PROPERTIES OF THE DUAL CONE OF MONOMIAL-POSITIVE IMMANANTS

ROHIT AGRAWAL, VLADIMIR SOTIROV

Abstract. We investigate a cone in the symmetric group algebra introduced by Stembridge [2]. It is dual to the cone of monomial-positive immanants of $n \times n$ matrices with indeterminate entries. We present a new set of relations between elements of the dual cone, and use these relations to show that the cone is finitely generated for $n = 6$, generalizing Stembridge's result for $n = 5$.

1. Introduction

1.1. Background.

Definition 1.1. Fix a positive integer n and for every pair of integers $1 \leq i \leq j \leq n$ define the *bracket* $[i, j] \in \mathbb{R}S_n$ as the sum of of all permutations σ of $\{1, \dots, n\}$ that fix integers outside the interval $[i, j] = \{i, i+1, \dots, j-1, j\}$. More formally, let $[i, j] = \sum_{\sigma \in S_{[i,j]}} \sigma$ where $S_{[i,j]} := \mathrm{Stab}(\{1, 2, \dots, i-1, j+1, \dots, n\}) \leq S_n$.

Definition 1.2 (Stembridge [2])**.** Let Π_n be the set of all finite products of brackets in $\mathbb{R}S_n$ (really in $\mathbb{Q}S_n$), and define the *cone* $\mathcal{C}(\Pi_n)$ to be the set of all non-negative linear combinations of elements in Π_n.

Remark 1.3. The significance of the cone $\mathcal{C}(\Pi_n)$ is that it is dual to the cone of monomial-positive immanants of $n \times n$ matrices with indeterminate entries. Briefly, a function $f\colon S_n \to \mathbb{R}$ gives rise to an *immanant* $A \to f[A] \in \mathbb{R}[a_{ij}]$ where $A = (a_{ij})$ is an $n \times n$ matrix with indeterminate entries, and the map is given by $A \mapsto \sum_{\sigma \in S_n} f(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \dots a_{n\sigma(n)}$. An immanant is said to be *monomial-positve* if $f[A]$ is a polynomial with non-negative coefficients. See [2] for details.

To each product of brackets we associate a *diagram*, which are constructed by drawing a line segment for each bracket, as in Figure 1 which shows the diagram for the product $[1, 2][3, 4][[2, 3][2, 5]$.

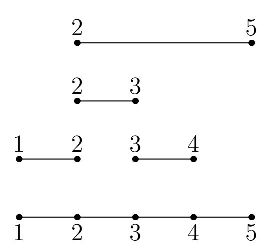

Figure 1. Diagram for $[1, 2][3, 4][2, 3][2, 5]$

Date: December 16, 2011.

We are preoccupied with the following conjecture:

Conjecture 1.4. *The cone $\mathcal{C}(\Pi_n)$ is finitely generated (i.e. polyhedral) for each n.*

The conjecture has been verified for $n \leq 5$ by Stembridge who used *Maple* to list all the extreme rays (generators) of the cone [2]. In this paper we present a more sophisticated approach to the problem, which makes verifying the conjecture for $n > 5$ computationally feasible. In particular, we verify the conjecture for $n = 6$, thus proving the following theorem.

Theorem 1.5. *The cone $\mathcal{C}(\Pi_6)$ is finitely generated with the 748 rays given in Appendix A.*

1.2. **Our approach.** We approach the problem by identifying sufficient "local" criteria for a ray of $\mathcal{C}(\Pi_n)$ to not be extreme, that is, to be *reducible*, and using these criteria to eliminate all but finitely many elements of Π_6 as possible extreme rays. Obviously, a sufficient 'global" condition for $\pi \in \Pi$ to not be an extreme ray of the cone $\mathcal{C}(\Pi_n)$ is that π can be written as a positive linear combination of products of brackets that are not positive multiples of π in the group algebra. In other words, it is enough to show that $\pi = \sum_{i=1}^{n} a_i \pi_i$ with $a_i > 0$ and $\pi_i \in \Pi_n$ where at least one of the π_i is not a positive scalar multiple of π in $\mathbb{R}S_n$.

Two questions arise:

(1) How do we find such decompositions of elements $\pi \in \Pi_n$?
(2) How do we determine if two products of brackets π_1 and π_2 are actually distinct in the group algebra?

The second question warrants some explanation: it is possible for a an element of Π_n living in the group algebra $\mathbb{R}S_n$ to have several different expressions as a product of brackets. For example, the two products of brackets [12][24][12] and [34][13][34] are in fact the same element of $\mathbb{R}S_n$. Furthermore, it is possible for two distinct products of brackets to be positive scalar multiples of each other, as for example $[12][13] = 2[13]$, or equivalently $[13] = \frac{1}{2}[12][13]$. Hence, question 2 raises a non-trivial concern.

A third, and more subtle question, arises when we try to phrase what it means for a reduction criterion to be "local" rather than "global". Naïvely, a "local" criterion would be a decomposable product of brackets π such that for any two products of brackets π' and π'', the product $\pi'\pi\pi''$ is also decomposable. The existence of such "local" criteria, however, is not at all clear:

(3) If π decomposes as $\sum_{i=1}^{k} a_i \pi_i$, how do we know that the decomposition $\pi'\pi\pi'' = \sum_{i=1}^{k} a_i \pi' \pi_i \pi''$ also has the property that at least one of the $\pi' \pi_i \pi''$ is not a positive scalar multiple of $\pi'\pi\pi''$?

It turns out that such a naïve"local" criterion in fact does not exist due to a general relation we call absorption.

Proposition 1.6 (Absorption)**.** *If a bracket $[i, j]$ is contained in a bracket $[i', j']$ in the sense that $i' \leq i \leq j \leq j'$ (in the language of diagrams, one segment is contained in one directly above or below it), then we have the identity:*

$$[i, j][i', j'] = (j - i + 1)![i', j'] = [i', j'][i, j].$$

Proof. It is clear that $[i', j']$, as the formal sum of all permutations in the subgroup $S_{[i',j']}$, is invariant under left- and right- multiplication by elements of that same subgroup. Since $S_{[i,j]} \leq S_{[i',j']}$ for $[i,j] \subset [i',j']$, it follows that for every $\sigma \in S_{[i,j]}$ we have $\sigma[i',j'] = [i',j'] = [i',j']\sigma$ and hence that

$$[i,j][i',j'] = \sum_{\sigma \in S_{[i,j]}} \sigma[i',j'] = \sum_{\sigma \in S_{[i,j]}} [i',j'] = (j-i+1)![i',j']$$

$$[i',j'][i,j] = [i',j'] \sum_{\sigma \in S_{[i,j]}} \sigma = \sum_{\sigma \in S_{[i,j]}} [i',j'] = (j-i+1)![i',j'] \qquad \square$$

Absorption makes naïve "local" criteria impossible because setting $\pi' = [1,n]$ will always kill the criterion, as in the following example. An easy computation tells us that $[12][23][12] = [12] + [13]$, yet by absorption we have $[13][12][23][12] = 8[1,3]$, $[13][12] = 2[1,3]$, and $[13][13] = 6[1,3]$, which means that the equation $[1,3][1,2][2,3][1,2] = [1,3][1,2] + [1,3][1,3]$ does *not* guarantee that the bracket product $[1,3][1,2][2,3][1,2]$ is not an extreme ray.

Nevertheless, we can address questions (2) and (3) in one fell swoop by defining the notion of weight.

Definition 1.7. Given a product of brackets $[i_1,j_1][i_2,j_2]\dots[i_k,j_k]$ we define its *weight* to be the sum $\sum_{r=1}^{k}(j_r - i_r)$. In the language of diagrams, the weight is the sum of the length of all the line segments (so the weight of Figure 1 is 6).

The weight satisfies two important properties. First: it is always non-negative. Second: it is additive in the sense that if π and π' are products of brackets, then the weight of $\pi\pi'$ is the sum of the weights of π and π'. Most importantly, the weight is a characteristic of the particular *expression* of a product of brackets, rather than of the corresponding element of the group algebra. These properties allow us to formulate a slightly different "local" condition, for which issues (2) and (3) do not arise.

Proposition 1.8. *A product of brackets π is not an expression of minimal weight for an extreme ray of the cone (is* reducible*) if*

$$\pi = \sum_{i=1}^{k} a_i \pi_i$$

where $a_i > 0$ and at least one of the π_i has weight lower weight than π.

Proof. Clearly, if π_i is of lower weight than π and π is an expression of minimal weight for an element of the group algebra, then π_i cannot be a positive multiple of π.

