply connected, have free abelian integral homology, and have homology occurring only in even dimensions, then the same is true for $E$. Furthermore*

$$\mathrm{Poin}(E, q) = \mathrm{Poin}(F, q)\,\mathrm{Poin}(B, q).$$

*Proof.* Note that Proposition 2.1 (ii) implies that the maps $d^i$ in the $E^i$ page of the Leray-Serre spectral sequence associated to $E \xrightarrow{p} B$ are all 0 for $i \geq 1$, so that it degenerates at the $E^1$ page. Also

$$E^1_{p,q} = H^p(B) \otimes_{\mathbb{Z}} H^q(F)$$

because $B$ is simply-connected, and the homology of $F$ is free abelian. The assertion follows. □

## 3. A presentation for the cohomology ring of subvarieties of $G/P_J$ defined by inclusions

We begin by establishing notation for subvarieties of $G/P_J$ defined by inclusions. Let $J \subseteq S$ index a partial flag manifold $G/P_J$, so that if $D = D(J)$, we can define

$$X_D := G/P_J = \{0 \subset V_{d_1} \subset V_{d_2} \subset \cdots \subset V_{d_s} \subset \mathbb{C}^n\}.$$

Given a triple $(D, L, U)$ where

- $D = D(J) = \{0 < d_1 < d_2 < \cdots < d_s < n\}$,
- $L$ is a set of ordered pairs of the form $(l_i, d_i)$ with $l_i \leq d_i$, and
- $U$ is a set of ordered pairs of the form $(d_i, u_i)$ with $d_i \leq u_i$,

define a subvariety $X_{(D,L,U)}$ of $X_D$ by

$$\begin{aligned} X_{(D,L,U)} = \{0 \subset V_{d_1} \subset V_{d_2} \subset \cdots \subset V_{d_s} \subset \mathbb{C}^n :& \\ \mathbb{C}^{l_i} \subset V_{d_i} \text{ if } (l_i, d_i) \in L& \\ V_{d_i} \subset \mathbb{C}^{u_i} \text{ if } (d_i, u_i) \in U\}& \end{aligned}$$

Clearly every subvariety $X$ of $G/P_J$ defined by inclusions has this form. Note that since the Borel subgroup stabilizes the subspaces $\mathbb{C}^i$, the subvarieties $X_{(D,L,U)}$ are always $B$-stable, and consequently Proposition 2.1 applies to them.

We review here Borel's presentation for $H^*(G/P_J)$ (see [3, §10.3, 10.6]). For $J = \emptyset$, so $D = D(J) = \{1, 2, \ldots, n-1\}$ and $X_D = G/B$, Borel showed

$$\mathbb{Z}[x_1, \ldots, x_n]/(e_1(1,n), \ldots, e_n(1,n)) \cong H^*(G/B, \mathbb{Z})$$

where we recall from the Introduction that $e_k(i,j) := e_k(x_i, x_{i+1}, \ldots, x_j)$ is the $k^{th}$ elementary symmetric function in the variables $x_i, x_{i+1}, \ldots, x_j$. The isomorphism sends $x_i$ to $-c_1(\mathcal{L}_i)$ where $\mathcal{L}_i$ denotes the line bundle on $G/B$ whose fiber over a complete flag

$$0 = V_0 \subset V_1 \subset \cdots V_{n-1} \subset V_n = \mathbb{C}^n$$

is the space $V_i/V_{i-1}$, and $c_1(\mathcal{L}_i)$ is its first Chern class. He also showed that for $D = D(J)$ arbitrary, the surjection $\pi : G/B \to G/P_J = X_D$ induces an inclusion

$$\pi^* : H^*(G/P_J) \hookrightarrow H^*(G/B)$$

sending the $k^{th}$ Chern class $c_k(V_{d_j}/V_{d_{j-1}})$ of the vector bundle $V_{d_j}/V_{d_{j-1}}$ to the elementary symmetric function $e_k(d_{j-1}+1, d_j)$. The image of this inclusion is exactly the invariant subring

$$\mathbb{Z}[x_1, \ldots, x_n]^{W_J}/(e_1(1,n), \ldots, e_n(1,n)).$$

Define for a triple $(D, L, U)$ as above

$$\begin{aligned} A_D &:= \mathbb{Z}[x_1, \ldots, x_n]^{W_J} \\ I_n = I_{d_{s+1}} &:= (e_1(1,n), \ldots, e_n(1,n)) \\ R_D &:= A_D/I_n \cong H^*(G/P_J) \\ I_{(D,L,U)} &:= I_n + (\{e_k(1,d_i) : (l_i, d_i) \in L,\ k > d_i - l_i\} \\ &\qquad \cup \{e_k(d_i+1, n) : (d_i, u_i) \in U,\ k > u_i - d_i\}) \\ R_{(D,L,U)} &:= A_D/I_{(D,L,U)} \end{aligned}$$

