1 On the Jacobi-Trudi formula for dual stable Grothendieck polynomials

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1.1 Review

We will first begin with a review of the facts that we already now about this problem.

Firstly, a semistandard Young tableau T is a Young tableau $\lambda = (\lambda_1, ..., \lambda_m)$ with positive integer entries which strictly increase in columns and weakly increase in rows.

Secondly, we will define a Schur function : a Schur function is a polynomial s_λ is defined as

$$s_\lambda = \sum_T x^T = \sum_T x_1^{t_1} x_2^{t_2} \cdots x_n^{t_n}, \tag{1}$$

where the summation is over all semistandard Young tableau T of shape λ; the exponents $t_1, \ldots, t_n$ represent the weight of the tableau, in other words the t_i counts the number of occurences of i in T.

Thirdly, a reverse plane partition is Young tableau with positive integer entries which increase weakly both in rows and columns.

Forthly,we introduce the dual-stable Grothendieck polynomials, defined as

$$g_\lambda = \sum_T x_T = \sum_T x_1^{t_1} \cdots x_n^{t_n}, \tag{2}$$

where the summation is over all reverse plane partitions T of shape λ; the exponents $t_1, \ldots, t_n$ represent the weight of the reverse plane partition, in other words the t_i counts the number of columns containing i in T.

We will note henceforth $\overline{\lambda} = (\overline{\lambda_1}, \ldots, \overline{\lambda_n})$ the conjugate of λ, a tableau with λ_i boxes on the $i-th$ column for all i.

We now introduce the **Jacobi-Trudi formulas**, or also known as the **Giambelli formulas**, expressing the schur functions in terms of elementary symmetric polynomials,

e_i, by having the formula

$$s_\lambda = det_{1\le i,j\le n}(e_{\overline{\lambda_i}-i+j}) \tag{3}$$

or

$$s_\lambda = \begin{vmatrix} e_{\overline{\lambda_1}} & e_{\overline{\lambda_1}+1} & \cdot & e_{\overline{\lambda_1}+n-1} \\ \cdot & e_{\overline{\lambda_2}} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ e_{\overline{\lambda_n}-n+1} & \cdot & \cdot & e_{\overline{\lambda_n}} \end{vmatrix} \tag{4}$$

Definition. An *elegant filling* (EF) of the skew shape λ/μ is a filling of λ/μ with the following conditions:

(1) the numbers weakly increase in rows and strictly increase in columns; and

(2) the numbers in the row i are in $[1, i-1]$. The number of EFs of λ/μ is denoted by f^μ_λ. In the case where μ is not included in λ we set $f^\mu_\lambda = 0$.

Theorem 1 (Lahm, Pylyavskyy [1]). Let λ be a partition. Then

$$g_\lambda = \sum_{\mu\subseteq\lambda} f^\mu_\lambda s_\mu. \tag{5}$$

1.2 Equivalent Relations

Now, we will prove some equivalences using the Jacobi-Trudi formulas.

Note, $\overline{\lambda} = (m_1, \ldots, m_r)$ with $m_1 \ge \ldots \ge m_r$ and also the symmetric polynomial $w_\lambda = w_{(m_1,\ldots,m_r)^T}$ defined by

$$w_\lambda = \begin{vmatrix} \binom{m_1-1}{m1-1}e_{m_1} + \ldots + \binom{m_1-1}{0}e_1 & \cdot & \cdot & \binom{m_1-1}{m_1-1}e_{m_1+r-1} + \ldots + \binom{m_1-1}{0}e_r \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \binom{m_r-1}{m_r-1}e_{m_r-r+1} + \ldots + \binom{m_r-1}{0}e_{2-r} & \cdot & \cdot & \binom{m_r-1}{m_r-1}e_{m_r} + \ldots + \binom{m_r-1}{0}e_1 \end{vmatrix} \tag{6}$$

or by writing the abbreviated formula, we have

$$w_\lambda = det\Bigg(\Big(\tbinom{m_i-1}{m_i-1}e_{m_i-i+j} + \ldots + \tbinom{m_i-1}{0}e_{j-i+1}\Big)_{1\le i,j\le r}\Bigg). \tag{7}$$

Note the linear vector $V_x = (e_x\ e_{x+1} \ldots\ e_{x+r-1})$ where $e_x = 0$ if $x \le 0$ and $e_0 = 1$.