Furthermore, for any two other products of brackets π' and π'' we have that $\pi'\pi\pi'' = \sum_{i=1}^{k} \pi'\pi_i\pi''$ and by additivity of the weight we have that the weight of $\pi'\pi_i\pi''$ is lower than the weight of $\pi'\pi\pi''$. Hence, the condition is "local". $\square$

For example, the equation $[13][12][23][12] = [13][12] + [13][13]$ expresses a product of brackets of weight 5 as the sum of products of brackets of weights 3 and 4, which automatically implies that $[13][12][23][12]$ is reducible. In fact, writing down the equation is unnecessary since we can simply observe that $[13][12][23][12]$ contains $[12][23][12]$ as a subproduct and that the equation $[12][23][12] = [12] + [13]$ is a "local" condition as it expresses a product of brackets of weight 3 as the sum of products of bracets of weights 1 and 2.

Now, having established a reasonable notion of what a "local" condition ought to look like, we can state the key result of our paper, which together with the absorption relation and certain symmetries of the group algebra is enough to show Theorem 1.5.

Proposition 1.9. *A triple of brackets* $[1,l][k,n][1,k]$ *for* $1 \leq n-l \leq k \leq l < n$ *and* $k \neq 1$ *(illustrated in Figure 2) decomposes as:*

$$\begin{aligned}[1,l][k,n][1,k] = {} & \frac{(k-1)!}{(n-l-1)!}[l+1,n][1,l+1][k+1,n] \\ & + (k-(n-l))(k-1)!\frac{(l-k+1)!}{(l-k)!}[1,l][k+1,n] \\ = {} & \frac{(k-1)!}{(n-l-1)!}[l+1,n][1,l+1][k+1,n] \\ & + (k-n+l)(l-k+1)(k-1)![1,l][k+1,n]\end{aligned}$$

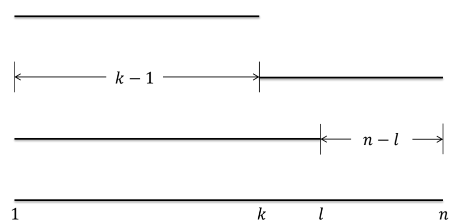

FIGURE 2. Diagram of Proposition 1.9

This proposition (actually, a slight generalization of it using absorption) is the only general relationship of brackets of products that we have been able to find. Notice that the term $[1,l][k+1,n]$ is of weight strictly less than $[1,l][k,n][1,k]$, which implies that for $k > n-l$ the above is in fact a "local" condition. For example, the easily computed relation $[12][23][12] = [12] + [13]$ follows immediately from setting $k = l = 2$ and $n = 3$. Similarly, the alluded to (and more tedious to compute) relation $[12][24][12] = [34][13][34]$ follows from setting $k = l = 2$ and $n = 4$, as does the relation $[12][25][13] = [35][14][45]$ from setting $k = 2$, $l = 3$, $n = 5$.

For $k = n-l$, however, the relation degenerates to stating that $[1,l][k,n][1,k]$ equals $[l+1,n][1,l+1][k+1,n]$ as elements of the group algebra. Since both expressions have the same weight, when $k = n-l$ we do not have a "local" condition. Nevertheless, we may use the relation when $k = n-l$ to decompose the product of brackets since even though a product of brackets $\pi'[1,l][k,n][1,k]\pi''$ may contain $[1,l][k,n][1,k]$ as its only triple, it is possible that the equivalent expression $\pi'[l+1,n][1,l+1][k+1,n]\pi''$ may contain additional triples that do decompose with terms of lower weight. In other words, decompositions which do not have terms of smaller weight may still be useful in showing a product of brackets is not an extreme ray if we allow ourselves to iteratively decompose until we do obtain a term of smaller weight (morally, this works because allowing ourselves to iteratively decompose is allowing ourselves to patch together several "local" decompositions or equivalences, which can give us stronger results).

For example, the product [12][24][12][35] of weight 6 equals (in $\mathbb{R}S_n$) the product [34][13][34][35] which by absorption is a scalar multiple of [34][13][35] which is of weight 5. Hence, we can conclude that [12][24][12][35] is not a minimal weight expression for an extreme ray, because it is a scalar multiple of the lower weight expression [34][13][35].

2. Proof of Theorem 1.5

In this section we prove Theorem 1.5. We begin with the single technical proof that involves the particular structure of the group algebra – the proof of Proposition 1.9. Then, by using the elementary property of absorption (Proposition 1.6), we amplify the Proposition to the more general Corollary 2.6. Finally, we encode the general triple relation and the property of absorption into a computationally-feasible algorithm that eliminates all products of brackets that are not minimal weight expressions of extreme rays of the cone. We run this algorithm for $n = 6$ and obtain a list of 750 distinct products of brackets which contains a superset of the extreme rays of $\mathcal{C}(\Pi_6)$. We then use the Parma Polyhedra Library to remove non-extremal rays.

Throughout this section, and the rest of the paper, we will make liberal use of the following symmetries of the group algebra, taking note that they preserve weight and relations between products of brackets, and hence boost four-fold any "local" condition that we prove.

(1) The standard anti-automorphism of the group algebra, which evidently transforms Π_n by the rule $[i_1, j_1] \dots [i_k, j_k] \to [i_k, j_k][i_{k-1}, j_{k-1}] \dots [i_1, j_1]$, effectively *reversing* the product of brackets. In the language of diagrams, this corresponds to reflection about a horizontal axis.

(2) The inner automorphism of the group algebra given by conjugating by $\sigma = (n, (n-1), \dots 2, 1)$ (one-line notation) which transforms Π_n according to the rule $[i_1, j_1] \dots [i_k, j_k] \to [\sigma(j_1), \sigma(i_1)] \dots [\sigma(j_k), \sigma(i_k)]$, effectively *reflecting* the products of brackets around $\frac{n}{2}$. In the language of diagrams, this corresponds to reflection about a vertical axis.

2.1. **Proof of Proposition 1.9.**

In order to prove Proposition 1.9, we will determine the coefficients of each permutation σ in the expansions of the three products of brackets $[1, l][k, n][1, k]$, $[l + 1, n][1, l + 1][k + 1, n]$, and $[1, l][k + 1, n]$ as elements of the group algebra $\mathbb{R}S_n$.

First, we consider the simplest term $[1, l][k + 1, n]$.

Lemma 2.1. *The coefficient of* σ *in* $[1, l][k + 1, n]$ *with* $1 < k \le l < n$ *is* 0 *if* $\sigma([1, k]) \cap [l + 1, n] \ne \emptyset$ *and* $(l - k)!$ *otherwise.*

Proof. First, note that any permutation σ occurring with non-zero coefficient in $[1, l][k + 1, n]$ can be written as a product $\alpha\beta$ where $\alpha \in S_{[1,l]}$ and $\beta \in S_{[k+1,n]}$. Evidently, α fixes point-wise the interval $[l+1, n]$ and β fixes point-wise the interval $[1, k]$, so $\alpha\beta([1, k]) \cap [l + 1, n] = \emptyset$.

Conversely, if $\sigma([1, k]) \cap [l+1, n] = \emptyset$, we will compute that there are $(l-k)! > 0$ ways of writing $\sigma = \alpha\beta$ where $\alpha \in S_{[1,l]}$ and $\beta \in S_{[k+1,n]}$.

Because β fixes point-wise the interval $[1, k]$ it follows that for any expression $\sigma = \alpha\beta$ we would have that $\sigma([1, k]) = \alpha([1, k])$. Hence, given a particular σ, the values of the possible $\alpha \in S_{[1,l]}$ on the interval $[1, k]$ are already determined which leaves exactly $(l-k)!$ possibilities for α. Each possibility for α gives rise to a unique possible β by the relation $\beta = \alpha^{-1}\sigma$. We need to check that β is in fact in $S_{[k+1,n]}$, but that is certainly true since for every $j \in [1,k]$ we have that $\sigma(j) = \alpha(j)$ and hence $\alpha^{-1}\sigma(j) = j$.

Hence, each σ with $\sigma([1,k]) \cap ([l+1,n])$ gives rise to exactly $(l-k)!$ expressions $\alpha\beta$ with $\alpha \in S_{[1,l]}$ and $\beta \in S_{[1,k+1]}$. □

Next, we compute explicitly the coefficients of the permutations σ in the expansion of the product $[1,l][k,n][1,k]$ for $1 \le k \le l \le n$.

