ON THE LATTICE STRUCTURE OF SHARD INTERSECTION ORDERS.

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Abstract. Garver and McConville recently introduced two lattices associated to a tree: the lattice of biclosed sets and the oriented flip graph. They showed that both of these lattices are congruence-uniform lattices, and thus both admit a lattice-theoretic shard intersection order. We show that the shard intersection order of the oriented flip graph is graded and rank-symmetric. We also construct as CU-labeling of the lattice of biclosed sets and use this to understand the lattice structure of its shard intersection order.

1. Introduction

The study of triangulated surfaces is important to understanding the combinatorics of and representation theory related to cluster algebras. However, not as much is understood about the connections between polygonal subdivisions of surfaces (partial triangulations) and similar combinatorial and representation-theoretic objects. In [3], the authors study this question, and obtain an isomorphism between a lattice of polygonal subdivisions, which they call an oriented flip graph, and a lattice of torsion pairs. The oriented flip graph is a congruence-uniform lattice, and thus we can define its lattice-theoretic shard intersection order in the sense of Reading. The authors in [3] showed that this new poset can be modeled by yet another combinatorial construction on the partially-triangulated surface. This construction is known as the **noncrossing tree partitions**, and generalizes the classical noncrossing set partitions.

This paper focuses on understanding the noncrossing tree partitions, $\mathrm{NCP}(T)$, and the biclosed sets, $\mathrm{Bic}(T)$, from a combinatorial perspective. In particular, we show that $\mathrm{NCP}(T)$ is always a graded and rank-symmetric lattice, and present a method for counting the number of maximal chains in this lattice. Moreover, the oriented flip graph is best understood through the lens of a different lattice arising from the geometric combinatorics of this partially-triangulated marked surface, a poset known as the lattice of **biclosed sets**. In terms of biclosed sets, we produce an explicit CU-labeling and then investigate the structure of the resulting shard intersection order, with the ultimate aim of proving that this poset is a lattice.

2. Preliminaries

2.1. **Lattices.** Let L be a finite, graded lattice. For $x, y \in L$, if $x < y$ and no $z \in L$ satisfies $x < z < y$, we write $x \lessdot y$. Let $\mathrm{Cov}(L) := \{(x, y) \in L^2 \mid x \lessdot y\}$ be the set of **covering relations** of L. A map $\lambda : \mathrm{Cov}(L) \to Q$, where $(Q, \leqslant_Q)$ is some poset is called an **edge labeling**. Given two maximal chains $C = c_1 < \cdots < c_n$ and $C' = c'_1 < \dots c'_n$ in L, we say C is **lexicographically smaller** than C' if $(\lambda(c_1, c_2), \dots, \lambda(c_{n-1}, c_n))$ lexicographically precedes $(\lambda(c'_1, c'_2), \dots, \lambda(c'_{n-1}, c'_n))$.

We call a labeling $\lambda : \mathrm{Cov}(L) \to Q$ an **EL-labeling** of L if for every interval $[x, y]$ of L,

(1) there is a unique increasing maximal chain C in $[x, y]$, and
(2) C is lexicographically smaller than any other maximal chain C' in $[x, y]$.

If L admits an EL-labeling, it is said to be **EL-shellable**.

We need the concepts of join and meet irreducibility to discuss another important type of labeling. We say that an element $j \in L$ is **join irreducible** if $j \neq \hat{0}$ and whenever $j = x \vee y$, either $j = x$ or $j = y$ holds. **Meet-irreducible** elements are defined dually. We denote the subset of join-irreducible (resp. meet-irreducible) elements by $\mathrm{JI}(L)$ (resp. $\mathrm{MI}(L)$). For j (resp. m) in $\mathrm{JI}(L)$ (resp. $\mathrm{MI}(L)$), we let j_* (resp. m^*) denote the unique element of L covered by (resp. that covers) j (resp. m). With this in hand, we arrive at the notion of a **CN-labeling**, which plays a prominent role in this paper.

Definition 2.1. A labeling $\lambda : \mathrm{Cov}(L) \to Q$ is a **CN-labeling** if L and its dual L^* satisfy the following: For elements $x, y, z \in L$ with $(z, x), (z, y) \in \mathrm{Cov}(L)$ and maximal chains C_1 and C_2in $[z, x \vee y]$ with $x \in C_1$ and $y \in C_2$,

(CN1) the elements $x' \in C_1, y' \in C_2$ such that $(x', x \vee y), (y', x \vee y) \in \mathrm{Cov}(L)$ satisfy

$$\lambda(x', x \vee y) = \lambda(z, y), \lambda(y', x \vee y) = \lambda(z, x);$$

(CN2) if $(u, v) \in \mathrm{Cov}(C_1)$ with $z < u, v < x \vee y$, then $\lambda(z, x), \lambda(z, y) <_Q \lambda(u, v)$;

(CN3) the labels on $\mathrm{Cov}(C_1)$ are pairwise distinct.

We say that λ is a **CU-labeling** if, in addition, it satisfies

(CU1) $\lambda(j_*, j) \neq \lambda(j'_*, j')$ for $j, j' \in \mathrm{JI}(L)$, $j \neq j'$, and

(CU2) $\lambda(m, m^*) \neq \lambda(m, m'^*)$ for $m, m' \in \mathrm{MI}(L)$, $m \neq m'$.

If L admits a CU-labeling, it is said to be **congruence-uniform** , see [3].

Given a lattice L with a CU-labeling, one can define a new poset, $\Psi(L)$, called the **shard intersection order** of L. Reading introduced this concept in [4].

Definition 2.2. Let L be a congruence-uniform lattice with CU-labeling $\lambda : \mathrm{Cov}(L) \to P$. Let $x \in L$ and let $y_1, \ldots y_k$ be the elements of L satisfying $(y_i, x) \in \mathrm{Cov}(L)$. We denote the set $\{\lambda(y_i, x)\}$ by $\lambda_{\downarrow}(x)$. Define the **shard intersection order** of L, denoted $\Psi(L)$, to be the collection of sets of the form

$$\psi(x) := \{\lambda(w, z) \mid \bigwedge_i^k y_i \leqslant w < z \leqslant x, (w, z) \in \mathrm{Cov}(L)\}$$

partially ordered by inclusion.

2.2. **Tree preliminaries.** A **tree** is a finite connected acyclic graph. Any tree may be embedded in a disk D^2 in such a way that a vertex is on the boundary if and only if it is a leaf. We will always assume that any tree comes equipped with such an embedding. We will refer to non-leaf vertices of a tree as **interior vertices**, and denote the set of interior vertices of a tree T by $V^o(T)$. We assume that any interior vertex of a tree has degree at least 3. Given trees T, T' embedded in D^2, we consider T and T' to be **equivalent** if there is an isotopy between the spaces $D^2 \backslash T$ and $D^2 \backslash T'$.

Let T be a tree embedded in D^2. The embedding of T in D^2 determines a collection of 2-dimensional regions in D^2 that we will refer to as **faces**. A **corner** of a tree is a pair (v, F) consisting of an interior vertex v and a 2-dimensional face F containing v. The embedding that accompanies T endows each interior vertex with a cyclic ordering. Given two corners $(u, F), (u, G)$, we say that (u, G) is **immediately clockwise** (resp. **immediately counterclockwise**) from (u, F) if $F \cap G \neq \varnothing$ and G is clockwise (resp. counterclockwise) from F according to the cyclic ordering at u.

An **acyclic path** supported by a tree T is a sequence $(v_0, \ldots, v_t)$ of vertices of T such that v_i and v_j are adjacent if and only if $|i - j| = 1$. We typically identify acyclic paths with their underlying vertex sets; that is, we do not distinguish between paths of the form $(v_0, \ldots, v_t)$ and $(v_t, \ldots, v_0)$.

An **arc** $p = (v_0, \ldots, v_t)$ is an acyclic path whose endpoints are leaves and for all i, the edges (v_{i-1}, v_i) and (v_i, v_{i+1}) are incident to a common face—another way of saying this is that p "turns sharply" at every vertex it contains. We say p **contains a corner** (v, F) if $v = v_i$ for for some $i = 0, 1, \ldots, t$ and F is the face that is incident to both (v_{i-1}, v_i) and (v_i, v_{i+1}). Since an arc p divides D^2 into two components, it determines two disjoint subsets of the set of faces of T that we will call **regions**.

A **segment** is an acyclic path consisting of at least two vertices and with the same incidence condition that is required of arcs, but whose endpoints are *not* leaves. Since trees have unique geodesics between any two vertices, if the endpoints of a segment or arc are v, w, we may denote the path by $[v, w]$. We say that two segments s and t are **composable** if they agree only at one endpoint and their concatenation at this endpoint is also a segment. We denote their composition by $s \circ t$.

2.3. **Noncrossing tree partitions.** We now introduce a partially ordered set defined on a tree, the **noncrossing tree partitions**, which is one of this paper's central objects of study. For the proof of all the results in this subsection, see [3].

Given a tree T, a **red admissible curve** $\gamma : [0, 1] \to D^2$ for a segment $s = [v_0, v_t]$ is a simple curve where

- its endpoints are v_0 and v_t,
- γ may only intersect edges of T of the form (v_{i-1}, v_i), where $i \in [t]$, and
- γ must leave v_0 (resp. v_t) on the face immediately clockwise from v_0 (resp. v_t), orienting T from v_0 to v_t.

Two segments are **noncrossing** if they admit red admissible curves that do not intersect. A **green admissible** curve is the same as a red admissible curve, replacing the word "clockwise" with "counterclockwise."

Given a collection of vertices $B \subseteq V^o(T)$, we say that B is **segment-connected** if for $v, w \in B$ there exists a sequence of segments $s_1 = [v_1 = v, v_2], s_2 = [v_2, v_3], \ldots, s_{n-1} = [v_{n-1}, v_n = w]$ such that $s_1 \circ \cdots \circ s_{n-1} = [v, w]$, with $v_i \in B$, and $[v_i, v_{i+1}] \cap (B \backslash \{v_i, v_{i-1}\}) = \varnothing$.

It was shown in [3] that given two such vertices u and v in a segmented-connected set $B \subseteq V^o(T)$, the sequence of segments described in the above definition is unique. The union of all segments appearing in the type of sequence described in the above definition is denoted by $\mathrm{Seg}_r(B)$. We are now ready to state the main definition of this section.

Definition 2.3. A **noncrossing tree partition** $\mathbf{B} = \{B_1, \ldots, B_k\}$ is a set partition of $V^o(T)$ such that each B_i is segment-connected and any segments $s_1 \in \mathrm{Seg}_r(B_i)$ and $s_2 \in \mathrm{Seg}_r(B_j)$ are noncrossing. For a noncrossing tree partition $\mathbf{B}$, we call each B_i a **block** of $\mathbf{B}$.

The following is a major result from [3] concerning the structure of the collection of noncrossing tree partitions of a given tree T that motivates much of the investigation of this paper.