Therefore equation (6) is becoming

$$w_\lambda = \begin{vmatrix} \binom{m_1-1}{m1-1}V_{m_1} + \ldots + \binom{m_1-1}{0}V_1 \\ \cdot \\ \cdot \\ \binom{m_r-1}{m_r-1}V_{m_r-r+1} + \ldots + \binom{m_r-1}{0}V_{2-r} \end{vmatrix}. \tag{8}$$

We can split the determinant by using the $n-linearity$ of the determinant like

$$\begin{vmatrix} R_1 + R_1' \\ R_2 \\ \cdot \\ \cdot \\ R_n \end{vmatrix} = \begin{vmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ R_n \end{vmatrix} + \begin{vmatrix} R_1' \\ R_2 \\ \cdot \\ \cdot \\ R_n \end{vmatrix} \tag{9}$$

therefore we get

$$g_\lambda = \sum_{i=1}^{r} \Big(\sum_{2-i \le k_i \le m_i - i + 1} \begin{vmatrix} \binom{m_i-1}{k_1+1-2}V_{k_1} \\ \cdot \\ \cdot \\ \cdot \\ \binom{m_r-1}{k_r+r-2}V_{k_r} \end{vmatrix} \Big) = \sum_{i=1}^{r} \Big(\sum_{2-i \le k_i \le m_i - i + 1} \prod_{i=1}^{r} \binom{m_i-1}{k_i+i-2} \begin{vmatrix} V_{k_1} \\ \cdot \\ \cdot \\ V_{k_r} \end{vmatrix} \Big). \tag{10}$$

We would like to compute now the coefficient of $\begin{vmatrix} V_{\alpha_1} \\ \cdot \\ \cdot \\ V_{\alpha_r} \end{vmatrix}$ where $\alpha_1 \ge \ldots \ge \alpha_r$ and also $2 - i \le \alpha_i \le m_i - i + 1$, $\forall 1 \le i \le r$.

Actually we can suppose that $\alpha_1 > \ldots > \alpha_r$, because if there are i, j such that $\alpha_i = \alpha_j$ then $\begin{vmatrix} \cdot \\ V_{\alpha_i} \\ \cdot \\ V_{\alpha_j} \\ \cdot \end{vmatrix} = 0$, therefore we can do the previous supposition.

Therefore the coefficient of $\begin{vmatrix} V_{\alpha_1} \\ \cdot \\ \cdot \\ V_{\alpha_r} \end{vmatrix}$ will be

$$\sum_{\sigma \in S_r} \prod_{i=1}^{r} \binom{m_i-1}{\alpha_{\sigma(i)+i-2}} \underbrace{\begin{vmatrix} V_{\alpha_{\sigma(1)}} \\ \cdot \\ \cdot \\ V_{\alpha_{\sigma(r)}} \end{vmatrix}}_{(-1)^{\epsilon(\sigma)} \begin{vmatrix} V_{\alpha_1} \\ \cdot \\ \cdot \\ V_{\alpha_r} \end{vmatrix}} = \sum_{\sigma \in S_r} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{r} \binom{m_i}{\alpha_{\sigma(i)+i-2}} \begin{vmatrix} V_{\alpha_1} \\ \cdot \\ \cdot \\ V_{\alpha_r} \end{vmatrix} \tag{11}$$