Lemma 2.2. *The coefficient of σ in $[1,l][k,n][1,k]$ for $1 \le k \le l \le n$, is $(k-i)!(l-k+1)!$ where $i = |\sigma([1,k]) \cap [l+1,n]| \le 1$, and zero otherwise.*

Proof. First, note that any permutation σ occurring with a non-zero coefficient in $[1,l][k,n][1,k]$ can be decomposed as a product $\sigma = \alpha\beta\gamma$ where $\alpha \in S_{[1,l]}$,$\beta \in S_{[k,n]}$,$\gamma \in S_{[1,k]}$.

Then $\sigma([1,k]) \cap [l+1,n] = \alpha\beta([1,k]) \cap [l+1,n] = \beta([1,k]) \cap [l+1,n]) = \beta([k,k]) \cap [l+1,n]$. Evidently then $|\sigma([1,k]) \cap [l+1,n]$ equals either 1 (in the case that $\beta(k) \in [l+1,n]$) or 0 (in the case that $\beta(k) \notin [l+1,n]$.

Conversely, suppose that $\sigma \in S_n$ is such that $|\sigma([1,k]) \cap [l+1,n]| \le 1$, and let $i = |\sigma([1,k]) \cap [l+1,n]$. We will show that the number of ways of writing $\sigma = \alpha\beta\gamma$ with $\alpha \in S_{[1,l]}$, $\beta \in S_{[k,n]}$, and $\gamma \in S_{[1,k]}$ is $(k-i)!(l-k+1)!$.

If $i = 1$, then σ sends exactly one element of $[1,k]$ to $[l+1,n]$. If that element is j, then because α fixes point-wise $[l+1,n]$, and β and γ fix point-wise $[1,k-1]$ and $[k+1,n]$ respectively, the only way this could happen if $\sigma = \alpha\beta\gamma$ is that $j \overset{\gamma}{\to} k \overset{\beta}{\to} \sigma(j) \overset{\alpha}{\to} \sigma(j)$. Hence, we have at most $(k-1)!$ choices for $\gamma \in S_{[1,k]}$. Then we need to compute the number of ways of write $\sigma\gamma^{-1} = \alpha\beta$ for $\alpha \in S_{[1,l]}$ and $\beta \in S_{[k,n]}$. For that we can use Lemma 2.1 since it is easy to see that $\sigma\gamma^{-1}([1,k-1]) \cap [l+1,n] = \emptyset$ as $\sigma\gamma^{-1}([1,k]) \cap [l+1,n] = \sigma([1,k]) \cap [l+1,n] = \{\sigma(j)\}$ and $\sigma\gamma^{-1}(k) = \sigma(j)$.

Hence, by Lemma 2.1 we have $(l-k+1)!$ ways of writing $\sigma\gamma^{-1} = \alpha\beta$ where $\alpha \in S_{[1,l]}$ and $\beta \in S_{[k,n]}$ for each of the $\gamma \in S_{[1,k]}$ such that $\gamma(j) = k$. Since there are $(k-1)!$ such γ, we obtain the desired coefficient of $(k-1)!(l-k+1)!$.

If $i = 0$, then σ sends no elements of $[1,k]$ to $[l+1,n]$, and we have no restrictions on γ and hence $k!$ choices. Again we can use Lemma 2.1 since it is again easy to see that $\sigma\gamma^{-1}([1,k-1]) \cap [l+1,n] = \emptyset$ as $\sigma\gamma^{-1}([1,k-1]) \cap [l+1,n] \subset \sigma\gamma^{-1}([1,k]) \cap [l+1,n] = \sigma([1,k]) \cap [l+1,n] = \emptyset$ by assumption. Hence, we have $(l-k+1)!$ ways of writing $\sigma\gamma^{-1} = \alpha\beta$ where $\alpha \in S_{[1,l]}$ and $\beta \in S_{[k,n]}$ for each of the $\gamma \in S_{[1,k]}$. Hence, we obtain the desired coefficient of $k!(l-k+1)!$. □

Third, we compute the coefficients of the permutations σ in the expansion of $[l+1,n][1,l+1][k+1,n]$ in a way compatible with the coefficients of $[1,l][k,n][1,k]$ computed in Lemma 2.2.

Lemma 2.3. *The coefficient of σ in $[l+1,n][1,l+1][k+1,n]$ for $1 < k \le l < n$ is $(n-l+1-i)!(l-k+1)!$ where $i = |\sigma([1,k]) \cap [l+1,n]| \le 1$, and zero otherwise.*

Proof. We will compute the coefficient of σ by applying the reflection and reversal symmetry of the group algebra.

First, we set $l' = n-l$, $k' = n-k$ and we observe that the reflection of $[l+1,n][1,l+1][k+1,n]$ is $[1,l'][k',n][1,k']$ with $1 < l' \le k' < n$. Then if σ' is the reflection of σ, then $\sigma([1,k]) \cap [l+1,n] = \sigma'([k'+1,n]) \cap [1,l']$. Hence

$i = |\sigma'([k'+1,n]) \cap [1,l']$ and the coefficient of σ' in $[1,l'][l',n][1,k']$ equals the coefficient of σ in $[l+1,n][1,l+1][k+1,n]$.

Next, observe that the anti-automorphism of the group algebra sends the product $[1,l'][l',n][1,k']$ to $[1,k'][l',n][1,l']$ and σ' to $\sigma'' = \sigma'^{-1}$. Since $i = |\sigma'(k'+1,n]) \cap [1,l']$ and $\sigma'' = \sigma^{-1}$, we have that $i = |\sigma''([1,l']) \cap [k'+1,n]$ and the coefficient of σ'' in $[1,k'][l',n][1,l']$ with $1 < l' \leq k' < n$ is the same as the coefficient of σ in $[1,l][k,n][1,k]$.

Applying Lemma 2.2 we see that the coefficient of σ'' in $[1,k'][l',n][1,l']$ with $1 < l' \leq k' < n$ such that $i = |\sigma''([1,l']) \cap [k'+1,n]| \leq 1$ is $(k'-i)!(l'-k'+1)!$ is $(l'-i)!(k'-l')!$. But this is also the same as the coefficient of σ in $[l+1,n][1,l+1][k+1,n]$ with $1 < k \leq l < n$ and $i = |\sigma([1,l]) \cap [k+1,n]|$. Rewritten in terms of k and l, we see that the coefficient is $(n-l-i)!(l-k+1)!$ as desired. □

Finally, we prove the key Proposition 1.9, restated below, using Lemmas 2.1, 2.2, and 2.3.

Proposition 2.4. *A triple of brackets* $[1,l][k,n][1,k]$ *for* $1 \leq n-l \leq k \leq l < n$ *and* $k \neq 1$ *decomposes as:*

$$\begin{aligned}
[1,l][k,n][1,k] &= \frac{(k-1)!}{(n-l-1)!}[l+1,n][1,l+1][k+1,n] \\
&\quad + (k-(n-l))(k-1)!\frac{(l-k+1)!}{(l-k)!}[1,l][k+1,n] \\
&= \frac{(k-1)!}{(n-l-1)!}[l+1,n][1,l+1][k+1,n] \\
&\quad + (k-n+l)(l-k+1)(k-1)![1,l][k+1,n]
\end{aligned}$$

Proof. Let σ be a permutation such that $i = |\sigma([1,l]) \cap [k+1,n]| \leq 1$.

Under the conditions of the proposition, Lemmas 2.2 and 2.3 tell us that such σ are the only ones that have non-zero coefficient in the products $[1,l][k,n][1,k]$ and $[l+1,n][1,l+1][k+1,n]$, and the the coefficients are $(k-i)!(l-k+1)!$ and $(n-l-i)!(l-k+1)!$, respectively.

Hence, the coefficients of σ in $\frac{(k-1)!}{(n-l-1)!}[l+1,n][1,l+1][k+1,n]$ are $(n-l)(k-1)!(l-k+1)!$ for $i=0$ and $(k-1)!(l-k+1)!$ for $i=1$. Subtracting from $[1,l][k,n][1,k]$, we are left with a coefficient $(k-(n-l))(k-1)!(l-k+1)!$ for $i=0$ and zero for $i=1$. This coefficient is non-negative only if $k \geq n-l$, which is guaranteed by our hypothesis.