Theorem 2.4. The set $\mathrm{NCP}(T) := \{\text{noncrossing partitions of } T\}$ partially ordered by refinement is a lattice.

2.4. **Oriented flip graphs.** We now discuss another lattice, the oriented flip graph, whose structure is closely related to the noncrossing tree partition lattice. All of the following results are proved in [3]. We have already defined what it means for two segments to cross; there is a similar notion for arcs.

We say that two arcs $p = (v_0, \ldots, v_t), q = (w_0, \ldots, w_s)$ are **crossing** along a segment $s = (u_0, \ldots, u_r)$ if

i) each vertex of s appears in both p and q and

ii) if R_p and R_q are regions defined by p and q, respectively, then $\mathrm{R}_p \not\subseteq \mathrm{R}_q$ and $\mathrm{R}_q \not\subseteq \mathrm{R}_p$.

We say they are **noncrossing** otherwise. The **noncrossing complex** $\Delta^{NC}(T)$ is defined to be the abstract simplicial complex whose simplices are pairwise noncrossing collections of arcs supported by a tree T.

If p is an arc whose vertices all lie on a common face, then p is non-crossing with every arc supported by T. We call such an arc a **boundary arc**. The **reduced noncrossing complex** $\widetilde{\Delta}^{NC}(T)$ is the abstract simplicial complex whose faces are the faces of $\Delta^{NC}(T)$ that contain no boundary arcs.

We now introduce a partial ordering on arcs that contain a particular corner of T. Let $\mathcal{F}$ be a face of $\Delta^{NC}(T)$ and let (v, F) be a corner that is contained in at least one arc of $\mathcal{F}$. The arcs of $\mathcal{F}$ that contain (v, F) are partially ordered in the following way: $p \leqslant_{(v,F)} q$ if and only if the region defined by p containing F is contained in that of q.

If $\mathcal{F}$ is a face of $\Delta^{NC}(T)$ and (v, F) is a corner contained in at least one arc of $\mathcal{F}$, then the partially ordered set $(\{p \in \mathcal{F} : \ p \text{ contains } (v, F)\}, \leqslant_{(v,F)})$ is a linearly ordered set. In particular, it has a unique maximal element, which we will denote by $p(v, F)$. We say that an arc p of $\mathcal{F}$ is **marked** at (v, F) if $p = p(v, F)$. The following theorem discussing marked corners is crucial to defining the oriented flip graph.

Theorem 2.5. Let $\mathcal{F}$ be a face of $\Delta^{NC}(T)$, let $p \in \mathcal{F}$, and let $\mathrm{Reg}_1, \mathrm{Reg}_2$ denote the regions defined by p.

1. The arc p is marked at a corner of T.
2. If p is not a boundary arc, then p is marked at a corner in Reg_1 and at a corner in Reg_2.
3. Assume that p is marked at two distinct corners $(v, F), (w, G) \in \mathrm{Cor}(T)$ and that F and G belong to the same region defined by p. Then there exists an arc $p' \notin \mathcal{F}$ that contains (v, F) and (w, G') where $G' \neq G$ and where $\mathcal{F} \cup \{p'\} \in \Delta^{NC}(T)$.
4. If $\mathcal{F}$ is a facet and $p \in \mathcal{F}$ is not a boundary arc, then there exists a unique arc $q \notin \mathcal{F}$ such that $(\mathcal{F} \backslash \{p\}) \cup \{q\}$ is a facet. Moreover, if p is marked at two distinct corners $(v, F), (u, G) \in \mathrm{Cor}(T)$, then $[v, u]$ is the unique longest segment along which p and q cross.

A corollary of this result is that the simplicial complex $\widetilde{\Delta}^{NC}(T)$ is **pure** (i.e. every facet has the same cardinality).

We refer to the operation $\mathcal{F} \longmapsto (\mathcal{F} \backslash \{p\}) \cup \{q\}$ sending facet $\mathcal{F}$ of $\widetilde{\Delta}^{NC}(T)$ to a new facet of $\widetilde{\Delta}^{NC}(T)$ as a **flip** of $\mathcal{F}$ at p, and denote it by μ_p. We define the **flip graph** of T, denoted $\mathrm{FG}(T)$, to be the graph whose vertices are facets of $\widetilde{\Delta}^{NC}(T)$ and such that two vertices are connected by an edge if and only if the corresponding facets can be obtained from each other by a single flip.

Given a facet $\mathcal{F}$ and an arc $p \in \mathcal{F}$ marked at (u, F) and (v, G), we call p **red** (resp. **green**) if F and G are the immediately clockwise (resp. counterclockwise) faces of their corresponding vertices, orienting from u to v; it is shown in [3] that for a facet $\mathcal{F}$, every arc in $\mathcal{F}$ is either green or red.

Definition 2.6. Let $\mathcal{F}_1, \mathcal{F}_2 \in FG(T)$ and assume that $\mathcal{F}_1$ and $\mathcal{F}_2$ are connected by an edge in $FG(T)$. Let $\mathcal{F}_2 = \mu_p \mathcal{F}_1$ and let q denote the unique arc produced by flipping p in $\mathcal{F}_1$. If $p = p(u, F) = p(v, G)$ and $q = p(u, F') = p(v, G')$, we orient the edge connecting $\mathcal{F}_1$ and $\mathcal{F}_2$ so that $\mathcal{F}_1 \longrightarrow \mathcal{F}_2$ if and only if the corner (u, F') (resp. (v, G')) is immediately clockwise from the corner (u, F) (resp. (v, G)) about vertex u (resp. v). We refer to the resulting directed graph as the **oriented flip graph** of T and denote it by $\overrightarrow{FG}(T)$.

The following result concerns the structure of $\overrightarrow{FG}(T)$, and allows to define the shard intersection of $\overrightarrow{FG}(T)$.

Theorem 2.7. The poset $\overrightarrow{FG}(T)$ is a congruence-uniform lattice.

With this in hand, we can consider $\Psi(\overrightarrow{FG}(T))$, and this is where one witnesses the connection between NCP(T) and $\overrightarrow{FG}(T)$.

Theorem 2.8. The poset $\Psi(\overrightarrow{FG}(T))$ is a lattice, and NCP$(T) \cong \Psi(\overrightarrow{FG}(T))$.

There exists a bijection $\rho : \overrightarrow{FG}(T) \to \mathrm{NCP}(T)$ given as follows. For a facet $\mathcal{F} \in \overrightarrow{FG}(T)$, each red arc in $\mathcal{F}$ defines a segment whose endpoints are the arc's two marked corners. Let S denote this set of segments. Recall that for a segment-connected block in NCP(T), there exists a unique set of segments realizing this segment-connectedeness. The authors showed in [3] that S defines an element of NCP(T) in this manner, and that ρ is in fact a bijection.

Accordingly, the isomorphism φ from NCP(T) to $\Psi(\overrightarrow{FG}(T))$ is given by mapping $\mathbf{B} \in \mathrm{NCP}(T)$ under ρ^{-1} and then letting $\varphi(\mathbf{B}) = \psi(\rho^{-1}(\mathbf{B}))$.

2.5. **Biclosed collections of segments.** All of the results in this subsection are proved in [3]. Let Seg(T) be the set of segments supported by a tree T. For $X \subseteq \mathrm{Seg}(T)$, we say X is **closed** if for segments $s, t \in \mathrm{Seg}(T)$, if $s, t \in X$ and $s \circ t \in \mathrm{Seg}(T)$ then $s \circ t \in X$. If X is any subset of Seg(T), its **closure** $\overline{X}$ is the smallest closed set containing X. Say X is **biclosed** if X and $\mathrm{Seg}(T)\backslash X$ are both closed. We let Bic(T) denote the poset of biclosed subsets of Seg(T), ordered by inclusion.

Theorem 2.9. The poset Bic(T) is a lattice.

We now discuss an important injective map ϕ from $\overrightarrow{FG}(T)$ to Bic(T) that will be used later in the report.

For an arc $p = (v_0, \dots, v_l)$ oriented from v_0 to v_l, let C_p be the set of segments $(v_i, \dots, v_j)$, $0 < i < j < l$ such that

- p turns right at v_i, and
- p turns left at v_j.

Given $\mathcal{F} \in \overrightarrow{FG}(T)$, we define $\phi(\mathcal{F})$ to be $\overline{\bigcup_{p \in \mathcal{F}} C_p}$.

Theorem 2.10. The map $\phi : \overrightarrow{FG}(T) \to \mathrm{Bic}(T)$ is an injective lattice map.

Corollary 2.11. The lattice $\overrightarrow{FG}(T)$ is isomorphic to its image under ϕ, and is thus a sublattice of Bic(T).

3. Lattice properties of NCP(T).

Throughout this section, we will let T denote a tree as defined in the previous section. It is known that NCP(T) is a lattice; in this section we will prove several results about the structure of NCP(T), as well as state future directions for exploration. We begin with results that discuss ways to decompose NCP(T) for a general tree.

3.1. Decomposing NCP(T).

Definition 3.1. Assume that T has an interior vertex w that is incident to at least two leaf vertices—we call such a vertex a **wall vertex**. Observe that there is a unique acyclic path of T connecting the leaves in these edges, and that this path divides T into two regions, R_1 and R_2. Given a wall vertex w, we can define two new trees, T_1 and T_2, where T_i is given by removing all vertices in R_j $(j \neq i)$ and all edges between these vertices, except for all vertices in R_j incident to w, which become boundary vertices of T_i (keeping also the edges that realize this incidence). It is clear from the definition that T_i is a tree with all interior vertices of degree at least three.

Lemma 3.2. Let w be a wall vertex of T. Then NCP$(T) \cong \mathrm{NCP}(T_1) \times \mathrm{NCP}(T_2)$.

Proof. We first define a bijection φ from NCP(T) to NCP$(T_1) \times \mathrm{NCP}(T_2)$. Let $\mathbf{B} \in \mathrm{NCP}(T)$. Note that if $u_1 \in R_1\backslash\{w\}$ and $u_2 \in R_2\backslash\{w\}$ are in the same block B of $\mathbf{B}$, then B contains w. We define φ as follows: for any block $B \in \mathbf{B}$ that does not contain w, we let it be a block of $\mathbf{B}_i \in \mathrm{NCP}(T_i)$, where B is contained in R_i. For the block B' of $\mathbf{B}$ that contains w, we create two new blocks, B_1 and B_2, where $B_i := B'\backslash(B' \cap (R_i\backslash\{w\}))$, and then let B_i be a block of $\mathbf{B}_i$. The pair $(\mathbf{B}_1, \mathbf{B}_2)$ is in NCP$(T_1) \times \mathrm{NCP}(T_2)$, and φ is a bijection.