Recall from Proposition 2.1 that for any $B$-stable subvariety $X$ of $X_D$, the cohomology $H^*(X)$ is free abelian, and is the image of the surjective restriction map

$H^*(X_D) \xrightarrow{i^*} H^*(X)$. Consequently, for any $X_{(D,L,U)}$, the composite ring homomorphism

$$\phi : A_D \longrightarrow A_D/I_n \cong H^*(X_D) \xrightarrow{i^*} H^*(X_{(D,L,U)})$$

will be surjective.

**Theorem 3.1.** *Let $X_{(D,L,U)}$ be a subvariety of $X_D$ defined by inclusions. Then the composite $\phi$ defined above has kernel $I_{(D,L,U)}$, and therefore induces a (grade-doubling) ring isomorphism*

$$\overline{\phi} : R_{(D,L,U)} \cong H^*(X_{(D,L,U)}).$$

The proof of this theorem occupies the remainder of this section.

We begin by showing that $I_{(D,L,U)} \subset \ker \phi$. Note that if $(l_i, d_i)$ is in $L$, then the fact that $\mathbb{C}^{l_i} \subset V_{d_i}$ on $X_{(D,L,U)}$ implies by the splitting principle that the vector bundle $V_{d_i}$ of rank $d_i$ has the same Chern classes as the direct sum $\mathbb{C}^{l_i} \oplus V_{d_i}/\mathbb{C}^{l_i}$, where $\mathbb{C}^{l_i}$ here represents a trivial bundle of rank $l_i$ over $X_{(D,L,U)}$. The Chern classes of the bundle $V_{d_i}$ on $X_{(D,L,U)}$ therefore coincide with those of $V_{d_i}/\mathbb{C}^{l_i}$, a bundle of rank $d_i - l_i$, so $c_k(V_{d_i})$ will vanish for $k > d_i - l_i$. Hence the $k^{th}$ elementary symmetric functions in the Chern roots $x_1, \ldots, x_{d_i}$ of the bundle $V_{d_i}$ must vanish in $H^*(X_{(D,L,U)})$ for $k > d_i - l_i$.

Similarly, if $(d_i, u_i)$ is in $U$, then $V_{d_i} \subset \mathbb{C}^{u_i}$ on $X_{(D,L,U)}$ implies the bundle $\mathbb{C}^n/V_{d_i}$ of rank $n - d_i$ has the same Chern classes as the direct sum $\mathbb{C}^n/\mathbb{C}^{u_i} \oplus \mathbb{C}^{u_i}/V_{d_i}$, where $\mathbb{C}^{n_i}/\mathbb{C}^{u_i}$ is a trivial bundle on $X_{(D,L,U)}$. Consequently, $c_k(\mathbb{C}^n/V_{d_i})$ vanishes for $k > u_i - d_i$, implying vanishing in $H^*(X_{(D,L,U)})$ of the $k^{th}$ elementary symmetric function in the Chern roots $x_{d_i+1}, \ldots, x_n$ of the bundle $\mathbb{C}^n/V_{d_i}$ for $k > u_i - d_i$.

Thus $I_{(D,L,U)} \subset \ker \phi$, so that $\phi$ induces a (grade-doubling) surjective ring homomorphism

$$\overline{\phi} : R_{(D,L,U)} \longrightarrow H^*(X_{(D,L,U)}).$$

For the rest of the theorem, we must show that this surjection $\overline{\phi}$ is an isomorphism. We observe that because $\overline{\phi}$ is a surjection, it will be an isomorphism if we can show either of two other equivalent conditions:

(C1) If $R_{(D,L,U)}$ has a $\mathbb{Z}$-spanning set of cardinality equal to the rank $r$ of the free abelian group $H^*(X_{(D,L,U)})$, then we would have a composite surjection of abelian groups

$$\mathbb{Z}^r \to R_{(D,L,U)} \xrightarrow{\overline{\phi}} H^*(X_{(D,L,U)}) \cong \mathbb{Z}^r$$

forcing $\overline{\phi}$ to be an isomorphism.