Applying Lemma 2.1, we see that we can collect the $(k-(n-l))!(l-k+1)!$ copies of those σ for which $\sigma([1,l]) \cap [k+1,n] = \emptyset$ as a number of copies of $[1,l][k+1,n]$. In particular, the σ in $[1,l][k+1,n]$ occur with coefficient $(l-k)!$, so we have exactly $(k-(n-l))(k-1)!\frac{(l-k+1)!}{(l-k)!}$ copies of $[1,l][k+1,n]$. □

2.2. **The general triple relation.** We now proceed to generalize Proposition 1.9. Since for our purposes we do not need the actual values of the non-zero coefficients in a decomposition, we will for our convenience use the notation $\pi \sim \sum_{i=1}^{k} \pi_i$ to mean that there exist some non-negative numbers a_i such that $\pi = \sum_{i=1}^{k} a_i \pi_i$. Note that $\sim$ is not symmetric, but is reflexive and transitive.

For example, the absorption property from Proposition 1.6 can be written as $[i,j][i',j'] \sim [i',j']$ for $i' \leq i \leq j \leq j'$. Note that it is also true that $[i',j'] \sim [i,j][i',j']$ for $i' \leq i \leq j \leq j'$.

Proposition 2.5. *For $1 \leq n-l \leq r \leq k \leq l < n$ and $r \neq 1$ we have that:*

$$[1,l][r,n][1,k] \sim \chi[1,l][k+1,n] + \sum_{i=1}^{\min\{n-l,k-r+1\}} [l+1,n][1,l+i][k+1,n]$$

where $\chi = 0$ if $n-l = r$ and $\chi = 1$ if $n-l < r$.

Proof. We begin by considering the bracket $[1,l][k,n][1,k+s]$ where $1 \leq n-l \leq k \leq l < n$ and $k \neq 1$ and $s \geq 0$. Clearly, this corresponds to the case in when $k-r = s$ in the notation of the proposition.

We unabsorb a bracket $[1,k]$ from $[1,k+s]$ to obtain the quadruple $[1,l][k,n][1,k+s] \sim [1,l][k,n][1,k][1,k+s]$. Evidently, Proposition 1.9 applies to the triple $[1,l][k,n][1,k]$ and gives us:

$$[1,l][k,n][1,k][1,k+s] \sim \chi[1,l][k+1,n][1,k+s] + [l+1,n][1,l+1][k+1,n][1,k+s]$$

where $\chi = 1$ if $n-l < k$ and $\chi = 0$ if $n-l = k$.

To ease the computations that will follow, we define $F(i,j) = [l+1,n][1,l+i][k+j,n][1,k+s]$ for integers i and j. Then the term $[l+1,n][1,l+1][k+1,n][1,k+s]$ is in fact $F(1,1)$. Naturally, we now consider the product of brackets $[1,l][k+j,n][1,k+s]$ and observe that because $1 \leq n-l \leq k < k+j \leq k+s \leq l < n$, we have the relation

$$\begin{aligned}[1,l][k+j,n][1,k+s] &\sim \chi[1,l][k+j+1,n][1,k+s] \\ &\quad + [l+1,n][1,l+1][k+j+1,n][1,k+s] \\ &= F(1,j+1) + \chi[1,l][k+j+1,n][1,k+s]\end{aligned}$$

Iterating on $[1,l][k,n][1,k+s]$ we see that in fact:

$$[1,l][k,n][1,k+s] \sim \sum_{j=1}^{s} F(1,j) + [1,l][k+s,n][k+s,n] \sim \sum_{j=1}^{s} F(1,j) + [1,l][k+s,n]$$

if $\chi = 1$, and

$$[1,l][k,n][1,k+s] \sim F(1,1)$$

if $\chi = 0$.

Hence, to determine the decomposition of $[1,l][k,n][1,k+s]$, we need only determine the decomposition of $F(1,j) = [l+1,n][1,l+1][k+j,n][1,k+s]$. Once again we can unabsorb from $[1,k+j]$ from $[1,k+s]$ and, as long as $i < n-l$ and $j < s+1$, apply Proposition 1.9 to obtain:

$$\begin{aligned}[l+1,n][1,l+i][k+j,n][1,k+s] &\sim [l+1,n][1,l+i][k+j,n][1,k+j][1,k+j][1,k+s] \\ &\sim [l+1,n][l+i+1,n][1,l+i+1][k+j+1,n][1,k+s] \\ &\quad + [l+1,n][1,l+i][k+j+1,n][1,k+s] \\ &\sim [l+1,n][1,l+i+1][k+j+1,n][1,k+s] \\ &\quad + [l+1,n][1,l+i][k+j+1,n][1,k+s]\end{aligned}$$

Hence, we have that $F(i,j) \sim F(i+1,j+1) + F(i,j+1)$ as long as $i < n-l$ and $j < s+1$. Letting $m = \min\{n-l, s+1\}$, we can iterate relation for $j < s+1$ to obtain $F(1,j) \sim \sum_{i=0}^{m-j} F(1+i, s+1)$.

Thus, we get that all the terms of $F(1,j)$ are terms of $F(1,1)$ and so $F(1,1) \sim \sum_{j=1}^{s} F(1,j)$. Thus, we have:

$$[1,l][k,n][1,k+s] \sim F(1,1) + \chi[1,l][k+s,n]$$

which is almost the relation we set out to prove.

For the final step, we compute $F(1,1)$ as follows: by absorbing $[1, k+s]$ in $[1, l+1+i]$ we get:

$$F(1+i, s+1) = [l+1, n][1, l+1+i][k+s+1, n][1, k+s] \sim [l+1, n][1, l+1+i][k+s+1, n]$$

and hence $F(1,1) \sim \sum_{i=1}^{m} F(1, s+1) \sim \sum_{i=1}^{m} [l+1, n][1, l+i][k+s+1, n]$. □

Note that when $\chi = 1$ this relation is in fact a "local" criterion for a product of brackets not to be an extreme ray because the weight of $[1, l][k+1, n]$ is surely lower than that of $[1, l][r, n][1, k]$. When $\chi = 0$, however, all the terms of the decomposition have weight than $[1, l][r, n][1, k]$.

Corollary 2.6. A product of brackets $[a, l][r, n][b, k]$ with $\max\{a, b\} < r \leq k \leq l < n$ with $n - l \leq r - \max\{a, b\} + 1$ (illustrated in Figure 3) decomposes as:

$$[a, l][r, n][b, k] \sim \chi[a, l][k+1, n][b, k] + \sum_{i=1}^{\min\{n-l, k-r+1\}} [a, l][l+1, n][1, l+i][k+1, n][b, k]$$

where $\chi = 0$ if $n - l = r - \max\{a, b\} + 1$, and $\chi = 1$ if $n - l < r - \max\{a, b\} + 1$.

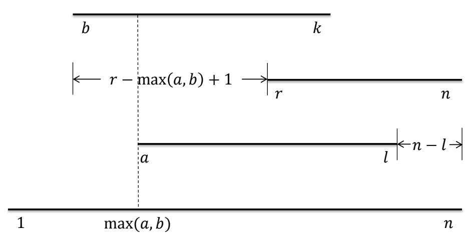

FIGURE 3. Diagram of Corollary 2.6

Proof. Let $s = \max\{a, b\}$ and unabsorb $[s, l]$ from $[a, l]$ and $[s, k]$ from $[b, k]$ to obtain $[a, l][r, n][b, k] \sim [a, l][s, l][r, n][s, k][b, k]$, and then use Proposition 2.5 on $[s, l][r, n][s, k]$, and the absorption $[a, l][1, l][k+1, n][b, k] \sim [a, l][k+1, n][b, k]$ to obtain the desired relation. □

Note that for $\chi = 1$ this general triple relation is also a "local" criterion since the weight of $[a, l][k+1, n][b, k]$ is certainly lower than the weight of $[a, l][r, n][b, k]$ as $r \leq k < k+1$.

2.3. **Computing the possible extreme rays for $\mathcal{C}(\Pi_n)$.** Combining Corollary 2.6 with the symmetries of the group algebra (which we have been calling reflection and reversal), we obtain four configurations of triples that we can decompose. With some cleverness, it is possible to use these triple relations together with absorption and unabsorpition to show by hand that for $n = 6$ any product of brackets consisting of more than a certain number of brackets is not a minimal weight expression for an extreme ray ([1]).

Since our interest, following Stembridge, lies in finding the actual extreme rays of the cone, we implemented an algorithm in C++ for testing if a product of triples

[1]the second author did in fact perform this check by hand

could be a minimal weight expression for an extreme ray according to the rules of absorption and triple reduction that we proved below.