To see that φ is order-preserving, suppose that $\mathbf{B} \leqslant \mathbf{B}'$ in NCP(T). Thus, every block of $\mathbf{B}$ is contained in a block of $\mathbf{B}'$. Let (B_1, B_2) be a pair of blocks in $\varphi(\mathbf{B})$. If $w \in B_i$, then there exists $B' \in \mathbf{B}'$ such that $w \in B'$ and $B_i \subseteq B'$, where B_i is seen as a collection of vertices in $R_i \subseteq V^o(T)$. It follows that $B_i \subseteq B'_i$. A similar argument holds when $w \notin B_i$. Hence, if $\varphi(\mathbf{B}') = (\mathbf{B}'_1, \mathbf{B}'_2)$, then there exist $B'_1 \in \mathbf{B}'_1$, $B'_2 \in \mathbf{B}'_2$ such that $B_1 \subseteq B'_1$ and $B_2 \subseteq B'_2$, so $\varphi(\mathbf{B}) \leqslant \varphi(\mathbf{B}')$. The proof that φ^{-1} is order-preserving is similar.

□

Next, we discuss a result concerning decomposing lower intervals of $\mathrm{NCP}(T)$ into the direct product of non-crossing tree partitions of smaller trees. As will be seen later in the report, such a decomposition has many consequences and applications.

Definition 3.3. The following construction is from [3]. Let B be a segmented-connected subset of $V^o(T)$, and let $S = \mathrm{Seg}_r(B)$. We define the **contracted tree** T_B such that

- B is the set of interior vertices of T_B,
- S is the set of interior edges of T_B, and
- for edges e with one endpoint u in B and the other endpoint not between two vertices of B, there is an edge from u to the boundary in the direction of e.

It is clear that T_B defines a tree with every interior vertex of degree at least three.

If $\mathbf{B} \in \mathrm{NCP}(T)$ and B is one of its blocks, then we can further partition B into sub-blocks, $\{B'_1, \ldots, B'_m\}$. If each sub-block of such a partition of B is segmented-connected and for any segments $s \in \mathrm{Seg}_r(B'_i), t \in \mathrm{Seg}_r(B'_j)$, s and t have non-crossing admissible curves, we say this partition is an element of the set that we call $\mathrm{NCP}(B)$, and define a partial-ordering on $\mathrm{NCP}(B)$ by refinement. It is clear that $\mathrm{NCP}(B)$ is isomorphic to $\mathrm{NCP}(T_B)$, since removing vertices in $V^o(T)\backslash B$ does not affect the crossing of admissible curves between elements of B.

Theorem 3.4. For $\mathbf{B} = \{B_1, \ldots, B_m\} \in \mathrm{NCP}(T)$, $[\hat{0}, \mathbf{B}] \cong \prod_{i=1}^m \mathrm{NCP}(T_{B_i})$.

Proof. In view of the above discussion, it suffices to show that $[\hat{0}, \mathbf{B}] \cong \prod_{i=1}^m \mathrm{NCP}(B_i)$. For any $\mathbf{B}' = \{B'_1, \ldots, B'_n\} \in [\hat{0}, \mathbf{B}]$, it is clear that $\mathbf{B}'$ defines a collection of elements in $\mathrm{NCP}(B_i)$, where for some $B_i \in \mathbf{B}$, we let $\mathbf{B}_i = \{B'_j \mid B'_j \subseteq B_i\}$. This map defines an order-preserving injection from $[\hat{0}, \mathbf{B}]$ to $\prod_{i=1}^m \mathrm{NCP}(B_i)$. It remains to show that it is surjective. To do this, we will define an injection from $\prod_{i=1}^m \mathrm{NCP}(B_i)$ to $[\hat{0}, \mathbf{B}]$.

An element of $\prod_{i=1}^m \mathrm{NCP}(B_i)$ naturally defines a collection of segment-connected blocks of interior vertices, $\mathbf{B}' = \{B'_1, \ldots, B'_\ell\}$. Thus, to complete the proof it suffices to show that any two segments $s, t \in \mathrm{Seg}_r(\mathbf{B}')$ are non-crossing. Let $s \in \mathrm{Seg}_r(B'_i)$ and $t \in \mathrm{Seg}_r(B'_j)$, $i \neq j$, where $B'_i \subseteq B_k \in \mathbf{B}$, $B'_j \subseteq B_r \in \mathbf{B}$, $k \neq r$; we want to show that s and t have non-crossing red-admissible curves. Observe that s is a composition of segments in $\mathrm{Seg}_r(B_k)$, $\{s_i\}_{i=1}^{n_i}$, and t is a composition of segments in $\mathrm{Seg}_r(B_r)$, $\{t_j\}_{j=1}^{n_j}$. Moreover, since $\mathbf{B} \in \mathrm{NCP}(T)$, given $\{s_i\}$, there exists a corresponding sequence of red-admissible curves, and none of these curves cross any of the curves in the analogous sequence for $\{t_j\}$. Thus, the composition of all the curves associated to $\{s_i\}$ and the composition of all the curves associated to $\{t_j\}$ define two new red admissible curves associated to s and t that do not cross, completing the proof.

□

This result, and the isomorphism constructed in its proof, implies that we can reduce every interval in $\mathrm{NCP}(T)$ to a product of upper intervals, as the following result shows.

Corollary 3.5. Every interval in $\mathrm{NCP}(T)$ is isomorphic to a product of upper intervals of noncrossing tree partitions.

Proof. Let $\mathbf{B} = \{B_1, \ldots, B_n\} \in \mathrm{NCP}(T)$ and let $\mathbf{B}' = \{B'_1, \ldots, B'_m\} \in \mathrm{NCP}(T)$, and $\mathbf{B} < \mathbf{B}'$. From the above Theorem, we know that $[\hat{0}, \mathbf{B}'] \cong \prod_{i=1}^m \mathrm{NCP}(T_{B'_i})$. Moreover, we know that under this isomorphism, $\mathbf{B}$ corresponds to $(x_1, \ldots, x_m)$, where $x_i = \{B_i^1, \ldots B_i^r\}$ is the collection of blocks in $\mathbf{B}$ that are contained in $B'_i \in \mathbf{B}'$. Moreover, since $\mathbf{B}'$ corresponds to $(\hat{1}_{\mathrm{NCP}(T_{B'_1})}, \ldots, \hat{1}_{\mathrm{NCP}(T_{B'_m})})$, we can conclude that $[\mathbf{B}, \mathbf{B}'] \cong \prod_{i=1}^m [x_i, \hat{1}_{\mathrm{NCP}(T_{B'_i})}]$. □

We conclude this subsection with a result concerning decomposing upper intervals in certain cases.

Definition 3.6. Let $\mathbf{B} \in \mathrm{NCP}(T)$, and $B \in \mathbf{B}$ be a block. If there exist two vertices $u, v \in B$ that are connected by an edge, we call the segment $[u, v]$ a **simple segment**. Given such a segment $s = [u, v]$, we can define a new tree, called the **reduced tree with respect to** s, given by identifying u with v, denoted T_s. The element $\mathbf{B}$ maps naturally to an element of $\mathrm{NCP}(T_s)$, denoted $\mathbf{B}_{T_s}$.

Lemma 3.7. Let $\mathbf{B} \in \mathrm{NCP}(T)$, and $B \in \mathbf{B}$ be a block that contains a simple segment s. Then $[\mathbf{B}, \hat{1}_{\mathrm{NCP}(T)}] \cong [\mathbf{B}_{T_s}, \hat{1}_{\mathrm{NCP}(T_s)}]$.

Proof. The result follows from the fact that for any $\mathbf{B}' \in [\mathbf{B}, \hat{1}_{\mathrm{NCP}(T)}]$, u and v are in the same block. Thus, sending an element of $[\mathbf{B}, \hat{1}_{\mathrm{NCP}(T)}]$ to its natural image in $[\mathbf{B}_{T_s}, \hat{1}_{\mathrm{NCP}(T_s)}]$ is an order-preserving injection. Moreover, given an element $\mathbf{B}^* \in [\mathbf{B}_{T_s}, \hat{1}_{\mathrm{NCP}(T_s)}]$, we can define an element $\mathbf{B}^*_T \in [\mathbf{B}, \hat{1}_{\mathrm{NCP}(T)}]$ by expanding the vertex at which u and v are identified according to the way u and v appear in T, and then keeping all blocks the same except for the block containing $u \sim v$, which we replace with u and v. Such a construction clearly creates a partition of $V^o(T)$ into segment-connected blocks. To see that this partition is noncrossing, note first that if neither s nor t in $\mathrm{Seg}_r(\mathbf{B}^*_T)$ is equal to $[u, v]$, then we can extend the admissible curves corresponding s and t in $\mathrm{Seg}_r(\mathbf{B}^*)$ to noncrossing curves for the images of s and t in T. Now if $s = [u, v]$, then in order for t to cross s, we know that t must pass either pass through v to reach u or vice versa; assume the former, and denote the subsegment of t approaching u from this direction by $[w, v]$. If t turns away from u at v, then it does not cross s, so we can assume it turns toward u at v. Note that in T_s the two sharp turns at $u \sim v$, orienting from w to v, are either the two sharp turns at v (with the same orientation in T) if the turn towards u is not one of these turns in T, or the one sharp turn at v away from u, and the first turn that we encounter by walking from v back to v, starting along $[u, v]$. In order for t to cross s, t must not make either of these turns, which is a contradiction, since then its corresponding segment in T_s does not turn sharply at $u \sim v$.

□

3.2. **Results concerning the oriented flip graph.** As discussed in the first section, $\mathrm{NCP}(T)$ is naturally isomorphic to the shard intersection order of the lattice called the oriented flip graph, $\overrightarrow{FG}(T)$. Thus, in order to prove results about $\mathrm{NCP}(T)$, it is helpful to investigate properties of $\overrightarrow{FG}(T)$. In this subsection, we will prove several results concerning $\overrightarrow{FG}(T)$, and then tie all of these results in with our results from the last subsection in order to discuss a proof for why $\mathrm{NCP}(T)$ is graded. Our first result concerns decomposing certain subsets of $\overrightarrow{FG}(T)$.

Definition 3.8. Let p be an arc of T. Let $S_p = \{\mathcal{F} \in \overrightarrow{FG}(T) \mid p \in \mathcal{F}\}$, where S_p inherits the partial order defined on $\overrightarrow{FG}(T)$. Let R_1 and R_2 be the regions defined by p. Let $V^o(p)$ denote the set of interior vertices contained in p. Note that for every element in S_p, every corner that p contains in R_i cannot be a marked corner of an arc in $R_i \backslash V^o(p)$, since the region containing any face in R_i cut out by p contains any such region cut out by an arc in $R_i \backslash V^o(p)$ by construction. Note further that for such a corner (v, F), there is an edge from v to two unique vertices in R_i, both contained in p, since if this were not the case, p would not contain (v, F).

For the remainder of this section, we work with a fixed tree T, and a fixed arc p of T.