and by noting $\binom{m_i-1}{\alpha_{\sigma(i)}+i-2} = a_{i\sigma(i)}$, it results that $\sum_{\sigma\in S_r}(-1)^{\epsilon(\sigma)}\prod_{i=1}^{r}\binom{m_i}{\alpha_{\sigma(i)}+i-2} = \sum_{\sigma\in S_r}(-1)^{\epsilon(\sigma)}\prod_{i=1}^{r} a_{i\sigma(i)} = det\Big((a_{ij})_{1\leq i,j\leq r}\Big) = det\Big(\binom{m_i-1}{\alpha_j+i-2}_{1\leq i,j\leq r}\Big)$

which yields that

$$(11) = det\Big(\binom{m_i-1}{\alpha_j+i-2}_{1\leq i,j\leq r}\Big)\begin{vmatrix} V_{\alpha_1} \\ \cdot \\ \cdot \\ V_{\alpha_r}\end{vmatrix} \tag{12}$$

and we will also note $\alpha_\lambda^\mu = det\Big(\binom{m_i-1}{\alpha_j+i-2}_{1\leq i,j\leq r}\Big)$ where $\mu = (\alpha_1,\ldots,\alpha_r + r-1)^T$.

Also, by using Theorem 1, we get that $\begin{vmatrix} V_{\alpha_1} \\ \cdot \\ \cdot \\ V_{\alpha_r}\end{vmatrix}$ can be written as the Schur polynomial $s_{(\alpha_1,\ldots,\alpha_r+r-1)^T} = s_\mu$. Therefore the coefficient of s_μ is in fact exactly α_λ^μ, hence $w_\lambda = \sum_{\mu\in B}\alpha_\lambda^\mu s_\mu$ for some set B of plane partitions.

We will now prove that $\mu \in B$ if and only if $\mu \subseteq \overline{\lambda}$ or the equivalent $\overline{\mu} \subseteq \lambda$.

Proof.

" $\implies$ " If $\mu = (\alpha_1,\ldots,\alpha_r + r-1)^T$, with $\alpha_i + i \geq \alpha_{i+1} + i + 1$, thus $\alpha_i \geq \alpha_{i+1} + 1$ and $2 - i \leq \alpha_i \leq m_i - i + 1$, therefore $1 \leq \underbrace{\alpha_i + i - 1}_{\overline{\mu}_i} \leq m_i = \lambda_i$, for all $1 \leq i \leq r$, hence $\mu \subseteq \overline{\lambda}$.

" $\impliedby$ " If $\overline{\mu} \subseteq \lambda$ then take $\alpha_i = \overline{\mu}_i - i + 1, \forall i$, and so $2-i \leq \alpha_i \leq m_i - i + 1$ and $\alpha_i = \overline{\mu}_i - i + 1 \geq \overline{\mu}_{i-1} - i + 1 = \alpha_{i-1} + 1$, so $\alpha_i > \alpha_{i-1}$, therefore $\mu \in B$.

This proves that

$$w_\lambda = \sum_{\mu\subseteq\lambda}\alpha_\lambda^\mu s_\mu. \tag{13}$$

Lemma:

The two following statements are equivalent:

a).For any plane partition λ, we have $w_\lambda = g_\lambda$;

b).For any $\mu \subseteq \lambda$, we have $f_\lambda^\mu = det\Big(\binom{\overline{\lambda}_i-1}{\overline{\mu}_j-j+i-1}_{1\leq i,j\leq r}\Big)$, where $\overline{\lambda} = (\overline{\lambda_1},\ldots,\overline{\lambda_r})$ and $\overline{\mu} = (\overline{\mu}_1,\ldots,\overline{\mu}_r)$ (we need that $\overline{\mu}$ has r columns, if not $f_\lambda^\mu = 0$).

Proof: " $\implies$ " If $g_\lambda = \sum_{\mu\subseteq\lambda} f_\lambda^\mu = \sum_{\mu\subseteq\lambda}\alpha_\lambda^\mu = w_\lambda$, and also knowing that the Schur functions form a basis in the space of symmetric polynomials, therefore it results that $f_\lambda^\mu = \alpha_\lambda^\mu = det\Big(\binom{\overline{\lambda}_i-1}{\overline{\mu}_j-j+i-1}_{1\leq i,j\leq r}\Big)$.