(C2) Given the decomposition

$$X_{(D,L,U)} = X = \coprod_{wW_J \in \mathcal{I}} X^o_{wW_J}$$

we know from Proposition 2.1 (v) that the cohomology classes $[X_{wW_J}]^*$ with $wW_J$ not in $\mathcal{I}$ generate the kernel of the map

$$A_D/I_n \cong H^*(G/P_J) \to H^*(X_{(D,L,U)}).$$

Consequently, $\overline{\phi}$ will be an isomorphism if we can show that each $[X_{wW_J}]^*$ with $wW_J$ not in $\mathcal{I}$, when considered as an element of $A_D/I_n \cong H^*(G/P_J)$, lies in the ideal $I_{(D,L,U)}/I_n$.

Our strategy consists of three steps:

**Step 1.** Use equivalent condition (C2) to reduce to the case where $|L|+|U|=1$, that is, when $X_{(D,L,U)}$ is defined by a *single* inclusion condition.

**Step 2.** Do an independent reduction that uses equivalent condition (C1) to reduce to the case of complete flags, i.e. $D=D_0:=\{0,1,2,\ldots,n-1,n\}$.

**Step 3.** Deal with the remaining very special case where $D=\{0,1,2,\ldots,n-1,n\}$ and $|L|+|U|=1$, using equivalent condition (C1) and one of the fibrations from Theorem 1.1 (iv).

**Step 1.**

We assume that $\overline{\phi}$ is an isomorphism whenever $|L|+|U|=1$, that is, when there is only one inclusion condition on the flags in $X_{(D,L,U)}$. Given an arbitrary $(D,L,U)$, to use condition (C2) we assume $wW_J$ is not in $\mathcal{I}$, so that at least one inclusion condition parametrized by $L$ or $U$ is not satisfied by all flags in $X_{wW_J}$. Let $X_{(D,L_0,U_0)}$ be the subvariety defined by this single inclusion condition, so that $|L_0|+|D_0|=1$. By our assumption, condition (C2) is satisfied for $X_{(D,L_0,U_0)}$, so that $[X_{wW_J}]^*$ lies in the ideal $I_{(D,L_0,U_0)}/I_n$ when we consider it as an element of $A_D/I_n$. Since $L_0\subseteq L$ and $U_0\subseteq U$ implies $I_{(D,L_0,U_0)}\subseteq I_{(D,L,U)}$, this implies $[X_{wW_J}]^*$ lies in the ideal $I_{(D,L,U)}/I_n$, as desired.

**Step 2.**

We assume that $\overline{\phi}$ is an isomorphism whenever $D=D_0$.

Given $X_{(D,L,U)}$, let $J\subset S$ satisfy $D=D(J)$. Then a fiber bundle as in Proposition 2.2, along with Proposition 2.3 imply that, as free abelian groups,

$$\operatorname{rank}_{\mathbb{Z}}H^*(X_{(D_0,L,U)})=r\cdot\operatorname{rank}_{\mathbb{Z}}H^*(X_{(D,L,U)}) \tag{6}$$

where $r$ is the rank of the cohomology of $\prod_{i=1}^{s+1}GL_{d_i-d_{i-1}}(\mathbb{C})/B$, that is

$$r=\prod_{i=1}^{s+1}(d_i-d_{i-1})!=|W_J|.$$

In light of (6), and since we have assumed $\overline{\phi}$ is an isomorphism when $D=D_0$, it would suffice to show that $R_{(D_0,L,U)}$ is a free $R_{(D,L,U)}$-module of rank $r$, since this would imply that $R_{(D,L,U)}$ must be free abelian and have the correct rank.

To this end, it is known that the inclusion

$$A_D=\mathbb{Z}[\mathbf{x}]^{W_J}\overset{\iota}{\hookrightarrow}\mathbb{Z}[\mathbf{x}]=A_{D_0}$$

makes $A_{D_0}$ a free $A_D$-module of rank equal to the same number $r=|W_J|$. Furthermore, there is a $A_D$-splitting

$$\pi_{w_0(J)}:A_{D_0}\to A_D$$

of this inclusion, where $w_0(J)$ is the longest permutation in $W_J$, and $\pi_{w_0(J)}$ is the *total $W_J$-symmetrizer* (or *isobaric divided difference*, *Demazure operator*) corresponding to $w_0(J)$ (see [8]). Since by definition $I_{(D_0,L,U)}$ is exactly the ideal in $A_{D_0}$ generated by $\iota(I_{(D,L,U)})$, the following lemma (whose straightforward proof we omit) completes Step 2:

**Lemma 3.2.** *Let $A\overset{\iota}{\hookrightarrow}B$ be an inclusion of rings which makes $B$ a free $A$-module of rank $r$, and which has an $A$-module splitting $\sigma:B\to A$, i.e. $\sigma\circ\iota=id_A$.*

*Then for any ideal $I$ in $A$, and $J$ the ideal in $B$ generated by $i(I)$, the induced map $A/I \overset{\iota}{\hookrightarrow} B/J$ is injective and makes $B/J$ into a free $A/I$-module of rank $r$.* $\square$

**Step 3.**

We must show that $\overline{\phi}$ is an isomorphism when $D = D_0$ and $|L| + |U| = 1$.

Assume without loss of generality that $L = \emptyset$ and $U = \{(d,u)\}$ is a singleton; the case where $L$ is a singleton and $U = \emptyset$ follows via the symmetry $x_i \longleftrightarrow x_{n+1-i}$ on $\mathbb{Z}[\mathbf{x}]$.

We will use condition (C1). A special case of the fiber bundles referred to in Theorem 1.1 (iv) is the map

$$X_{(D_0,\emptyset,(d,u))} \to X_{(\{0,1,2,\ldots,d,u\},\emptyset,\emptyset)}$$

which sends a complete flag $0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n$ satisfying $V_d \subset \mathbb{C}^u$ to the partial flag $0 \subset V_1 \subset \cdots \subset V_d \subset \mathbb{C}^u$, and has fiber $GL_{n-d}(\mathbb{C})/B$. Since the fiber and base are partial flag manifolds whose cohomology rings are free abelian of rank $(n-d)!$ and $u(u-1)\cdots(u-d+1)$, respectively, one concludes using Proposition 2.3 that $H^*(X_{(D_0,\emptyset,(d,u))})$ is free abelian of rank

$$r := u(u-1)\cdots(u-d+1)\cdot(n-d)!.$$

To show that $R_{(D_0,\emptyset,(d,u))}$ has a spanning set of this cardinality $r$, we borrow a trick from [3, p. 163]. Recall $I_n = (e_i(1,n) : i = 1,2,\ldots,n)$, so that we have the following identities in $R_{D_0}[t] = A_{D_0}/I_n[t]$:

$$\begin{aligned}\prod_{i=1}^{n}(1-x_it) &= \sum_{s\geq 0}(-1)^s e_s(1,n)t^s \\ &= 1 \\ \prod_{i=1}^{m}(1-x_it)^{-1} &= \prod_{i=m+1}^{n}(1-x_it)\end{aligned}$$

and consequently

$$\sum_{r\geq 0} h_r(1,m)t^r = \sum_{s=0}^{n-m}(-1)^s e_s(m+1,n)t^s \tag{7}$$

where $h_r(i,j)$ is the (*complete*) *homogeneous symmetric function* of degree $r$ in the variables $x_i, x_{i+1}, \ldots, x_j$, i.e. the sum of all monomials of degree $r$ in these variables. We conclude that in $R_{D_0}$ (and hence also in the further quotient $R_{(D_0,\emptyset,(d,u))}$), we have for any $m$ that

$$h_r(1,m) = 0 \text{ for } r > n-m. \tag{8}$$

But we can say more in $R_{(D_0,\emptyset,(d,u))}$, since $I_{(D_0,\emptyset,(d,u))}$ contains also the elements $e_s(d+1,n)$ for $s \geq u-d+1$. Using the fact that

$$e_s(m,n) = e_s(m+1,n) + x_m e_{s-1}(m+1,n)$$

we conclude that $I_{(D_0,\emptyset,(d,u))}$ also contains $e_s(m+1,n)$ for every $m \leq d$ and $s \geq u-m+1$. Consequently, using (7) we conclude that in $R_{(D_0,\emptyset,(d,u))}$

$$h_r(1,m) = 0 \text{ for } m \leq d \text{ and } r \geq u-m+1. \tag{9}$$

Since $h_r(1,m)$ equals $x_m^r$ plus a sum of monomials which are lower in a lexicographic ordering with $x_n > \cdots > x_1$, one can use such relations to systematically rewrite polynomials that use powers $x_m^s$ with $s > r$ in terms of monomials involving lower powers of $x_m$ and arbitrary powers of $x_1, \ldots, x_{m-1}$. Therefore the relations (8), (9) imply that $R_{(D_0,\emptyset,(d,u))}$ is spanned by all monomials

$$x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d} x_{d+1}^{i_{d+1}} x_{d+2}^{i_{d+2}} \cdots x_n^{i_n}$$

satisfying $i_m \geq 0$ for each $m$ and

$$\begin{aligned} &i_1 \leq u-1,\ i_2 \leq u-2,\ \cdots,\ i_d \leq u-d \\ &i_{d+1} \leq n-d-1,\ i_{d+2} \leq n-d-2,\ \cdots,\ i_{n-1} \leq 1,\ i_n \leq 0. \end{aligned}$$