Similar to our definition of the contracted tree earlier, we can define two new trees, T_1 and T_2. We define T_i by the following procedure:

(1) Delete all vertices in $R_j \backslash p$ and edges between these vertices. This gives a new tree, T_i'; denote the image of p in T_i' by p'.
(2) Note that if p' contains a corner (v, F), then F is either the image of a face in R_i, or is the unique face in T_i' that is incident only to vertices in p'. If the former, delete v and any edges incident to it. As discussed in definition 3.8, such a v is incident to two unique vertices in R_i. After deleting v, connect these two vertices by a new edge. We obtain a new tree.
(3) Repeat the second step with the resulting tree until no corners in the image of R_i remain. The resulting tree is T_i.

It is clear that T_i is a tree with every interior vertex of degree at least three. We now provide an example of the construction of T_1 and T_2.

Example 3.9. Figure 1 shows the above procedure. We start with our tree T on the far left, with arc p outlined in green. Each new picture represents a step in the procedure, where here $i = 1$. Figure 2 shows the result of applying the procedure to T when $i = 2$.

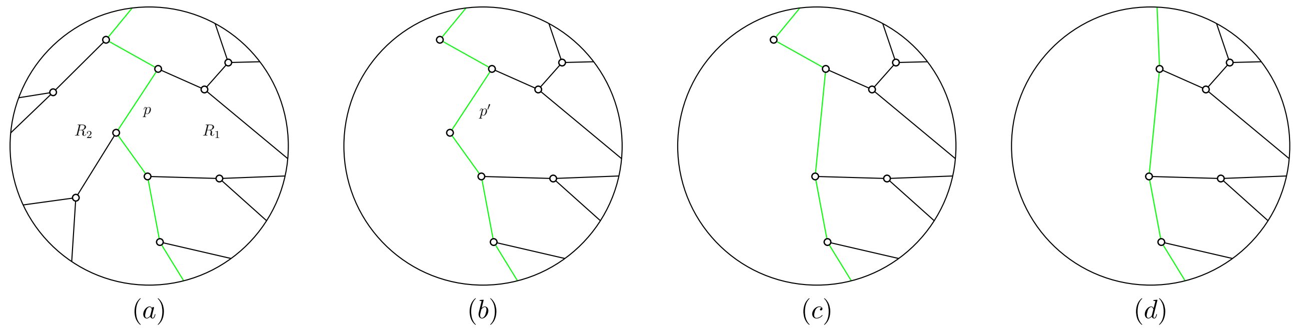


FIGURE 1. The steps outlined in the above procedure.

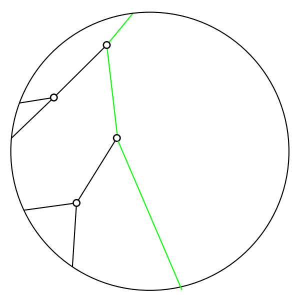


FIGURE 2. The tree T_2, obtained by following the same procedure but with R_2.

The following results show that the poset S_p has a very interesting structure.

Theorem 3.10. Letting S_p be the same poset defined above, $S_p \cong \overrightarrow{FG}(T_1) \times \overrightarrow{FG}(T_2)$.

Proof. There is a natural bijection φ from S_p to $\overrightarrow{FG}(T_1) \times \overrightarrow{FG}(T_2)$. For $\mathcal{F}' \in S_p$ and $i \in \{1, 2\}$, we can remove all arcs in $\mathcal{F}'$ containing any vertices in $R_j \backslash p$ and then remove vertices and contract edges according to the construction of T_i. We claim that the remaining collection of arcs, $\mathcal{F}'_i$, is in $\overrightarrow{FG}(T_i)$. To see that $\mathcal{F}'_i$ is a facet of $\widetilde{\Delta}^{NC}(T_i)$, note that by construction the total rank of T_1 and T_2 is the rank of T, and since we split any $\mathcal{F}'$ into $\mathcal{F}'_1$ and $\mathcal{F}'_2$ such that the total number of arcs in both of these facets is the number of arcs in $\mathcal{F}$ minus one (p becomes a boundary arc in both facets), we cannot add any noncrossing arc to $\mathcal{F}'_i$. This is because every member of $V^o(T)$ has a unique image in exactly one T_i, and so the total number of arcs in any facet of $\widetilde{\Delta}^{NC}(T_1)$ and any facet of $\widetilde{\Delta}^{NC}(T_2)$ sums to $|V^o(T)| - 2 = |\mathcal{F}| - 1$.

We construct $\mathcal{F}'_i$ for $i = 1$ and $i = 2$, thus producing a pair $(\mathcal{F}'_1, \mathcal{F}'_2) \in \overrightarrow{FG}(T_1) \times \overrightarrow{FG}(T_2)$, so that φ is well-defined. It is clear that φ is injective, and surjectivity follows from the fact that a pair of facets, one in $\overrightarrow{FG}(T_1)$ and the other in $\overrightarrow{FG}(T_2)$, determine an element of $\overrightarrow{FG}(T)$ by taking the union of the image in T of each arc in $\mathcal{F}_i$ for $i = 1, 2$ and then adding in p—this operation is clearly injective, since if $\mathcal{F}_1$ contains an arc that $\mathcal{F}'_1 \in \overrightarrow{FG}(T_1)$ does not, any element of $\overrightarrow{FG}(T)$ corresponding to $\mathcal{F}'_1$ will also lack that arc.

We will first show that φ^{-1} is order-preserving; it suffices to show that φ^{-1} preserves the markings of every arc in $\mathcal{F}'_i$, since if every marking of $\mathcal{F}'_i$ is preserved under φ^{-1}, if a facet $\mathcal{F}''_i \in \overrightarrow{FG}(T_i)$ is obtained from $\mathcal{F}'_i$ by a flip at arc t, then $\mathcal{F}''$ can be obtained from $\mathcal{F}'$ by a flip at the arc in $\overrightarrow{FG}(T)$ corresponding to t. Observe that the image of any arc t in some $\mathcal{F}'_i$ under φ^{-1} can never be marked at a corner that p contains in R_i, since the region cut out by p containing the corresponding face is always larger. Thus, the markings of the image of t depend only on the other arcs in $\mathcal{F}'_i$, and φ^{-1} does not alter these markings, so the result follows.

The proof that φ is order-preserving is more difficult.

The bijection φ naturally induces a surjection $\varphi_1 : \overrightarrow{FG}(T) \to \overrightarrow{FG}(T_1)$. Suppose by contradiction that we have some $\mathcal{F} <_{\overrightarrow{FG}(T)} \mathcal{F}'$ but $\mathcal{F}_1 := \varphi_1(\mathcal{F}) \nless_{\overrightarrow{FG}(T_1)} \mathcal{F}'_1 := \varphi_1(\mathcal{F}')$. Recall the injective lattice map ϕ from $\overrightarrow{FG}(T)$ to $\mathrm{Bic}(T)$. We have that $\phi(\mathcal{F}_1) \nsubseteq \phi(\mathcal{F}'_1)$ in $\mathrm{Bic}(T_1)$ and $\phi(\mathcal{F}) \subseteq \phi(\mathcal{F}')$ in $\mathrm{Bic}(T)$. Since in this proof we will frequently be switching between T and T_1, if s is a segment of T_1, we write it as s^1, and its image in T under the process of obtaining T from T_1 by s, similarly with arcs.

Since $\phi(\mathcal{F}_1) \nsubseteq \phi(\mathcal{F}'_1)$, there exists $s^1 \in \phi(\mathcal{F}_1)\backslash\phi(\mathcal{F}'_1)$. The first claim is that s^1 must overlap p^1 along a segment s^1_p in T_1. If s^1 does not overlap p^1 at all, then s does not overlap p at all, and thus when we pass from $\mathcal{F}$ to $\mathcal{F}'$, we cannot obtain s, since no segment from C_p or C_q for $q \in R_2$ can be composed with other segments to form s. Suppose that s^1 just overlaps with p^1 at a vertex v^1. Since $s^1 \notin \phi(\mathcal{F}_1)$, there is no composition of segments in C_{q^1} for $q^1 \in \mathcal{F}'_1$ that can form s^1. Note that v is an interior vertex in T, and hence s only overlaps with p at v in T. It follows that no segment in C_p or in C_q for $q \in R_2$ can be used in a composition of s, since all segments in s are not subsegments of p. Thus, s^1 overlaps p^1 along a segment $s^1_p = [u^1_1, u^1_2]$ contained in p^1. Note that s_p is a segment contained in p. We can assume that $s^1 \in C_{q^1}$ for $q^1 \in \mathcal{F}_1$, since if we show that all such segments are contained in some C_{r^1} for $r^1 \in \mathcal{F}'_1$, then any composition of such segments is as well.

Since $s^1 \notin \phi(\mathcal{F}'_1)$, there must be some $t = [v_1, v_2] \in C_p$ or C_q for $q \in R_2$ that is part of a decomposition of s, specifically of s_p, in $\phi(\mathcal{F}')$. Let q be any such arc containing t. Note that since $t \in C_q$, orienting q from v_1 to v_2 (without loss of generality) must turn right at v_1 and left at v_2. We claim that one of v_1 or v_2 is a corner of p on the R_1 side. If $q = p$, then since p must turn a different direction at v_1 than at v_2, one of these turns implies that v_i is part of a corner (v_i, F) of p in R_1. If $q \in R_2$ and by contradiction neither v_1 or v_2 is a part of corner of q in R_1, then both are parts of corners of q in R_2. However, this implies that q turns at the same direction at v_1 and v_2, which is a contradiction—thus, the claim is true. Now since one endpoint of t is v_i such that (v_i, F) is a corner of p in R_1 and v^1_i is not an interior vertex of T_1, the segment s_p cannot have v_i as an endpoint, since both of its endpoints are vertices of T that are also interior vertices of T_1. Thus, we must have some other segment $t' = [v_i, v_3]$ in some $C_{q'}$, where now q' is any arc in $\mathcal{F}'$. Note that since t' only agrees with t at v_i, q' must turn a different direction than q at v_i, since if it turns the same direction, the turn at v_i is always either an initial left turn, or a terminal right turn, neither of which allows t' to be in $C_{q'}$. It follows that $q' \in R_2$, since (v_i, F) is a corner in R_1, and the corner that q' contains at v_i is a corner in R_2. Hence, the other corner of q', (v_3, F'), cannot be in R_2, and is thus a corner of p in R_1. But now v_3 cannot be an endpoint of s_p either, since v^1_3 is not an interior vertex in T_1. It is clear that this process will continue indefinitely, a contradiction. □

Corollary 3.11. The poset S_p a lattice.

Corollary 3.12. Let p be an arc in some tree T. If there exists a facet $\mathcal{F} \in \overrightarrow{FG}(T)$ such that p is green in $\mathcal{F}$ and all other arcs are red, then $\mathcal{F}$ is the unique facet of $\widetilde{\Delta}^{NC}(T)$ with this property.