Before we describe the algorithm, we should give illustrate a key procedure which we label "creative unabsorption". Consider the product [13][35][23][46][35][13]. It contains no actual triples that can be reduced. Yet we can unabsorb [12] from the bottom and from the top [13], thus getting that

$$[13][35][23][46][35][13] \sim [13][35]([12][23][12])[46][35][13].$$

In this way, we have created a triple that can be decomposed, namely [12][23][12], one of whose terms after decomposition is simply [12]. Thus, one of the terms of [13][35][23][46][35][13] after such a decomposition is [13][35][12][46][35][13] $\sim$ [13][35][46][35][13], which is of strictly smaller weight and thus we can conclude that [13][35][23][46][35][13] is not a minimal weight expression for some extreme ray.

The algorithm for reducing $\pi = [i_1, j_1] \dots [i_k, j_k]$ is then the following, which quits at any point where the weight of π drops below the weight we started with.

(1) Transform π by either absorbing every bracket below the s^{th} bracket $[i_s, j_s]$ of π into $[i_s, j_s]$, or by absorbing $[i_s, j_s]$ into some bracket below, if possible.
(2) For every bracket $[i_s, j_s]$ find the largest brackets $[i_{s'}, j_{s'}]$ and $[i_{s''}, j_{s''}]$ that can be unabsorbed from brackets below and above $[i_s, j_s]$ so that $[i_{s'}, j_{s'}][i_s, j_s][i_{s''}, j_{s''}]$ is a triple that can be decomposed first by Corollary 2.6, and then by each of the corollary's symmetric variants.
(3) Replace that triple with a term of the decomposition and go back to the first step, unless we are stuck in an infinite loop, in which case proceed with the next term or next triple.
(4) If we run out of triples, then the product of brackets could be a minimal weight expression for an extreme ray.

Evidently, the above algorithm will terminate on any product of brackets and return whether or not using the triple relation and unabsorption is sufficient to determine that the product of brackets is not a minimal weight expression for an extreme ray.

We thus use a depth first search of all possible products of brackets, noting that any branch of the search terminates when the algorithm decides that a node is reducible, since if π is reducible, then so is $\pi\pi'$ for any product π'.

Hence, if the triple relation and unabsorption is sufficient to show that the $\mathcal{C}(\Pi_n)$ is finitely generated, the procedure described above terminates, providing a list that surely containing a possible superset of the extreme rays. For $n = 6$ we obtain 750 distinct products of brackets that cannot be decomposed using the triple relation and unabsorption. Using the Parma Polyhedra Library [1], we check and find that all the returned rays save two are extreme. The two rays which were not extreme are notable since the top term in the Bruhat order of their values in $\mathbb{R}S_n$ is the permutation $123456 \mapsto 654321$.

3. Miscellaneous results

Proposition 3.1 (Reducibility of alternating towers). *Consider a non-trivial product of brackets $[i_1, j_1][i_2, j_2][i_3, j_3], \dots, [i_k, j_k]$ inside $\mathcal{C}(\Pi_n)$ which alternates in the sense that for every s the product $[i_{s-1}, j_{s-1}][i_s, j_s][i_{s+1}, j_{s+1}]$ we have either $i_{s-1}, i_{s+1} < i_s \leq j_{s-1}, j_{s+1} < j_s$ or $i_s < i_{s-1}, i_{s+1} \leq j_s < j_{s-1}, j_{s+1}$ (illustrated in Figure 4).*

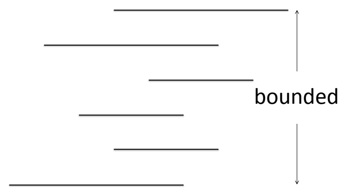

FIGURE 4. Diagram of Proposition 3.1

We claim that such products in $\mathcal{C}(\Pi_n)$, which we call alternating towers, are of bounded height.

Proof. Suppose that we have an irreducible alternating tower of brackets

$$[i_1, j_1][i_2, j_2][i_3, j_3], \dots, [i_k, j_k]$$

in $\mathcal{C}(\Pi_n)$. If there does not exist an s such that $i_s = 1$, the tower is equivalent to a tower inside $\mathcal{C}(\Pi_{n-1})$ which must already be of bounded height.

If there does exist an s such that $i_s = 1$, then it is surely enough to show that the irreducible subtower $[i_s, j_s][i_{s+1}, j_{s+1}], \dots, [i_k, j_k]$ is of bounded height. Thus, without loss of generality we may assume that $i_1 = 1$.

Define a triple $[i_{s-1}, j_{s-1}][i_s, j_s][i_{s+1}, j_{s+1}]$ to be *left-supported* if it satisfies $i_{s-1}, i_{s+1} < i_s \leq j_{s-1}, j_{s+1} < j_s$ and *right-supported* if it satisfies $i_s < i_{s-1}, i_{s+1} \leq j_s < j_{s-1}, j_{s+1}$. Since an alternating tower by definition is such that every triple is either left- or right-supported, we see that inside a tower left-supported triples are those for which $i_{s-1} < i_s$ (equivalently $i_{s+1} < i_s$), and right-supported triples are those for which $i_s < i_{s-1}$ (equivalently $i_s < i_{s+1}$). It follows that

$$[i_{s-1}, j_{s-1}][i_s, j_s][i_{s+1}, j_{s+1}]$$

is left-supported, i.e. $i_{s+1} < i_s$, if and only if the next triple

$$[i_s, j_s][i_{s+1}, j_{s+1}][i_{s+2}, j_{s+2}]$$

is right-supported.

Now, surely the triple $[i_1, j_1][i_2, j_2][i_3, j_3]$ inside the tower is such that $i_1 = 1 \leq i_2$. Hence triples of the form $[i_{2s-1}, j_{2s-1}][i_{2s}, j_{2s}][i_{2s+1}, j_{2s+1}]$ are left-supported, and triples of the form $[i_{2s}, j_{2s}][i_{2s+1}, j_{2s+1}][i_{2(s+1)}, j_{2(s+1)}]$ are right-supported. We will refer to left-supported triples as *even* and to right-supported triples as *odd* (based on the parity of the index of their middle term).

By Corollary 2.6 it follows that an even triple $[i_{2s-1}, j_{2s-1}][i_{2s}, j_{2s}][i_{2s+1}, j_{2s+1}]$ is irreducible only if $j_{2s} - \max\{j_{2s-1}, j_{2s+1}\} > i_{2s} - \max\{i_{2s-1}, i_{2s+1}\}$. Similarly, an odd triple $[i_{2s}, j_{2s}][i_{2s+1}, j_{2s+1}][i_{2(s+1)}, j_{2(s+1)}]$ is irreducible only if $\min\{i_{2s}, i_{2(s+1)}\} - i_{2s+1} > \min\{j_{2s}, j_{2(s+1)}\} - j_{2s+1}$.

Next, we show that for an irreducible quadruple

$$[i_{2s-1}, j_{2s-1}][i_{2s}, j_{2s}][i_{2s+1}, j_{2s+1}][i_{2(s+1)}, j_{2(s+1)}],$$

we have that $i_{2s-1} \leq i_{2s+1}$ implies $j_{2s} > j_{2(s+1)}$.

Plugging $i_{2s-1} \leq i_{2s+1}$ into the irreducibility inequality for the even triple, we obtain $j_{2s} - j_{2s+1} \geq j_{2s} - \max\{j_{2s-1}, j_{2s+1}\} > i_{2s} - \max\{i_{2s-1}, i_{2s+1}\} = i_{2s} - i_{2s+1}$.

Plugging in into the irreducibility inequality for the odd triple, we obtain $i_{2s} - i_{2s+1} \geq \min\{i_{2s}, i_{2(s+1)}\} - i_{2s+1} > \min\{j_{2s}, j_{2(s+1)}\} - j_{2s+1}$. Since $i_{2s} - i_{2s+1} < j_{2s} - j_{2s+1}$ it follows that $j_{2s} > j_{2(s+1)}$.

By reflection, we also have that $j_{2(s+1)} \leq j_{2s}$ implies $i_{2(s+1)-1} < i_{2(s+1)+1}$. Consequently, since our initial triple $[i_1, j_1][i_2, j_2][i_3, j_3]$ is such that $i_1 = 1 \leq i_3$, it follows that all even triples $[i_{2s-1}, i_{2s}, i_{2s+1}]$ for $s > 1$ satisfy $i_{2(s+1)-1} < i_{2(s+1)+1}$ and all odd triples $[i_{2s}, j_{2s}][i_{2s+1}, j_{2s+1}][i_{2(s+1)}, j_{2(s+1)}]$ satisfy $j_{2(s+1)} < j_{2s}$.