" $\Longleftarrow$ " It is clear from (13) and Theorem 1.

We also get a consequence from the lemma: $det\left(\binom{a_i}{b_j-j+i-1}_{1\le i,j\le r}\right) \ge 0$, where $a_1,\ldots,a_r,b_1,\ldots b_r$ are integers and $a_1 \ge \ldots \ge a_r \ge 0$, $b_1 \ge \ldots \ge b_r \ge 0$.

Now we will state the main conjecture of my REU project:

Conjecture. For any plane partition λ we have $w_\lambda = g_\lambda$.

We will prove this conjecture for some special cases.

1.3 Proof of the conjecture in some special cases

Note (i) = column with i boxes and (i,j) = two column plane partition with the first column having i boxes and the second one having j columns.

1.3.1 One column case

Case I : one column case, $\lambda = (r)$. From Theorem 1, $g_{(r)} = \sum_{i=1}^{r} f_{(r)}^{(i)} s_{(i)}$. We will prove that $f_{(r)}^{(i)} = \binom{r-1}{i-1}$, and by the lemma we get our result.

Let $a_{i+1},\ a_{i+2},\ldots,\ a_r$ the numbers filled in the elegant filling of the skew-shape $(r)/(i)$, the j-th box containing a_{j+i}. By the definition of the elegant filling we have that $1 \le a_{i+1}\ <\ a_{i+2}\ < \cdots < a_r$ (condition 1) and all $a_j \in [1,\ j-1]$ for all $j = \overline{i+1,\ r}$ (condition 2). But actually this is equivalent to pick any $r-i$ distinct numbers in the interval $[1, r-1]$, as by simply doing that both conditions will be satisfied. The number of ways of picking $r-i$ numbers from 1 to r is obviously $\binom{r-1}{r-i} = \binom{r-1}{i-1}$, therefore getting that $f_{(r)}^{(i)} = \binom{r-1}{i-1}$, and the conclusion follows.

We can prove this result through other method also:

Note $S_k^{k+m} = \binom{m}{m}e_{k+m} + \ldots + \binom{m}{0}e_k$, where $k, m \ge 0$ and we can easily prove that $S_{k-1}^{k+m} - S_k^{k+m} = S_{k-1}^{k+m-1}$. By induction we get $S_k^{k+m} = \binom{k-1}{k-1}(-1)^{(k-1)-(k-1)}g_{(k+m)} + \ldots + \binom{k-1}{0}(-1)^{k-1-0}g_{(m+1)}$, therefore by plugging in $k = 1$ we get $S_1^{m+1} = g_{(m+1)} = \binom{m}{m}e_{m+1} + \ldots + \binom{m}{0}e_1$, hence $g_{(r)} = \binom{r-1}{r-1}e_r + \ldots + \binom{r-1}{0}e_1$.

1.3.2 Two columns case

Case II : $\lambda = (r, s)$ with $r \geq s$.

By using the lemma, we need to prove

$$f^{(i,j)}_{(r,s)} = \begin{vmatrix} \binom{r-1}{i-1} & \binom{r-1}{j-2} \\ \binom{s-1}{i} & \binom{s-1}{j-1} \end{vmatrix} = \binom{r-1}{i-1}\binom{s-1}{j-1} - \binom{r-1}{j-2}\binom{s-1}{i} \tag{14}$$

with $i \geq j,\ s \geq j,\ r \geq i$.

We will prove this in multiple steps.

Step1. If $j = 0$ then obviously $f^{(i,j)}_{(r,s)} = 0$, i.e. there is no *elegant filling*.