Proof. Let $\mathcal{F}$ be such a facet. As described in the proof of Theorem 3.10, $\mathcal{F}$ defines facets $\mathcal{F}_1 \in \overrightarrow{FG}(T_1)$ and $\mathcal{F}_2 \in \overrightarrow{FG}(T_2)$. Moreover, the proof of Theorem 3.10 also implies that $\mathcal{F}_i = \hat{1}_{\overrightarrow{FG}(T_i)}$, since the color of every arc is preserved. If there exists another $\mathcal{F}' \in \overrightarrow{FG}(T)$ containing p as a green arc and the rest red, then $\mathcal{F}'_i \neq \mathcal{F}_i$ for some $i \in \{1, 2\}$. This implies that $\overrightarrow{FG}(T_i)$ has two distinct maximal elements, which contradicts the fact that $\overrightarrow{FG}(T_i)$ is a lattice. Thus, $\mathcal{F}$ is unique.

□

The following result is more general, and addresses how arcs can turn from red to green in a facet of $\overrightarrow{FG}(T)$, and is important in light of the result that will follow it.

Proposition 3.13. Let $\mathcal{F} \in \overrightarrow{FG}(T)$. If $p = [u_1, u_2]$ is a green arc in $\mathcal{F}$, then p cannot change color due to a sequence of flips that do not involve p.

Proof. Suppose by contradiction that this can occur. Then there must be a facet $\mathcal{F}''$ in which p is green, and it must be separated by a single flip from facet $\mathcal{F}'$, in which p is red. We can assume without loss of generality that $\mathcal{F}'' = \mathcal{F}$. Suppose that p is marked at (v_1, F_1) and (v_2, F_2). We will use the Figure 3 (a) throughout the proof as a reference:

In order to turn p red, we need to change at least one marking of p. It is easy to see that our flip cannot change both markings. In order to do this, we p must no longer be marked at these corners, which it will be unless a new arc in $\mathcal{F}'$ inherits these markings. Thus, the arc that we obtain after the flip to reach $\mathcal{F}'$ is marked at both (v_1, F_1) and (v_2, F_2), making it green, a contradiction, since such an arc is always red.

Now suppose that p can be made red by changing one of its markings. Assume without loss of generality that we change the marking (v_1, F_1). In order to do this, the new arc $t' = [w_1, w_2]$ that we obtain in $\mathcal{F}'$ after flipping

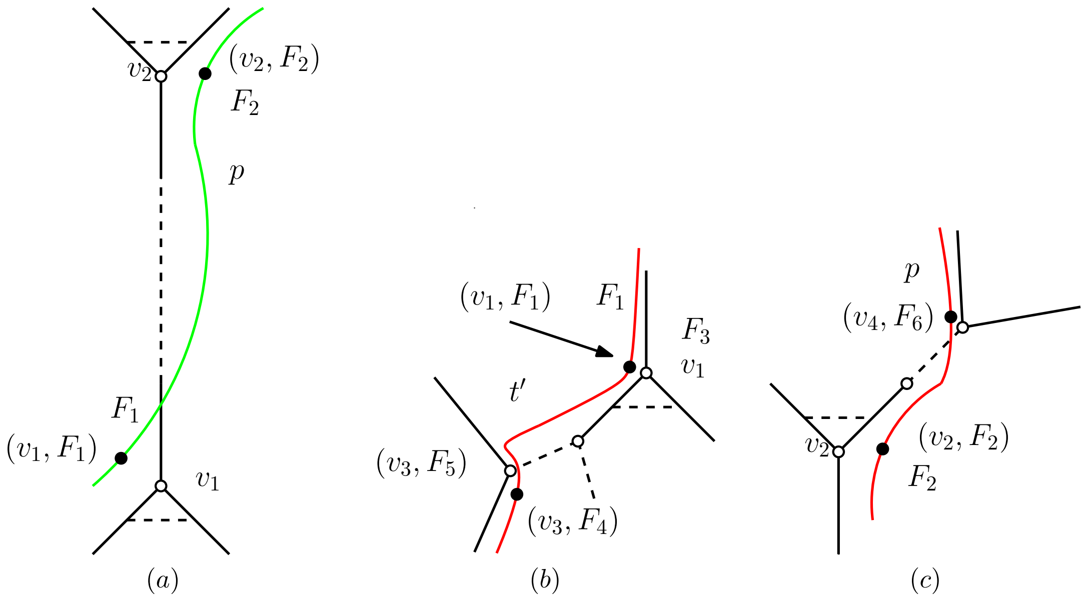


FIGURE 3.

an arc $t = [u_3, u_4]$ in $\mathcal{F}$ must be marked at (v_1, F_1). Since t' is red in $\mathcal{F}'$, the fact that it is marked at (v_1, F_1) implies that the other marking of t' must lie on $[w_1, v_1]$, where w_1 is the endpoint of t' whose path to v_2 passes through v_1; see Figure 3 (b). Call this other marking of t' (v_3, F_4). Note that t is marked at (v_3, F_5) and (v_1, F_3), where F_5 is the face paired with F_4 during the flip.

The key observation now is that in order for p to become red, its new marking in $\mathcal{F}'$, (v_4, F_6), must lie on $[v_2, u_2]$; see Figure 3 (c). However, p is not marked at (v_4, F_6), although it contains the corner. Since the only change from $\mathcal{F}$ to $\mathcal{F}'$ is the position of t, it follows that no arc except t can be marked at (v_4, F_6), since if some other arc r were marked there, p would not be marked there in $\mathcal{F}'$, since r does not change from $\mathcal{F}$ to $\mathcal{F}'$. Thus, t is marked at (v_4, F_6), which contradicts the fact that one of its markings, (v_3, F_5), must lie on $[w_1, v_1]$, which is disjoint from $[v_2, u_2]$. It follows that p cannot be turned red with a flip.

□

We now come to a result that follows from the above theorem and Theorem 3.10 that discusses when an arc cannot be contained in certain facets of $\overrightarrow{FG}(T)$.

Theorem 3.14. Let $\mathcal{F} \in \overrightarrow{FG}(T)$ and p a green arc in $\mathcal{F}$. Let $\mathcal{F}'$ be the facet obtained by flipping p in $\mathcal{F}$. Then $[\mathcal{F}', \hat{1}] \cap S_p = \varnothing$.

Proof. By Theorem 3.10 if $\mathcal{F}' < \mathcal{F}''$ for some $\mathcal{F}'' \in S$ where p is red, then $\mathcal{F}''$ can be reached from $\mathcal{F}$ by a sequence of flips not involving p. This contradicts the result of Proposition 3.13, since p is green in $\mathcal{F}$ and red in $\mathcal{F}''$.

□

Corollary 3.15. For a fixed arc p, S_p is a sublattice of $\overrightarrow{FG}(T)$.

Proof. It suffices to show that $S_p = [\hat{0}_{S_p}, \hat{1}_{S_p}]$, where this is an interval in $\overrightarrow{FG}(T)$. First, it is clear that S_p is contained in this interval. Suppose by contradiction that we have some facet $\mathcal{F}$ that is larger than $\hat{0}_{S_p}$ and smaller than $\hat{1}_{S_p}$ but is not in S_p; i.e., $\mathcal{F}$ does not contain p. Since $\mathcal{F} > \hat{0}_{S_p}$, we can reach $\mathcal{F}$ from $\hat{0}_{S_p}$ via a sequence of flips, and since $p \in \hat{0}_{S_p}$, this sequence of flips must involve a flip at p. However, we can also reach $\hat{1}_{S_p}$ from $\mathcal{F}$ via a sequence of flips, which is a contradiction, since we have already flipped p by the time we reach $\mathcal{F}$ from $\hat{0}_{S_p}$, contradicting Theorem 3.14, as $\hat{1}_{S_p} \in S_p$.

□

The result of this theorem can be used alongside the results of the past two subsections to prove that the lattice $\mathrm{NCP}(T)$ is graded, as follows.

Theorem 3.16. The lattice $\mathrm{NCP}(T)$ is graded of rank $\#V^o(T) - 1$.

Proof. Let $\mathbf{B}, \mathbf{B}' \in \mathrm{NCP}(T)$, and $n = \#V^o(T)$. We will show that if we can merge m blocks of $\mathbf{B}$, where $3 \leqslant m \leqslant n$ to obtain $\mathbf{B}'$, then we can merge $m-1$ of these blocks to obtain an element $\mathbf{B}'' \in \mathrm{NCP}(T)$. Let B' be the union of m blocks of $\mathbf{B}$. By Theorem 3.4, we can assume without loss of generality that $m = n$, since if this equality does not hold, we can consider $\mathbf{B}$ as an element $\mathrm{NCP}(T_{B'})$.

For a facet $\mathcal{F} \in \overrightarrow{FG}(T)$, define $I_{\mathcal{F}} := [\bigwedge_{i=1}^{m}\{\mathcal{F}_i \in \overrightarrow{FG}(T) \mid (\mathcal{F}_i, \mathcal{F}) \in \mathrm{Cov}(\overrightarrow{FG}(T))\}$. Now let $\mathcal{F} \in \overrightarrow{FG}(T)$ be the unique facet corresponding to $\mathbf{B}$. By Theorem 2.8, this facet satisfies $\lambda_{\downarrow}(\mathcal{F}) = \mathrm{Seg}(\mathbf{B})$ and $\psi(\mathcal{F}) = \overline{\mathrm{Seg}(\mathbf{B})}$. Recall that $\psi(\mathcal{F})$ is the set of all labels appearing in $I_{\mathcal{F}}$. We will construct a facet $\mathcal{F}' \in \overrightarrow{FG}(T)$ where $I_{\mathcal{F}} \subset I_{\mathcal{F}'}$ and $\#\lambda_{\downarrow}(\mathcal{F}') = n-2$.

By assumption, $\mathcal{F}$ contains at least two green arcs, p and q. By Corollary 3.15, S_p is a sublattice of $\overrightarrow{FG}(T)$, and hence $\bigvee_{\overrightarrow{FG}(T)} S_p = \hat{1}_{S_p} \in S_p$; from now on all meets and joins involving S_p are taken in $\overrightarrow{FG}(T)$ unless otherwise specified. We claim that $\bigvee S_p$ is our desired facet. By Theorem 3.14, p is green in $\bigvee S_p$, since $\mathcal{F} \leqslant \bigvee S_p$, and $p \in \bigvee S_p$. This implies that $\bigvee S_p$ is the unique facet in $\overrightarrow{FG}(T)$ with p green and all other arcs red, since if $\bigvee S_p$ contained any other green arcs, it would be strictly less than another element of S. Hence, we know that $\#\lambda_{\downarrow}(\bigvee S_p) = n-2$, as desired.