Now, if $i_s = 1$ only for $s = 1$, the remainder of our tower composed of triples with $s > 1$ fits inside $\mathcal{C}(\Pi_n)$ and is thus of bounded height, so we might as well assume that our initial triple in fact satisfies $i_1 < i_3$.

But now we have that our tower must satisfy $1 \leq i_1 < i_3 < \dots i_{2k+1} < \cdots \leq n$ and $n \geq j_2 > j_4 > \cdots > j_{2k} > \cdots \geq 1$, hence it must be bounded above by $2n$. □

References

1. R. Bagnara, P. M. Hill, and E. Zaffanella, *The Parma Polyhedra Library: Toward a complete set of numerical abstractions for the analysis and verification of hardware and software systems*, Science of Computer Programming **72** (2008), no. 1–2, 3–21.
2. J. Stembridge, *Some conjectures for immanants*, Can. J. Math **44** (1992), no. 5, 1079–1099.

Appendix A. Extreme rays of $\mathcal{C}(\Pi_6)$

```
1. 1
2. [16]
3. [15]
4. [15][26]
5. [15][36]
6. [15][46]
7. [15][56]
8. [26]
9. [26][15]
10. [26][14]
11. [26][13]
12. [26][12]
13. [14]
14. [14][26]
15. [14][26][12]
16. [14][25]
17. [14][25][36]
18. [14][25][46]
19. [14][25][56]
20. [14][36]
21. [14][36][23]
22. [14][35]
23. [14][35][46]
24. [14][35][56]
25. [14][46]
26. [14][46][34]
27. [14][45]
28. [14][45][56]
29. [14][56]
```

30. [14] [56] [25]
31. [14] [56] [25] [56]
32. [14] [56] [35]
33. [14] [56] [45]
34. [25]
35. [25] [14]
36. [25] [14] [56]
37. [25] [14] [56] [25]
38. [25] [14] [56] [25] [56]
39. [25] [14] [56] [35]
40. [25] [14] [56] [45]
41. [25] [36]
42. [25] [13]
43. [25] [13] [46]
44. [25] [13] [46] [25]
45. [25] [13] [46] [24]
46. [25] [13] [46] [35]
47. [25] [13] [46] [35] [23]
48. [25] [13] [46] [34]
49. [25] [13] [56]
50. [25] [46]
51. [25] [12]
52. [25] [12] [36]
53. [25] [12] [36] [25]
54. [25] [12] [36] [25] [12]
55. [25] [12] [36] [24]
56. [25] [12] [36] [23]
57. [25] [12] [46]
58. [25] [12] [56]
59. [25] [56]
60. [36]
61. [36] [15]
62. [36] [14]
63. [36] [14] [45]
64. [36] [25]
65. [36] [25] [14]
66. [36] [25] [13]
67. [36] [25] [12]
68. [36] [13]
69. [36] [13] [34]
70. [36] [24]
71. [36] [24] [13]
72. [36] [24] [12]
73. [36] [23]
74. [36] [23] [12]
75. [13]
76. [13] [26]
77. [13] [26] [13]

78. [13][26][12]
79. [13][25]
80. [13][25][36]
81. [13][25][46]
82. [13][25][12]
83. [13][25][12][46]
84. [13][25][12][56]
85. [13][25][56]
86. [13][36]
87. [13][36][13]
88. [13][36][24]
89. [13][36][24][12]
90. [13][36][23]
91. [13][24]
92. [13][24][36]
93. [13][24][36][23]
94. [13][24][36][23][12]
95. [13][24][35]
96. [13][24][35][46]
97. [13][24][35][56]
98. [13][24][46]
99. [13][24][46][34]
100. [13][24][45]
101. [13][24][45][56]
102. [13][35]
103. [13][35][46]
104. [13][35][23]
105. [13][35][23][46]
106. [13][35][23][46][34]
107. [13][35][23][56]
108. [13][35][56]
109. [13][46]
110. [13][46][25]
111. [13][46][24]
112. [13][46][35]
113. [13][46][35][23]
114. [13][46][34]
115. [13][34]
116. [13][34][46]
117. [13][34][46][34]
118. [13][34][46][34][23]
119. [13][34][45]
120. [13][34][45][56]
121. [13][45]
122. [13][45][24]
123. [13][45][24][56]
124. [13][45][24][56][35]
125. [13][45][24][56][45]

126. [13] [45] [34]
127. [13] [45] [34] [56]
128. [13] [45] [34] [56] [45]
129. [13] [45] [56]
130. [13] [56]
131. [13] [56] [25]
132. [13] [56] [25] [12]
133. [13] [56] [25] [12] [56]
134. [13] [56] [25] [56]
135. [13] [56] [24]
136. [13] [56] [24] [35]
137. [13] [56] [24] [45]
138. [13] [56] [35]
139. [13] [56] [35] [23]
140. [13] [56] [34]
141. [13] [56] [34] [45]
142. [13] [56] [45]
143. [13] [56] [45] [24]
144. [13] [56] [45] [34]
145. [24]
146. [24] [36]
147. [24] [36] [13]
148. [24] [36] [13] [34]
149. [24] [36] [23]
150. [24] [36] [23] [12]
151. [24] [13]
152. [24] [13] [46]
153. [24] [13] [46] [25]
154. [24] [13] [46] [24]
155. [24] [13] [46] [35]
156. [24] [13] [46] [35] [23]
157. [24] [13] [46] [34]
158. [24] [13] [45]
159. [24] [13] [45] [24]
160. [24] [13] [45] [24] [56]
161. [24] [13] [45] [24] [56] [35]
162. [24] [13] [45] [24] [56] [45]
163. [24] [13] [45] [34]
164. [24] [13] [45] [34] [56]
165. [24] [13] [45] [34] [56] [45]
166. [24] [13] [45] [56]
167. [24] [35]
168. [24] [35] [46]
169. [24] [35] [56]
170. [24] [46]
171. [24] [46] [34]
172. [24] [12]
173. [24] [12] [36]

174. [24] [12] [36] [25]
175. [24] [12] [36] [25] [12]
176. [24] [12] [36] [24]
177. [24] [12] [36] [23]
178. [24] [12] [35]
179. [24] [12] [35] [24]
180. [24] [12] [35] [24] [56]
181. [24] [12] [35] [24] [56] [35]
182. [24] [12] [35] [24] [56] [45]
183. [24] [12] [35] [46]
184. [24] [12] [35] [23]
185. [24] [12] [35] [23] [46]
186. [24] [12] [35] [23] [46] [35]
187. [24] [12] [35] [23] [46] [34]
188. [24] [12] [35] [23] [56]
189. [24] [12] [35] [56]
190. [24] [12] [46]
191. [24] [12] [46] [34]
192. [24] [12] [46] [34] [23]
193. [24] [12] [45]
194. [24] [12] [45] [56]
195. [24] [45]
196. [24] [45] [56]
197. [24] [56]
198. [24] [56] [13]
199. [24] [56] [13] [45]
200. [24] [56] [13] [45] [24]
201. [24] [56] [13] [45] [34]
202. [24] [56] [35]
203. [24] [56] [12]
204. [24] [56] [12] [35]
205. [24] [56] [12] [35] [24]
206. [24] [56] [12] [35] [23]
207. [24] [56] [12] [45]
208. [24] [56] [45]
209. [35]
210. [35] [14]
211. [35] [14] [46]
212. [35] [14] [46] [34]
213. [35] [14] [56]
214. [35] [14] [56] [25]
215. [35] [14] [56] [25] [56]
216. [35] [14] [56] [35]
217. [35] [14] [56] [45]
218. [35] [13]
219. [35] [13] [46]
220. [35] [13] [46] [25]
221. [35] [13] [46] [24]

222. [35] [13] [46] [35]
223. [35] [13] [46] [35] [23]
224. [35] [13] [46] [34]
225. [35] [13] [34]
226. [35] [13] [56]
227. [35] [13] [56] [34]
228. [35] [13] [56] [34] [45]
229. [35] [24]
230. [35] [24] [13]
231. [35] [24] [12]
232. [35] [24] [56]
233. [35] [24] [56] [13]
234. [35] [24] [56] [13] [45]
235. [35] [24] [56] [13] [45] [24]
236. [35] [24] [56] [13] [45] [34]
237. [35] [24] [56] [35]
238. [35] [24] [56] [12]
239. [35] [24] [56] [12] [35]
240. [35] [24] [56] [12] [35] [24]
241. [35] [24] [56] [12] [35] [23]
242. [35] [24] [56] [12] [45]
243. [35] [24] [56] [45]
244. [35] [46]
245. [35] [23]
246. [35] [23] [46]
247. [35] [23] [46] [35]
248. [35] [23] [46] [12]
249. [35] [23] [46] [12] [35]
250. [35] [23] [46] [12] [35] [24]
251. [35] [23] [46] [12] [35] [23]
252. [35] [23] [46] [12] [34]
253. [35] [23] [46] [12] [34] [23]
254. [35] [23] [46] [34]
255. [35] [23] [12]
256. [35] [23] [56]
257. [35] [23] [56] [12]
258. [35] [56]
259. [46]
260. [46] [15]
261. [46] [15] [56]
262. [46] [14]
263. [46] [14] [35]
264. [46] [14] [45]
265. [46] [25]
266. [46] [25] [14]
267. [46] [25] [13]
268. [46] [25] [12]
269. [46] [24]