Suppose that $j = 1$. For the second skew-column $(s)/(j)$ there is only one possibility to have an elegant filling, i.e. starting up to down with 1 till $s - 1$. Then every *elegant filling* of the first column taken separately, will provide an elegant filling of the skew-shape $(r, s)/(i, j)$, therefore the number of elegant fillings of $(r, s)/(i, 1)$ is equal to the number of elegant fillings of $(r)/(i)$, which we computed in the previous case to be $\binom{r-1}{i-1}$, therefore we proved that $f^{(i,1)}_{(r,s)} = \binom{r-1}{i-1}$. Thus, from now on we can suppose that $j \geq 2$.

Step2. Suppose that $s - 1 \leq i$, then $\binom{s-1}{i} = 0$. As there will be no rows with two boxes, any elegant filling of the first skew-column $(r)/(i)$ together with any elegant filling of the skew-column $(s)/(j)$ will make a good elegant filling of $(r, s)/(i, j)$, therefore the number is equal to $f^{(i,j)}_{(r,s)} = f^{(i)}_{(r)} f^{(j)}_{(s)} = \binom{s-1}{j-1}\binom{r-1}{i-1}$. From now on we can suppose that $s - 1 \geq i$.

Step3.

Definition. A $non-elegant\ filling$ (NEF) of a skew-shape $(r, s)/(i, j)$ with two columns such that:

1). strictly increases in columns

2). there exists at least one row containing two boxes which are strictly decreasing in row

3). every number on the $i - th$ row is between 1 and $i - 1$.

We denote the number of $non - elegant\ fillings$ with $n^{(i,j)}_{(r,s)}$.

Definition. A $semi-elegant\ filling$ (SEF) of a skew-shape $(r,s)/(i,j)$ with two columns such that:

1). the numbers strictly increase in columns

2). every number on the $i-th$ row is between 1 and $i-1$.

We denote the number of $semi-elegant\ filling$ with $s_{(r,s)}^{(i,j)}$. We can see that in fact these conditions means that every column separately is filled in an $elegant$ way. Hence, we can actually compute the number of $semi-elegant\ fillings$, this being $s_{(r,s)}^{(i,j)} = \binom{r-1}{i-1}\binom{s-1}{j-1}$.

We can obviously see that a $semi-elegant\ filling$ can be either a $non-elegant\ filling$ or an $elegant\ filling$, therefore we get that $f_{(r,s)}^{(i,j)} + n_{(r,s)}^{(i,j)} = s_{(r,s)}^{(i,j)}$. If we suppose that $f_{(r,s)}^{(i,j)} = \binom{r-1}{i-1}\binom{s-1}{j-1} - \binom{s-1}{j}\binom{r-1}{j-2}$ then this will give us that $n_{(r,s)}^{(i,j)} = \binom{s-1}{i}\binom{r-1}{j-2}$. This means that we need to prove now that the number of NEFs is $\binom{s-1}{i}\binom{r-1}{j-2}$.

Main Theorem. The number of the $NEFs$ of the skew-shape $(r,s)/(i,j)$ is $\binom{s-1}{i}\binom{r-1}{j-2}$.

Proof:

1). First we prove if $s-1=i$. We consider a NEF for the skew-shape $(r,s)/(s-1,j)$, and let $b_s,\ldots,b_r,a_{j+1},\ldots,a_s$ with b_l being the $l-th$ number on the first skew-column and a_l being the $l-th$ number on the second skew-column. Being a NEF gives us that $a_{j+1} < \ldots < a_s$, $a_m \in [1,\ m-1]$ for all $m = \overline{j+1,s}$, $b_s < \ldots < b_r$, $b_l \in [1,\ l-1]$ for all $l = \overline{s,r}$, and also $a_s < b_s \leq s-1$. But the latter condition gives us that in fact $a_s \leq s-2$, therefore $a_m \leq m-2$ for all $m = \overline{j+1,s}$, making the numbers $a_{j+1} < \ldots < a_s < b_s < \ldots < b_r$ an $elegant\ filling$ of a skew-shape $(r)/(j-1)$. Therefore this implies $n_{(r,s)}^{(i,j)} = f_{(r)}^{(j-1)} = \binom{r-1}{j-2} = \binom{r-1}{j-2}\binom{s-1}{i}$, hence the conclusion.