Observe now that every element that $\bigvee S_p$ covers in $\overrightarrow{FG}(T)$ is contained in S_p. Moreover, $\bigwedge_{i=1}^{\ell}\{\mathcal{F}_i \in \overrightarrow{FG}(T) \mid (\mathcal{F}_i, \bigvee S_p) \in \mathrm{Cov}(\overrightarrow{FG}(T))\} = \bigwedge S_p$, since

$$\{\mathcal{F}_i \in \overrightarrow{FG}(T) \mid (\mathcal{F}_i, \bigvee S_p) \in \mathrm{Cov}(\overrightarrow{FG}(T))\} = \varphi^{-1}(S_{T_1}), S_{T_1} := \{(\mathcal{F}'_i, \hat{1}_{\overrightarrow{FG}(T_2)}), \mathcal{F}'_i \in \overrightarrow{FG}(T_1) \mid (\mathcal{F}'_i, \hat{1}_{\overrightarrow{FG}(T_1)}) \in \mathrm{Cov}(\overrightarrow{FG}(T_1))\} \sqcup \varphi^{-1}(S_{T_2}), S_{T_2} := \{(\hat{1}_{\overrightarrow{FG}(T_1)}, \mathcal{F}''_i), \mathcal{F}''_i \in \overrightarrow{FG}(T_2) \mid (\mathcal{F}''_i, \hat{1}_{\overrightarrow{FG}(T_2)}) \in \mathrm{Cov}(\overrightarrow{FG}(T_2))\},$$

and hence $\bigwedge_{i=1}^{\ell}\{\mathcal{F}_i \in \overrightarrow{FG}(T) \mid (\mathcal{F}_i, \bigvee S) \in \mathrm{Cov}(\overrightarrow{FG}(T))\} = \bigwedge(\varphi^{-1}(S_{T_1}) \cup \varphi^{-1}(S_{T_2})) = \varphi^{-1}(\bigwedge(\pi_1(S_{T_1}) \cup \pi_1(S_{T_2})), \bigwedge(\pi_2(S_{T_1}) \cup \pi_2(S_{T_2}))) = \varphi^{-1}(\hat{0}_{\overrightarrow{FG}(T_1)}, \hat{0}_{\overrightarrow{FG}(T_2)}) = \bigwedge S_p$, where π_i is the canonical projection; the fact that the meet in $\overrightarrow{FG}(T_i)$ of all elements covered by $\hat{1}_{\overrightarrow{FG}(T_i)}$ is $\hat{0}_{\overrightarrow{FG}(T_i)}$ follows from the fact that $\overrightarrow{FG}(T_i)$ is a CU-lattice.

Hence, we have that $S_p = [\bigwedge S_p, \bigvee S_p] = I_{\bigvee S_p}$. To finish the proof, it suffices to show that $I_{\mathcal{F}} \subseteq S_p$. Since p is green in $\mathcal{F}$, all of the facets that $\mathcal{F}$ covers in $\overrightarrow{FG}(T)$ are in S_p, and therefore their meet is in S_p. Thus, if any member of $I_{\mathcal{F}}$ is not in S_p, then p must be flipped in some facet strictly between the minimal element of $I_{\mathcal{F}}$ and $\mathcal{F}$, which by Theorem 3.14 cannot occur, since $p \in \mathcal{F}$.

Now if $\mathbf{B} \lessdot \mathbf{B}'$ in $\mathrm{NCP}(T)$, we must combine two blocks of $\mathbf{B}$ to obtain $\mathbf{B}'$, or else by the above result there is at least one element strictly between $\mathbf{B}$ and $\mathbf{B}'$. The theorem follows. □

3.3. **Further lattice properties.** In this subsection, we show that, unlike the classical noncrossing partition lattice $\mathrm{NC}(n)$, in general the lattice $\mathrm{NCP}(T)$ is not self-dual. We investigate a bijection defined on $\mathrm{NCP}(T)$ in [3] called the Kreweras Complement and show that $\mathrm{NCP}(T)$ is rank-symmetric.

Definition 3.17. For a tree T, a **red-green tree** $\mathcal{T}$ is a collection of pairwise noncrossing colored segments such that every pair of vertices in $V^o(T)$ is connected by a sequence of curves in $\mathcal{T}$. The segments in $\mathcal{T}$ are allowed to be red or green.

Theorem 3.18. Let $\mathbf{B}$ be a noncrossing tree partition. There exists a unique red-green tree $\mathcal{T}$ whose set of red segments is $\mathrm{Seg}(\mathbf{B})$.

See [3] for the proof of this result.

Definition 3.19. Let $\mathbf{B}$ be a noncrossing tree partition. Let $\mathcal{T}$ be the tree corresponding to $\mathbf{B}$ as discussed in the above theorem. Garver and McConville show in [3] that the green segments in $\mathcal{T}$ define a new noncrossing tree partition of T. We call this partition the **Kreweras Complement** of $\mathbf{B}$, and denote it by $\mathrm{Kr}(\mathbf{B})$.

Theorem 3.20. [3] The map $\mathrm{Kr} : \mathrm{NCP}(T) \to \mathrm{NCP}(T)$ is a bijection.

The following result discusses what we can deduce about $\mathrm{Kr}(\mathbf{B})$ given what we know about $\mathbf{B}$.

Theorem 3.21. If $\mathbf{B} \in \mathrm{NCP}(T)$ has m blocks, then $\mathrm{Kr}(\mathbf{B})$ has $\#V^o(T) - m + 1$ blocks.

Proof. In order to obtain a red-green tree, there must be a path consisting of green segments from every block of $\mathbf{B}$ to every other block. For a block $B \in \mathbf{B}$, if a vertex in B is connected by a segment in $\mathrm{Seg}(\mathrm{Kr}(\mathbf{B}))$ to another block of $\mathbf{B}$, then we call this vertex a **connecting vertex**. Otherwise, we call it an **isolated vertex**. Observe that every isolated vertex in a block of $\mathbf{B}$ corresponds to a singleton block in $\mathrm{Kr}(\mathbf{B})$.

We claim that a maximal sequence of green segments $[w_1, w_2], [w_2, w_3], \ldots, [w_{n-1}, w_n]$, where w_i is a connecting vertex, defines a block of $\mathrm{Kr}(\mathbf{B})$, and that the two types of blocks of $\mathrm{Kr}(\mathbf{B})$ that we have described make up all of its blocks. It is clear that the above sequence is at least contained in a block of $\mathrm{Kr}(\mathbf{B})$. To see that in fact it is the entire block, note that if this block contained any more elements, they would be other connecting vertices, contradicting the maximality of our construction. The fact that the two types of blocks previously described encompass all of the blocks of $\mathrm{Kr}(\mathbf{B})$ is clear. Note that in the above sequence of green segments, for $i \neq j$, w_i and w_j cannot be in the same block of $\mathbf{B}$, or else $\mathcal{T}$ contains a cycle. Thus, it suffices to show that the number of isolated vertices and maximal sequences of connecting vertices, call this sum $S(\mathbf{B})$, is always $\#V^o(T) - m + 1$.

By the above remarks, the blocks of $\mathbf{B}$ form the vertices of a tree, where each edge in this tree is given by a segment of a block in $\mathrm{Kr}(\mathbf{B})$. Hence, there are $m - 1$ such segments. Note that the upper bound for the number of blocks of $\mathrm{Kr}(\mathbf{B})$ is $\#V^o(T)$. We claim that every one of the green segments in $\mathcal{T}$ between blocks of $\mathbf{B}$ reduces the numbers of blocks in $\mathrm{Kr}(\mathbf{B})$ by 1. If we add a green segment s between two vertices u and v, we have three cases:

(1) u and v are connecting vertices prior to adding s. In this case, the number of blocks in $\mathrm{Kr}(\mathbf{B})$ decreases by 1, since we combine two separate blocks of sequences of green segments;
(2) (without loss of generality) v is a connecting vertex prior to adding s, but u is not. In this case, we lose 1 block of $\mathrm{Kr}(\mathbf{B})$, since u is no longer an isolated vertex;
(3) neither u nor v is a connecting vertex prior to adding s. In this case, we lose two blocks of $\mathrm{Kr}(\mathbf{B})$, since both u and v are no longer isolated vertices, but gain a block of $\mathrm{Kr}(\mathbf{B})$, since we now have the beginning of a sequence of green segments.

The result follows.

□

In light of this result, we see that the noncrossing tree partition lattice has the following additional structure:

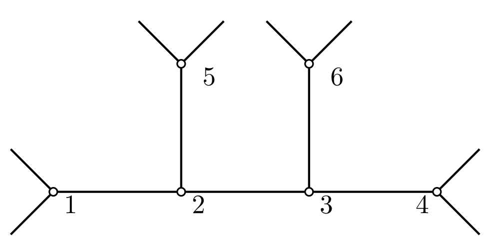
Corollary 3.22. The noncrossing tree partition lattice is rank-symmetric.

The name "Kreweras Complement" comes from a similar bijection defined on the $\mathrm{NC}(n)$ lattice, which has the important property of being order-reversing. In fact, it is modified in [5] to prove that the $\mathrm{NC}(n)$ lattice is self-dual. We now show that in general there does not exist an order-reversing bijection on the noncrossing tree partition lattice. In particular, unlike this classical bijection, the Kreweras Complement is not order-reversing.

Theorem 3.23. In general, there does not exist an ordering-reversing bijection on the noncrossing partition lattice.

Notation 3.24. When we write a noncrossing tree partition $\mathbf{B}$, we only write the blocks of $\mathbf{B}$ that are not singletons.

Proof. We employ the following tree as a counterexample:



Suppose by contradiction that there exists some order-reversing bijection on $\mathrm{NCP}(T)$. Consider $\mathbf{B} = \{1,2,5\}\{3,4\} \in \mathrm{NCP}(T)$. Note that $\mathbf{B}$ covers four elements of $\mathrm{NCP}(T)$: $\{1,2,5\}, \{1,5\}\{3,4\}, \{2,5\}\{3,4\}$, and $\{1,2\}\{3,4\}$. Moreover, $\mathbf{B}$ is covered by three elements: $\{1,2,5,6\}\{3,4\}, \{1,2,3,4,5\}$, and $\{1,2,5\}\{3,4,6\}$. In order for an order-reversing bijection to exist, $\mathrm{NCP}(T)$ must contain an element that covers three elements and is covered by four elements. We claim there does not exist such a partition. First, it is trivial that no element consisting of five blocks can satisfy this property—these are the atoms. Similarly, no elements consisting of two blocks can satisfy this property. Moreover, this element cannot contain three blocks either: for a partition three blocks, there are a maximum of three elements covering it.

We now explain why no four-block element of $\mathrm{NCP}(T)$ can satisfy the desired properties. Note that we have two types of four-block elements, either two pairs of two vertices and the rest singletons, or a triple of vertices and the rest singletons. Our element cannot be of the first type, as such partitions cover exactly two elements. By symmetry, it suffices to check that none of $\{1,2,3\}, \{1,2,5\}, \{1,2,6\}, \{1,2,4\}, \{2,3,5\}, \{1,3,4\}$, and $\{1,3,6\}$ are covered by four elements, which is a straight-forward computation. □

Corollary 3.25. In general, the noncrossing tree partition lattice is not self-dual.