Since this spanning set has cardinality $r = u(u-1)\cdots(u-d+1)\cdot(n-d)!$, we are done with Step 3, and with the proof of Theorem 3.1.

**Remark 3.3.**
In the special case where

$$\begin{aligned} &D = \{0,1,2\ldots,l,n\}, \\ &L = \emptyset, \\ &U = \{(1,\lambda_1),\ldots,(l,\lambda_l)\} \end{aligned}$$

for some number partition $\lambda = (\lambda_1 \leq \cdots \leq \lambda_l)$, the varieties $X_{(D,L,U)}$ are all smooth Schubert subvarieties of $X_D$, which were studied by K. Ding [2] under the name of *partition varieties*. Among other things, he observed that their Schubert cell decomposition is indexed by rook placements on the Ferrers board corresponding to the partition $\lambda$.

In this special case, the proof of Step 3 in Theorem 3.1 actually showed that

$$R_{(D,L,U)} \cong \mathbb{Z}[x_1,\ldots,x_l]/(h_{\lambda_i - i+1}(1,i) : i = 1,2,\ldots,l) \tag{10}$$

and that $\{h_{\lambda_i-i+1}(1,i) : i = 1,2,\ldots,l\}$ forms a Gröbner basis for the ideal that it generates with respect to a lexicographic ordering of the variables in which $x_1 < \cdots < x_l$ (cf. [10, §1.2]).

## 4. A pattern avoidance characterization for Schubert varieties defined by inclusions

In this section we give a characterization of the Schubert varieties defined by inclusions (Theorem 4.2), making use of Fulton's notion of the essential set of a permutation $w$.

The essential set was introduced to parametrize a minimal set of Schubert conditions which imply all of the other Schubert conditions on $X_{wW_J}$. We present here a version slightly different from his original definition that is adapted to our conventions, and which may be inferred from the discussion in [4, pp. 20-21]. Given $w = w_1 w_2 \cdots w_n$, we first describe an adapted version of its *diagram*: First draw an $n$-by-$n$ matrix of *bubbles*. For each $i$, place an $\times$ in the $(i, w_i)$ entry, and erase all bubbles weakly to its left in the same row, and all bubbles weakly below it in the same column. The bubbles remaining define the diagram.

Note that the number of bubbles in this diagram is *not* the length $l(w)$, but rather $l(w_0 w)$ where $w_0$ is the longest element in $W = S_n$. This corresponds to the fact that this diagram is actually a mirror image of the *usual* diagram $D(w_0 w)$ associated to $w_0 w$ [4, p. 11], [8, p. 8].

The *essential set* is defined to be the set of bubbles $(i, j)$ for which there is neither a bubble in $(i+1, j)$ nor in $(i, j-1)$. Assuming that $w = \max wW_J$, for each such bubble there is a Schubert condition on the flags in $X_{wW_J}$ asserting

$$\dim V_i \cap \mathbb{C}^{j-1} \geq r \tag{11}$$

where $r$ is the number of bubbles in column $j$ lying weakly above the bubble $(i, j)$. We will call such Schubert conditions *essential*. The essential Schubert conditions imply all of the other Schubert conditions on flags in $X_{wW_J}$ and no essential Schubert condition is implied by the other Schubert conditions. For example, if $w = 56238417$, then the diagram of $w$ is:

$$\begin{array}{ccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & & & & & \times & \bullet & \circ & \circ \\
2 & & & & & & \times & \circ & \circ \\
3 & & \times & \bullet & \circ & & & \circ & \circ \\
4 & & & \times & \bullet & & & \bullet & \circ \\
5 & & & & & & & & \times \\
6 & & & & \times & & & \circ & \\
7 & \times & & & & & & \bullet & \\
8 & & & & & & & \times &
\end{array}$$