2). Now we suppose that $s \geq i+2$.

Note $N_\lambda^\mu = \{$ all the $NEFs$ of the shape λ/μ $\}$ and $E_\lambda^\mu = \{$ all the EFs of the shape λ/μ $\}$, with $\mu \subseteq \lambda$.

We will construct a bijection between $N_{(r,s)}^{(i,j)}$ and $E_{(r)}^{(j-1)} \times E_{(s)}^{(i+1)}$.

i). We define the bijection h. Take $A \in N_{(r,s)}^{(i,j)}$. Note $x_{i+1}, \ldots, x_r$ the numbers in the first column and $y_{j+1}, \ldots y_s$ the numbers in the second one. Let $k = min\ \{l \mid x_l > y_l,\ i+1 \leq l \leq s\ \}$ (k exists because A is a NEF).

We have that $y_k < x_k \leq k-1$, hence $y_l \leq l-2$ for all $l = \overline{j+1, k}$. Because $x_m \in [1, m-1]$ for all $m \in [i+1, r]$, $y_l \in [1, l-2]$ for all $l \in [j+1, s]$ and also $y_{j+1} < \ldots < y_k < x_k < \ldots < x_r$, we get that $y_{j+1}, \ldots, y_k, x_k, x_{k+1}, \ldots, x_r$ can be an *elegant filling* for a skew-shape $(r)/(j-1)$ which belongs to $E_{(r)}^{(j-1)}$. We note this filling by B_A. Also, because $x_m \in [1, m-1] \subset [1, m]$ for all $m = \overline{i+1, k-1}$ and $y_l \in [1, l-1]$ and also $x_{i+1} < \ldots < x_{k-1} < y_{k+1} < \ldots < y_s$, we get that $x_{i+1}, \ldots, x_{k-1}, y_{k+1}, \ldots, y_s$ can be an *elegant filling* for a skew-shape $(s)/(i+1)$ which belongs to $E_{(s)}^{(i+1)}$. We note this filling with C_A.

Now, we will define the bijection in the following way : $h: N_{(r,s)}^{(i,j)} \rightarrow E_{(r)}^{(j-1)} \times E_{(s)}^{(i+1)}$ and $h(A) = (B_A, C_A) \in E_{(r)}^{(j-1)} \times E_{(s)}^{(i+1)}$.

ii). We will prove that h is well defined. Suppose that there is an A in $N_{(r,s)}^{(i,j)}$ and $h(A) = (B_A, C_A) = (B'_A, C'_A)$ which implies that $x_m = x'_m$ for all $m = \overline{i+1, r}$ and $y_l = y'_l$ for all $l = \overline{j+1, s}$, hence $B_A = B'_A$ and $C_A = C'_A$ which proves that h is well defined.

iii). At this step we will prove that h is indeed a bijection. As it is clear that the sets $N_{(r,s)}^{(i,j)}$ and $E_{(r)}^{(j-1)} \times E_{(s)}^{(i+1)}$ are finite, it is sufficient to prove that h is surjective.

Take any $B \in E_{(r)}^{(j-1)}$ and $C \in E_{(s)}^{(i+1)}$. We note the numbers in B to be $\beta_j, \ldots, \beta_r$ with $\beta_l \in [1, l-1]$ for all $l = \overline{j, r}$ and $\beta_j < \ldots < \beta_r$, and we also note the numbers in C to be $\alpha_{i+2}, \ldots, \alpha_s$ with $\alpha_m \in [1, m-1]$ for all $m = \overline{i+2, s}$ and $\alpha_{i+2} < \ldots < \alpha_s$.