The fact that the noncrossing tree partition lattice is not self-dual marks a significant difference from the classical noncrossing partition lattice. However, we conjecture that the following structural similarity does hold:

Conjecture 3.26. The noncrossing tree partition lattice is rank-unimodal.

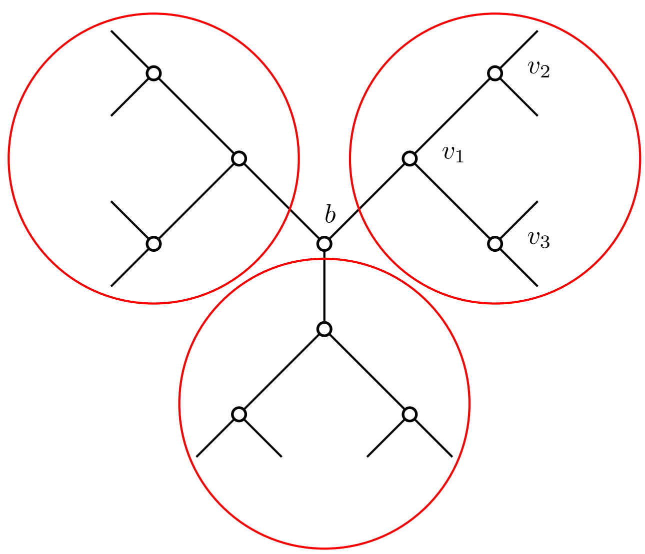
4. Shellability

In [1], the author shows that the lattice $\mathrm{NC}(n)$ is EL-shellable. In this short section, we discuss the way in which a similar labeling system to that employed in [1] fails in general for the noncrossing tree partition lattice.

Definition 4.1. For a tree T with $\#V^o(T) = n$, assume that we label the interior vertices with $\{1, \ldots, n\}$, with each number appearing as only one label. We can define a labeling of $\mathrm{NCP}(T)$, where if $(\mathbf{B}, \mathbf{B}') \in \mathrm{Cov}(\mathrm{NCP}(T))$, then $\lambda(\mathbf{B}, \mathbf{B}') = \max(\min B_1, \min B_2)$, where B_1 and B_2 are the blocks of $\mathbf{B}$ that are merged to obtain $\mathbf{B}'$. We call this labeling an **ordered vertex labeling**. Note that for $\mathrm{NC}(n)$, this same function λ is an EL-labeling.

Proposition 4.2. In general, there does not exist an ordered vertex labeling of the noncrossing tree partition lattice that is an EL-labeling.

Proof. We employ the following tree as a counterexample:



The three circled subgraphs in our picture are called the **outer triples** of T. In a given outer triple, the lone vertex with three edges to other interior vertices of T is called the **center vertex** of the triple, and the other two vertices in the triple are called the **edge vertices** of the triple. We call the very center vertex the **base vertex** of the tree, denoted by b. Note that the center vertex of one outer triple must receive a label that is greater than or equal to 3; denote this vertex by v_1, its triple by G, and the edge vertices of G by v_2 and v_3.

More generally, let $[\mathbf{B}, \mathbf{B}']$ be an interval in NCP(T). Note that any maximal chain in this interval is also a maximal chain in the lattice of set partitions of $[n]$, $\Pi(n)$, since every covering relation in NCP(T) is given by combining two blocks. It is known that λ gives an EL-labeling for $\Pi(n)$, and so there exists a unique increasing lexicographically-minimal maximal chain in $[\mathbf{B}, \mathbf{B}'] \subseteq \Pi(n)$, and this must be the same chain in NCP(T). Moreover, this maximal chain is given by combining two blocks that yield the smallest possible covering label under λ at every step. Denote the smallest possible covering relation label in $[\mathbf{B}, \mathbf{B}']$ involving $\mathbf{B}$ by $\pi(\mathbf{B}, \mathbf{B}')$. Hence, if we can show that in every case, there is such an interval where the smallest possible covering label under λ can never be achieved by an element covering $\mathbf{B}$ in NCP(T), then we are done.

Suppose by contradiction that some ordered vertex labeling of NCP(T) is an EL-labeling. Denote the label of a vertex $v \in V^o(T)$ by $\ell(v)$. Note that one edge vertex in G must have a label greater than that of v_1; if both of their labels are smaller, then $\mathbf{B} = \{v_2, b\}\{v_3\}$ and $\mathbf{B}' = \{v_1, v_2, v_3, b\}$ are both in NCP(T) with $\mathbf{B} < \mathbf{B}'$. We have that $\pi(\mathbf{B}, \mathbf{B}')$ is achieved by merging $\{v_2, b\}$ and $\{v_3\}$. However, the resulting set partition is not in NCP(T), so that we cannot have an increasing chain from $\mathbf{B}$ to $\mathbf{B}'$.

Suppose first that $\ell(v_1) > 3$. Observe that two of $\{1, 2, 3\}$ must be labels of vertices outside of G; call the two vertices with these labels w_1 and w_2 (there may be three of such vertices—in that case just pick the ones with labels 1 and 2). Let p_1 be the path in T from w_1 to v_2, and p_2 the path in T from w_2 to v_3 (including the endpoints). Then $\mathbf{B} = \{v_2, w_1\}\{v_3, w_2\}$ and $\mathbf{B}' = \{V^o(p_1), V^o(p_2)\}$; we remark that the roles of w_1 and w_2 may have to be interchanged in order to ensure that $\mathbf{B} \in$ NCP(T)— the important thing to note is that no matter where w_1 and w_2 are located in T, (at least) one of $\{v_2, w_1\}\{v_3, w_2\}$ and $\{v_2, w_2\}\{v_3, w_1\}$ is in NCP(T), and we assume without loss of generality that the configuration given above is in NCP(T). Thus, we have that $\mathbf{B}$ and $\mathbf{B}'$ are in NCP(T), with $\mathbf{B} < \mathbf{B}'$. Moreover, $\pi(\mathbf{B}, \mathbf{B}')$ is given by merging $\{v_2, w_1\}$ and $\{v_3, w_2\}$ in $\mathbf{B}$, which gives a set partition that is not in NCP(T).

Hence, we can assume that $\ell(v_1) = 3$. If $\ell(v_2), \ell(v_3) > 3$, then the identical argument used in the paragraph above can be used to show that there exist $\mathbf{B}, \mathbf{B}' \in$ NCP(T) with $\mathbf{B} < \mathbf{B}'$ and no increasing maximal chain in $[\mathbf{B}, \mathbf{B}'] \subseteq$ NCP(T). Therefore, assume further that $\ell(v_2) \in \{1, 2\}$. However, this means that a center vertex of one of the two other outer triples has a label strictly greater than 3, so that again the same argument from the above paragraph carries over to provide the same contradiction. It follows that our ordered vertex labeling cannot be an EL-labeling.

□

5. A CU-labeling of Bic(T)

In [3], the authors prove that $Bic(T)$ is congruence-uniform. Thus it admits a CU-labeling. In this section, we will explicitly construct such a labeling.

Definition 5.1. A lattice is *congruence-uniform* if it can be constructed from the element lattice by a finite sequence of interval doublings. Alternatively, the set of its covering relations permits a CU-labeling.

Definition 5.2. For a tree T, segments s_1, s_2 are composable if $s_1 \circ s_2 \in Seg(T)$. A subset $B \subset T$ is closed if for $s_1, s_2 \in B, s_1 \circ s_2 \in B$. B is biclosed if both B and its complement, B^C, are closed.

Definition 5.3. We say a segment s is a *split* of segment t if s is contained in t, and s and t share exactly one endpoint.
A *break* of a segment $[a, c]$ is a pair of splits $([a, b], [b, c])$, of $[a, c]$ for some vertex b of segment $[a, c]$ lying between a and c. We say that b is the *faultline* of the break $([a, b], [b, c])$.

Definition 5.4. Define the poset P whose elements are of the form $s_{\{f_1, f_2, \ldots, f_m\}}$, where s is a segment of T with m breaks and each f_i is a split of s with no two $f_i' s$ from the same break. The ordering is given by $[a, c]_{\{s_1, s_2, \cdots, s_k\}} \geqslant [d, e]_{\{m_1, m_2, \cdots, m_l\}}$ if $[a, c]$ contains $[d, e]$.

Theorem 5.5. Bic(T) has a CU-labeling, $\widetilde{\lambda} : Cov(Bic(T)) \to (Seg(T) \times 2^{Seg(T)})$, given by: $\widetilde{\lambda}(S \lessdot (S \cup \{[a, c]\})) = [a, c]_{\{s_1, s_2, \cdots, s_k\}}$ where $s_1, s_2, \cdots, s_k$ are the splits of $[a, c]$ which are contained in S.

Example 5.6. Consider the tree $T = P_4$, consisting of a path of 4 vertices numbered 1,2,3,4 in order. Consider the biclosed sets $B = \{[1,2],[1,3]\}$ and $C = \{[1,2],[1,3],[1,4]\}$.

Then $\widetilde{\lambda}(B \lessdot C) = [1,4]_{\{[1,2],[1,3]\}}$.

Proof. of Theorem 5.5

We will verify axioms (CN1)-(CN3), as well as (CU1) and (CU2) [3]. For convenience, we will let $\lambda : Cov(Bic(T)) \to Seg(T)$ denote the first coordinate function of $\widetilde{\lambda}$. Note that $\lambda(B \lessdot B \cup \{s\}) = s$.

(CN1): Given C and D which both cover B where $C = B \cup [a,c]$ and $D = B \cup [d,e]$. The biclosed set $C \vee D \supset \{[a,c],[d,e]\}$ Say $C \vee D$ covers $C' \geqslant C$ and $D' \geqslant D$. If $C' \supset \{[d,e]\}$, then $C' \geqslant C \geqslant B$ so $C' \geqslant D$ meaning that $C \vee D$ is not the least upper bound of C and D, a contradiction. So C' does not contain $[d,e]$ and similarly, D' does not contain $[a,c]$. Thus $\widetilde{\lambda}(C' \lessdot C \vee D)$ is $[d,e]$ with some subscript, while $\widetilde{\lambda}(D' \lessdot C \vee D)$ is $[a,c]$ with some subscript.

The label $\widetilde{\lambda}(B \lessdot C)$ is $[a,c]_{s_1,s_2,\cdots,s_k}$ and $\widetilde{\lambda}(B \lessdot D)$ is $[d,e]_{m_1,m_2,\cdots,m_l}$. Thus C contains $s_1, s_2, \cdots, s_k$, so C' does as well. The set D contains $m_1, m_2, \cdots, m_l$, so D' does as well. Thus the subscripts for$\widetilde{\lambda}(C' \lessdot C \vee D)$ are $s_1, s_2, \cdots, s_k$ and the subscripts for $\widetilde{\lambda}(D' \lessdot C \vee D)$ are $m_1, m_2, \cdots, m_l$. So $\widetilde{\lambda}(C' \lessdot C \vee D) = \widetilde{\lambda}(B \lessdot D)$ while $\widetilde{\lambda}(D' \lessdot C \vee D) = \widetilde{\lambda}(B \lessdot C)$.