270. [46][24][13]
271. [46][24][12]
272. [46][24][12][45]
273. [46][24][45]
274. [46][35]
275. [46][35][14]
276. [46][35][13]
277. [46][35][13][34]
278. [46][35][24]
279. [46][35][24][13]
280. [46][35][24][12]
281. [46][35][23]
282. [46][35][23][12]
283. [46][34]
284. [46][34][13]
285. [46][34][23]
286. [46][34][23][12]
287. [12]
288. [12][26]
289. [12][26][14]
290. [12][26][13]
291. [12][26][12]
292. [12][25]
293. [12][25][36]
294. [12][25][13]
295. [12][25][13][56]
296. [12][25][46]
297. [12][25][12]
298. [12][25][12][36]
299. [12][25][12][36][25]
300. [12][25][12][36][24]
301. [12][25][12][36][23]
302. [12][25][12][46]
303. [12][25][12][56]
304. [12][25][56]
305. [12][36]
306. [12][36][25]
307. [12][36][25][12]
308. [12][36][24]
309. [12][36][23]
310. [12][24]
311. [12][24][36]
312. [12][24][36][13]
313. [12][24][36][13][34]
314. [12][24][36][23]
315. [12][24][36][23][12]
316. [12][24][35]
317. [12][24][35][46]

318. [12][24][35][56]
319. [12][24][46]
320. [12][24][46][34]
321. [12][24][12]
322. [12][24][12][46]
323. [12][24][12][46][34]
324. [12][24][12][46][34][23]
325. [12][24][12][45]
326. [12][24][12][45][56]
327. [12][24][45]
328. [12][24][45][56]
329. [12][35]
330. [12][35][24]
331. [12][35][24][56]
332. [12][35][24][56][35]
333. [12][35][24][56][45]
334. [12][35][46]
335. [12][35][23]
336. [12][35][23][46]
337. [12][35][23][46][35]
338. [12][35][23][46][34]
339. [12][35][23][56]
340. [12][35][56]
341. [12][46]
342. [12][46][25]
343. [12][46][25][13]
344. [12][46][25][12]
345. [12][46][24]
346. [12][46][24][12]
347. [12][46][24][12][45]
348. [12][46][24][45]
349. [12][46][35]
350. [12][46][35][24]
351. [12][46][35][23]
352. [12][46][23]
353. [12][46][23][35]
354. [12][46][23][34]
355. [12][46][34]
356. [12][46][34][23]
357. [12][23]
358. [12][23][36]
359. [12][23][36][24]
360. [12][23][36][24][13]
361. [12][23][36][24][12]
362. [12][23][36][23]
363. [12][23][36][23][12]
364. [12][23][35]
365. [12][23][35][46]

366. [12] [23] [35] [23]
367. [12] [23] [35] [23] [12]
368. [12] [23] [35] [23] [56]
369. [12] [23] [35] [23] [56] [12]
370. [12] [23] [35] [56]
371. [12] [23] [34]
372. [12] [23] [34] [46]
373. [12] [23] [34] [46] [34]
374. [12] [23] [34] [46] [34] [23]
375. [12] [23] [34] [46] [34] [23] [12]
376. [12] [23] [34] [45]
377. [12] [23] [34] [45] [56]
378. [12] [34]
379. [12] [34] [46]
380. [12] [34] [46] [24]
381. [12] [34] [46] [24] [12]
382. [12] [34] [46] [24] [12] [45]
383. [12] [34] [46] [24] [45]
384. [12] [34] [46] [34]
385. [12] [34] [46] [34] [23]
386. [12] [34] [23]
387. [12] [34] [23] [46]
388. [12] [34] [23] [46] [35]
389. [12] [34] [23] [46] [34]
390. [12] [34] [23] [45]
391. [12] [34] [23] [45] [34]
392. [12] [34] [23] [45] [34] [56]
393. [12] [34] [23] [45] [34] [56] [45]
394. [12] [34] [23] [45] [56]
395. [12] [34] [45]
396. [12] [34] [45] [56]
397. [12] [34] [56]
398. [12] [34] [56] [23]
399. [12] [34] [56] [23] [45]
400. [12] [34] [56] [23] [45] [34]
401. [12] [34] [56] [45]
402. [12] [45]
403. [12] [45] [24]
404. [12] [45] [24] [46]
405. [12] [45] [24] [46] [34]
406. [12] [45] [24] [12]
407. [12] [45] [24] [12] [46]
408. [12] [45] [24] [12] [46] [34]
409. [12] [45] [24] [12] [46] [34] [23]
410. [12] [45] [24] [56]
411. [12] [45] [24] [56] [35]
412. [12] [45] [24] [56] [12]
413. [12] [45] [24] [56] [12] [45]

414. [12][45][24][56][45]
415. [12][45][23]
416. [12][45][23][34]
417. [12][45][23][56]
418. [12][45][23][56][34]
419. [12][45][23][56][34][45]
420. [12][45][34]
421. [12][45][34][23]
422. [12][45][34][56]
423. [12][45][34][56][23]
424. [12][45][34][56][23][45]
425. [12][45][34][56][23][45][34]
426. [12][45][34][56][45]
427. [12][45][56]
428. [12][56]
429. [12][56][25]
430. [12][56][25][13]
431. [12][56][25][13][56]
432. [12][56][25][12]
433. [12][56][25][12][56]
434. [12][56][25][56]
435. [12][56][24]
436. [12][56][24][35]
437. [12][56][24][12]
438. [12][56][24][12][45]
439. [12][56][24][45]
440. [12][56][35]
441. [12][56][35][24]
442. [12][56][35][23]
443. [12][56][23]
444. [12][56][23][35]
445. [12][56][23][35][23]
446. [12][56][23][35][23][12]
447. [12][56][23][34]
448. [12][56][23][34][45]
449. [12][56][23][45]
450. [12][56][23][45][34]
451. [12][56][45]
452. [12][56][45][24]
453. [12][56][45][24][12]
454. [12][56][45][34]
455. [12][56][45][34][23]
456. [23]
457. [23][36]
458. [23][36][14]
459. [23][36][14][45]
460. [23][36][13]
461. [23][36][13][34]

462. [23][36][24]
463. [23][36][24][13]
464. [23][36][24][12]
465. [23][36][23]
466. [23][36][23][12]
467. [23][35]
468. [23][35][13]
469. [23][35][13][46]
470. [23][35][13][46][25]
471. [23][35][13][46][24]
472. [23][35][13][46][35]
473. [23][35][13][46][35][23]
474. [23][35][13][46][34]
475. [23][35][13][34]
476. [23][35][13][56]
477. [23][35][13][56][34]
478. [23][35][13][56][34][45]
479. [23][35][46]
480. [23][35][23]
481. [23][35][23][12]
482. [23][35][23][56]
483. [23][35][23][56][12]
484. [23][35][56]
485. [23][46]
486. [23][46][35]
487. [23][46][35][13]
488. [23][46][35][13][34]
489. [23][46][12]
490. [23][46][12][35]
491. [23][46][12][35][24]
492. [23][46][12][35][23]
493. [23][46][12][34]
494. [23][46][12][34][23]
495. [23][46][34]
496. [23][12]
497. [23][12][36]
498. [23][12][36][25]
499. [23][12][36][25][12]
500. [23][12][36][24]
501. [23][12][36][23]
502. [23][12][35]
503. [23][12][35][24]
504. [23][12][35][24][56]
505. [23][12][35][24][56][35]
506. [23][12][35][24][56][45]
507. [23][12][35][46]
508. [23][12][35][23]
509. [23][12][35][23][46]