Suppose that there exists k such that $k+1 = min\{l \mid \beta_{l-2} < \alpha_l,\ l \in [i+2, s]\}$. We have that $\alpha_k \leq \beta_{k-2} < \beta_k \leq k-1$, hence $\alpha_l \in [1, l-2]$ for all $l = \overline{i+2, k}$. We define two sequences $x_{i+1}, \ldots, x_r$ and $y_{j+1}, \ldots, y_s$ in the following way:

$$x_l = \alpha_{l+1}\ for\ all\ l = \overline{i+1, k-1}\ and\ x_l = \beta_l\ for\ all\ l = \overline{k, r} \tag{15}$$

and

$$y_l = \beta_{l-1}\ for\ all\ l = \overline{j+1, k}\ and\ y_l = \alpha_l\ for\ all\ l = \overline{k+1, s}. \tag{16}$$

We take a skew-shape $(r,s)/(i,j)$ and we fill it out with $x_{i+1},\dots,x_l$ on the first column and with $y_{j+1},\dots,y_s$ on the second column and we note this filling $A_{B,C}$. We have that $x_{i+1} < \dots < x_r$ and $y_{j+1} < \dots < y_s$, $x_m \in [1, m-1]$ for all $m \in [i+1, r]$, $y_l \in [1, l-1]$ for all $l \in [j+1, s]$ and on the $k-th$ row we have $x_k > y_k$, all these conditions prove that $A_{B,C}$ is a $N_{(r,s)}^{(i,j)}$. Now, we can immediately observe that (B,C) is the image through h of $A_{B,C}$ (just apply the algorithm defined in **i).**).

Now, suppose that there is no k such that $k+1 = min\{l|\ \beta_{l-2} < \alpha_l,\ l \in [i+2, s]\}$. Hence $\beta_{l-2} \geq \alpha_l, \forall\, i+2 \leq l \leq s$. This implies that $\alpha_l \in [1, l-3] \subset [1, l-2]$ for all $l = \overline{i+2, s}$. We define two sequences $x_{i+1},\dots,x_r$ and $y_{j+1},\dots,y_s$ in the following way:

$$x_l = \alpha_{l+1}\ for\ all\ l = \overline{i+1, s-1}\ and\ x_l = \beta_l\ for\ all\ l = \overline{s, r} \tag{17}$$

and

$$y_l = \beta_{l-1}\ for\ all\ l = \overline{j+1, s}. \tag{18}$$

we take the skew-shape $(r,s)/(i,j)$ and we fill it out with $x_{i+1},\dots,x_r$ on the first column and with $y_{j+1},\dots,y_s$ on the second one and we note this filing $A'_{B,C}$. We have that $x_{i+1} < \dots < x_r$ and $y_{j+1} < \dots < y_s$, $x_l \in [1, l-2] \subset [1, l-1]$ for all $l \in [i+1, r]$, $y_l \in [1, l-2] \subset [1, l-1]$ for all $l \in [j+1, s]$ and on the $k-th$ row we have $x_k > y_k$, all these conditions prove that $A'_{B,C}$ is a $N_{(r,s)}^{(i,j)}$. Again, we see immediately that, in this case also, (B,C) is the image through h of $A'_{B,C}$.

Hence, the conclusion. Therefore, h is surjective, thus also bijective. h is indeed a bijection between $N_{(r,s)}^{(i,j)}$ and $E_{(r)}^{(j-1)} \times E_{(s)}^{(i+1)}$, which implies that the cardinals of $N_{(r,s)}^{(i,j)}$ and $E_{(r)}^{(j-1)} \times E_{(s)}^{(i+1)}$ are equal, so $|N_{(r,s)}^{(i,j)}| = |E_{(r)}^{(j-1)}| \times |E_{(s)}^{(i+1)}| = f_{(r)}^{(j-1)} f_{(s)}^{(i+1)} = \binom{r-1}{j-2}\binom{s-1}{i}$, hence $n_{(r,s)}^{(i,j)} = \binom{r-1}{j-2}\binom{s-1}{i}$, therefore $f_{(r,s)}^{(i,j)} = \binom{r-1}{i-1}\binom{s-1}{j-1} - \binom{s-1}{j}\binom{r-1}{j-2}$.

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