Here the bubbles in the essential set are shown darkened and the essential Schubert conditions on $X_{wW_J}$ are:

$$\begin{aligned}
&\dim V_1 \cap \mathbb{C}^5 \geq 1, \text{ i.e. } V_1 \subset \mathbb{C}^5 \\
&\dim V_3 \cap \mathbb{C}^2 \geq 1 \\
&\dim V_4 \cap \mathbb{C}^6 \geq 4, \text{ i.e. } V_4 \subset \mathbb{C}^6 \\
&\dim V_4 \cap \mathbb{C}^3 \geq 2 \\
&\dim V_7 \cap \mathbb{C}^6 \geq 6, \text{ i.e. } \mathbb{C}^6 \subset V_7.
\end{aligned}$$

The following observation is immediate from the construction of the diagram of $w$.

**Proposition 4.1.** *The essential set of $w$ contains an element in row $i$ if and only if $w_i < w_{i+1}$.*

*In particular, if $w = \max wW_J$, then the essential set of $w$ is contained entirely in the rows indexed by $D(J)$.* □

Note that the second assertion of Proposition 4.1 implies that the Schubert conditions on complete flags in $X_w$ are essentially the same as those on partial flags in $X_{wW_J}$, proving Proposition 2.2.

The following theorem gives a combinatorial characterization of the Schubert varieties $X_{wW_J}$ defined by inclusions.

**Theorem 4.2.** *Let $w = \max wW_J$. Then $X_{wW_J}$ is defined by inclusions if and only if $w$ does not contain a subsequence with the same relative order as*

$$4231,\ 35142,\ 42513,\ \textit{or } 351624. \tag{12}$$

*Proof.* We begin with two facts about an essential Schubert condition (11). Both follow immediately from the way the diagram is constructed.

**Fact 1**: $r \leq i$, with strict inequality if and only if there exists some $\times$ in the diagram strictly to the northeast of $(i, j)$.
**Fact 2**: $r \leq j-1$, with strict inequality if and only if there exists some $\times$ in the diagram strictly to the southwest of $(i, j)$.

Given these facts, we now prove the theorem. Assume $X_{wW_J}$ is not defined by inclusions, so there is an essential bubble $(i, j)$ in the diagram of $w$ that does not correspond to an inclusion condition. Consequently the conditions in Facts 1 and 2 must hold, i.e. there must exist some $\times$ located in $(i', w_{i'})$ with $i' < i$ and $w_{i'} > j$, and also some $\times$ located in $(i'', w_{i''})$ with $i'' > i$ and $w_{i''} < j-1$. Call these two $\times$'s *witnesses*. We will further assume, without loss of generality, that the southwest witness located in $(i'', w_{i''})$ is chosen with $i''$ minimal.

We show that at least one of the four sequences in (12) will then appear, depending upon whether the rectangular box in the diagram of $w$ bounded by rows $i', i''$ and columns $w_{i''}, w_{i'}$ contains an $\times$ strictly to the left of column $j$, and an $\times$ strictly below row $i$. If one can find

- both, then combining these with the two witnesses produces a subsequence in $w$ with the same relative order as 4231,

- the former, but not the latter, then combining the two witnesses with the former and with the two $\times$'s in $(w^{-1}(j), j), (i+1, w_{i+1})$ produces a subsequence in $w$ with the same relative order as 42513,

- the latter, but not the former, then combining the two witnesses with the latter and with the two $\times$'s in $(w^{-1}(j-1), j-1), (i, w_i)$ produces a subsequence in $w$ with the same relative order as 35142,

- neither, then combining the two witnesses with the four $\times$'s in
$$(w^{-1}(j-1), j-1),\ (w^{-1}(j), j),\ (i, w_i),\ (i+1, w_{i+1})$$
produces a subsequence in $w$ with the same relative order as 351624.

The converse is easier to see. First suppose that $u$ is one of the four permutations in (12). Then it is easy to exhibit a (unique!) essential bubble that corresponds to a non-inclusion Schubert condition in $X_u$: it is the bubble at position

- $(2,3)$ for $u = 4231$,
- $(3,4)$ for $u = 35142$,
- $(2,3)$ for $u = 42513$,
- $(3,4)$ for $u = 351624$.

Then if $w$ contains a subsequence with the same relative order as $u$, there will be a "region" containing some bubbles within the diagram for $w$ that corresponds to the above essential bubble in $u$. By Facts 1 and 2, it is easy to see that any essential bubble in that region will give rise to a non-inclusion Schubert condition on $X_{wW_J}$. $\square$

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