510. [23] [12] [35] [23] [46] [35]
511. [23] [12] [35] [23] [46] [34]
512. [23] [12] [35] [23] [56]
513. [23] [12] [35] [56]
514. [23] [12] [34]
515. [23] [12] [34] [46]
516. [23] [12] [34] [46] [24]
517. [23] [12] [34] [46] [24] [12]
518. [23] [12] [34] [46] [24] [12] [45]
519. [23] [12] [34] [46] [24] [45]
520. [23] [12] [34] [46] [34]
521. [23] [12] [34] [46] [34] [23]
522. [23] [12] [34] [23]
523. [23] [12] [34] [23] [46]
524. [23] [12] [34] [23] [46] [35]
525. [23] [12] [34] [23] [46] [34]
526. [23] [12] [34] [23] [45]
527. [23] [12] [34] [23] [45] [34]
528. [23] [12] [34] [23] [45] [34] [56]
529. [23] [12] [34] [23] [45] [34] [56] [45]
530. [23] [12] [34] [23] [45] [56]
531. [23] [12] [34] [45]
532. [23] [12] [34] [45] [56]
533. [23] [34]
534. [23] [34] [46]
535. [23] [34] [46] [34]
536. [23] [34] [46] [34] [13]
537. [23] [34] [46] [34] [23]
538. [23] [34] [46] [34] [23] [12]
539. [23] [34] [45]
540. [23] [34] [45] [56]
541. [23] [45]
542. [23] [45] [12]
543. [23] [45] [12] [34]
544. [23] [45] [12] [34] [23]
545. [23] [45] [12] [34] [56]
546. [23] [45] [12] [34] [56] [23]
547. [23] [45] [12] [34] [56] [23] [45]
548. [23] [45] [12] [34] [56] [23] [45] [34]
549. [23] [45] [12] [34] [56] [45]
550. [23] [45] [12] [56]
551. [23] [45] [34]
552. [23] [45] [34] [56]
553. [23] [45] [34] [56] [45]
554. [23] [45] [56]
555. [23] [56]
556. [23] [56] [35]
557. [23] [56] [35] [13]

558. [23][56][35][13][34]
559. [23][56][35][23]
560. [23][56][35][23][12]
561. [23][56][12]
562. [23][56][12][35]
563. [23][56][12][35][24]
564. [23][56][12][35][23]
565. [23][56][12][34]
566. [23][56][12][34][23]
567. [23][56][12][34][23][45]
568. [23][56][12][34][23][45][34]
569. [23][56][12][34][45]
570. [23][56][12][45]
571. [23][56][12][45][34]
572. [23][56][12][45][34][23]
573. [23][56][34]
574. [23][56][34][45]
575. [23][56][45]
576. [23][56][45][34]
577. [34]
578. [34][13]
579. [34][13][36]
580. [34][13][36][24]
581. [34][13][36][23]
582. [34][13][35]
583. [34][13][35][46]
584. [34][13][35][23]
585. [34][13][35][23][46]
586. [34][13][35][23][46][34]
587. [34][13][35][23][56]
588. [34][13][35][56]
589. [34][13][46]
590. [34][13][46][25]
591. [34][13][46][24]
592. [34][13][46][35]
593. [34][13][46][35][23]
594. [34][13][46][34]
595. [34][13][45]
596. [34][13][45][24]
597. [34][13][45][24][56]
598. [34][13][45][24][56][35]
599. [34][13][45][24][56][45]
600. [34][13][45][34]
601. [34][13][45][34][56]
602. [34][13][45][34][56][45]
603. [34][13][45][56]
604. [34][46]
605. [34][46][14]

606. [34] [46] [14] [35]
607. [34] [46] [14] [45]
608. [34] [46] [24]
609. [34] [46] [24] [13]
610. [34] [46] [24] [12]
611. [34] [46] [24] [12] [45]
612. [34] [46] [24] [45]
613. [34] [46] [34]
614. [34] [46] [34] [13]
615. [34] [46] [34] [23]
616. [34] [46] [34] [23] [12]
617. [34] [23]
618. [34] [23] [46]
619. [34] [23] [46] [35]
620. [34] [23] [46] [35] [13]
621. [34] [23] [46] [35] [13] [34]
622. [34] [23] [46] [12]
623. [34] [23] [46] [12] [35]
624. [34] [23] [46] [12] [35] [24]
625. [34] [23] [46] [12] [35] [23]
626. [34] [23] [46] [12] [34]
627. [34] [23] [46] [12] [34] [23]
628. [34] [23] [46] [34]
629. [34] [23] [12]
630. [34] [23] [45]
631. [34] [23] [45] [12]
632. [34] [23] [45] [12] [34]
633. [34] [23] [45] [12] [34] [23]
634. [34] [23] [45] [12] [34] [56]
635. [34] [23] [45] [12] [34] [56] [23]
636. [34] [23] [45] [12] [34] [56] [23] [45]
637. [34] [23] [45] [12] [34] [56] [23] [45] [34]
638. [34] [23] [45] [12] [34] [56] [45]
639. [34] [23] [45] [12] [56]
640. [34] [23] [45] [34]
641. [34] [23] [45] [34] [56]
642. [34] [23] [45] [34] [56] [45]
643. [34] [23] [45] [56]
644. [34] [45]
645. [34] [45] [56]
646. [34] [56]
647. [34] [56] [13]
648. [34] [56] [13] [35]
649. [34] [56] [13] [35] [23]
650. [34] [56] [13] [45]
651. [34] [56] [13] [45] [24]
652. [34] [56] [13] [45] [34]
653. [34] [56] [23]

654. [34] [56] [23] [12]
655. [34] [56] [23] [45]
656. [34] [56] [23] [45] [12]
657. [34] [56] [23] [45] [12] [34]
658. [34] [56] [23] [45] [12] [34] [23]
659. [34] [56] [23] [45] [34]
660. [34] [56] [45]
661. [45]
662. [45] [14]
663. [45] [14] [36]
664. [45] [14] [36] [23]
665. [45] [14] [46]
666. [45] [14] [46] [34]
667. [45] [14] [45]
668. [45] [14] [45] [56]
669. [45] [14] [56]
670. [45] [14] [56] [25]
671. [45] [14] [56] [25] [56]
672. [45] [14] [56] [35]
673. [45] [14] [56] [45]
674. [45] [24]
675. [45] [24] [13]
676. [45] [24] [46]
677. [45] [24] [46] [34]
678. [45] [24] [12]
679. [45] [24] [12] [46]
680. [45] [24] [12] [46] [34]
681. [45] [24] [12] [46] [34] [23]
682. [45] [24] [56]
683. [45] [24] [56] [13]
684. [45] [24] [56] [13] [45]
685. [45] [24] [56] [13] [45] [24]
686. [45] [24] [56] [13] [45] [34]
687. [45] [24] [56] [35]
688. [45] [24] [56] [12]
689. [45] [24] [56] [12] [35]
690. [45] [24] [56] [12] [35] [24]
691. [45] [24] [56] [12] [35] [23]
692. [45] [24] [56] [12] [45]
693. [45] [24] [56] [45]
694. [45] [34]
695. [45] [34] [13]
696. [45] [34] [23]
697. [45] [34] [23] [12]
698. [45] [34] [56]
699. [45] [34] [56] [13]
700. [45] [34] [56] [13] [35]
701. [45] [34] [56] [13] [35] [23]

702. [45][34][56][13][45]
703. [45][34][56][13][45][24]
704. [45][34][56][13][45][34]
705. [45][34][56][23]
706. [45][34][56][23][12]
707. [45][34][56][23][45]
708. [45][34][56][23][45][12]
709. [45][34][56][23][45][12][34]
710. [45][34][56][23][45][12][34][23]
711. [45][34][56][23][45][34]
712. [45][34][56][45]
713. [45][56]
714. [56]
715. [56][15]
716. [56][15][46]
717. [56][15][56]
718. [56][25]
719. [56][25][14]
720. [56][25][14][56]
721. [56][25][14][56][25]
722. [56][25][14][56][35]
723. [56][25][14][56][45]
724. [56][25][13]
725. [56][25][13][56]
726. [56][25][12]
727. [56][25][12][56]
728. [56][25][56]
729. [56][35]
730. [56][35][14]
731. [56][35][13]
732. [56][35][13][34]
733. [56][35][24]
734. [56][35][24][13]
735. [56][35][24][12]
736. [56][35][23]
737. [56][35][23][12]
738. [56][45]
739. [56][45][14]
740. [56][45][14][45]
741. [56][45][14][45][56]
742. [56][45][24]
743. [56][45][24][13]
744. [56][45][24][12]
745. [56][45][34]
746. [56][45][34][13]
747. [56][45][34][23]
748. [56][45][34][23][12]