(CN2): Let C and D cover B and both $C' > C$ and $D' > D$ be covered by $C \vee D$. Labels of the form $\widetilde{\lambda}(G \lessdot H)$ where $C \leqslant G, H \leqslant C'$ or $D \leqslant G, H \leqslant D'$ consist of elements of $C \vee D$ which are not elements of B, with some subscript. $C \vee D$ is the closure of $C \cup D$. It consists of four types of segments:

(1) segments in B
(2) a composition of $[a,c]$ with segments in B, which is necessarily in C.
(3) a composition of $[d,e]$ with segments in B, which is necessarily in D.
(4) a composition of both $[a,c]$, $[d,e]$ and perhaps some segments in B.

A segment of type 1 is in B and thus cannot be $\lambda(G \lessdot H)$. A segment of type 2 cannot be used as a label between C and C' since it is contained in C. It cannot be $\lambda(G \lessdot H)$ where $D \leqslant G \lessdot H \leqslant D'$ because it is a segment including $[a,c]$ and thus not included in D'. A segment of type of type 3 cannot be used as a label between D and D' since it is contained in D. It cannot be $\lambda(G \lessdot H)$ where $C \leqslant G \lessdot H \leqslant C'$ because it is a segment including $[d,e]$ and thus not included in C'. Hence any label $\widetilde{\lambda}(G \lessdot H)$ is a segment of type 4) with some subscript.

A segment of type 4) contains both $[a,c]$ and $[d,e]$ so $\widetilde{\lambda}(G \lessdot H) \geqslant \widetilde{\lambda}(B \lessdot C), \widetilde{\lambda}(B \lessdot D)$.

(CN3): Within a maximal chain of the interval $[x,y]$, the label given to the covering relation $S \subset T$ is the element of T not contained in S, (possibly) along with some subscripts. For $S_1 \subset T_1$ and $S_2 \subset T_2$ where $S_1 < S_2$, we have $T_1 \leqslant S_2$ so S_2 already contains the element of T not contained in S and thus such an element cannot be involved in the label of $S_2 \subset T_2$. So all labels in a maximal chain are unique.

(CU2): Consider elements M and M' of Bic(T) which are uniquely covered by M^* and M'^* respectively. Assume for the sake of contradiction that $\widetilde{\lambda}(M \lessdot M^*) = \widetilde{\lambda}(M' \lessdot M'^*)$. Thus $M^* = M \cup \{[a,c]\}$ and $M'^* = M' \cup \{[a,c]\}$ for some segment $[a,c]$. The sets M and M' contain the same split $[a,b]$ or $[b,c]$ for each break $([a,b],[b,c])$ of $[a,c]$.

Also $M \vee M' \geqslant M^*, M'^*$. Thus the closure of $M \cup M'$ contains $[a,c]$. So there exist some composable set of segments $[a,b_1],[b_1,b_2],[b_2,b_3],\cdots,[b_{n-1},b_n],[b_n,c]$ in $M \cup M'$. Without loss of generality, $[a,b_1] \in M$, $[b_1,b_2] \in M'$, $[b_2,b_3] \in M$, etc. Since $[a,b_1] \in M$, then $[a,b_1] \in M'$. $[b_1,b_2] \in M'$ as well so $[a,b_2] \in M'$ via composition. Thus, $[a,b_2] \in M$. $[b_2,b_3] \in M$, so $[a,b_3] \in M$ via composition. Thus $[a,b_3] \in M'$. Continuing in this manner, we see that $[a,c] \in M$, which is a contradiction. Thus $M \lessdot M^*$ and $M' \lessdot M'^*$ have different labels.

(CU1): Consider elements J and J' of Bic(T) which uniquely cover J_* and J'_* respectively. Thus $J_*, J'_* \geqslant J \wedge J'$. Assume for the sake of contradiction that $\widetilde{\lambda}(J_* \lessdot J) = \widetilde{\lambda}(J'_* \lessdot J')$. Say that this label is $[a,c]$ with some subscripts. Thus J and J' both contain $[a,c]$, so $[a,c] \in J \vee J'$. J_* and J'_* do not contain $[a,c]$ so $J \wedge J'$ does not either. The set $J \wedge J'$ is the complement of $J^C \vee J'^C$ so $J^C \vee J'^C$ contains $[a,c]$. So $[a,c]$ is in the closure of the union of the biclosed sets J^C and J'^C. For each break of $[a,c]$, J_* and J'_* and thus J and J' contain the same split. Thus J^C and J'^C contain the same split. We then apply argument used for (CU2). ☐

6. Shard Intersection Poset

Throughout this section, we let $\widetilde{\lambda} : Cov(Bic(T)) \to (Seg(T) \times 2^{Seg(T)})$ denote the CU-labeling constructed in Section 5 and $\lambda : Cov(Bic(T)) \to Seg(T)$ denote the first coordinate function of $\widetilde{\lambda}$.

Definition 6.1. For an element B of the poset of biclosed sets, define $\psi(B)$ as $\{\lambda(u,v) \mid R \leqslant u < v \leqslant B\}$ where R is the meet of the biclosed sets covered by B.

Definition 6.2. $\Psi(T)$ is the poset whose elements are $\psi(B)$ for biclosed sets B of T, partially ordered by inclusion.

Conjecture 6.3. $\Psi(T)$ is a lattice.

Lemma 6.4. The shard intersection order of $\mathrm{Bic}(T)$ has a unique maximal element.

Proof. The top element, X of $\mathrm{Bic}(T)$ is the set of all segments of T. It covers every set consisting of all segments of T except $[a,b]$ where a and b are adjacent in T: Such a set is necessarily closed because it contains any segment $[c,e]$ that is formed from composing two segments of T. Its complement is necessarily closed since it contains only one segment. It is covered by X because it contains all but one segment of T.

Thus the meet of the elements covered by X cannot contain any segments $[a,b]$ where a,b are adjacent in T. Thus the complement of the meet contains all segments $[a,b]$ of T where a,b are adjacent. By composition, the complement of the meet must contain all segments in T so the meet contains no segments. Thus $\psi(X)$ consists of the set of all labels. □

Given a biclosed set B, which covers $B_1, B_2, \ldots, B_k$, with $\widetilde{\lambda}(B_i, B) = \lambda_i$ for $i = 1, 2, \ldots k$, we let $S = \{s_1, s_2, \ldots, s_k\}$ where s_i is the segment part of λ_i, given by $\lambda(B_i, B)$. The set of compositions of segments in S is denoted by $\overline{S}$.

Definition 6.5. Let $s \in \overline{S}$ be a segment expressed as $s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_w}$ with each $s_{i_j} \in S$. Let t be a split of s such that there exists a break (t, t') or (t'', t) of s. We say that t is a *faultline split* if the faultline of such a break occurs at the endpoint of some s_{i_j}. Otherwise, we say that t is a *non-faultline split.*

Proposition 6.6. Given a biclosed set B, which covers $B_1, B_2, \ldots, B_k$, with $\lambda(B_i, B) = s_{i\delta_i}$ for $i = 1, 2, \ldots k$, we can describe $\psi(B)$ as consisting of all labels of the following form:

$$(s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_w})_\delta$$

where $[a,c] = s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_w} \in \overline{S}$ and δ is a list of segments containing one split per break of s with any of the 2^{w-1} possible choices for the faultline splits and with the non-faultline splits determined as follows: for b, a vertex within $[x,y] = s_{i_j}$ for some $1 \leqslant j \leqslant w$, where x is the endpoint of s_{i_j} closer to a, then δ contains $[a,b]$ if δ_{i_j} contains $[x,b]$ and δ contains $[b,c]$ if δ_{i_j} contains $[b,y]$.

Example 6.7. Consider the tree $T = P_4$ given by a path of 4 vertices numbered 1,2,3,4 in order. Consider the biclosed set $B = \{[1,2],[2,3],[1,3],[1,4]\}$. In $Bic(T)$, B covers $B_1 = \{[1,2],[1,3],[1,4]\}$ and $B_2 = \{[1,2],[2,3],[1,3]\}$. The meet of the elements covered by B is $B_1 \wedge B_2 = \{[1,2],[1,3]\}$.
Thus, the set $\psi(B)$ is given by $\{[2,3]_{\{\}}, [1,4]_{\{[1,2],[1,3]\}}\}$.

Proof. of Proposition 6.6

Lemma 6.8. For a biclosed set B and any (B_i, B) and (B_j, B) in Cov(Bic(T)), the segment $\lambda(B_i, B)$ is not a split of $\lambda(B_j, B)$.

Proof. Suppose for the sake of contradiction that there exist some biclosed sets B_1, B_2 such that $\lambda(B_1, B)$ is a split of $\lambda(B_2, B)$. Then $B_1 = B - \{[a,b]\}$ and $B_2 = B - \{[a,c]\}$ where $[a,b]$ is a split of $[a,c]$.

Consider $L = B - \{[a,b],[a,c]\}$. Since $L \subset B_1, B_2$, a composition of composable elements in L lies in B_1 and B_2. Thus it lies in $B_1 \cap B_2 = L$ so L is closed. A composition of two elements in L^C is a composition of two elements in $B_1^C \cap B_2^C$ and perhaps $[a,c]$ and $[a,b]$. A composition of two elements in B_1^C is in B_1^C and thus in L^C. A composition of two elements in B_2^C is in B_2^C and thus in L^C. The only other possible composition of elements would be $[a,b]$ and $[a,c]$ but those are not composable, so L^C is closed. Thus L is biclosed.

Consider the segment $[b,c]$. If it is not contained in B, then B_1 contains $[a,c]$ without containing $[a,b]$ or $[b,c]$, meaning it is not biclosed. If $[b,c]$ is contained in B, then it is contained in B_2, which then contains $[a,b]$ and $[b,c]$ without containing $[a,c]$. But B_1 and B_2 are biclosed, so this is a contradiction.

□

Lemma 6.9. Let $B_1, \cdots, B_k \in Bic(T)$ denote the biclosed sets covered by B and define $s_i = \lambda(B_i, B)$ for $i = 1, \cdots, k$. Then

$$\bigwedge_{i=1}^{k} B_i = B - \overline{\{s_1, \cdots, s_k\}}$$

.

Proof.

$$\begin{aligned}\bigwedge_{i=1}^{k} B_i &= \wedge\{B - \{s\} \mid s \in S\} \\ &= (\vee\{B - \{s\} \mid s \in S\})^C \\ &= (\overline{\cup\{B^C \cup \{s\} \mid s \in S\}})^C\end{aligned}$$

The set $\overline{\cup\{B^C \cup \{s\} \mid s \in S\}}$ contains B^C and $\overline{S}$. Any composition of two composable elements of B^C is in B^C and any composition of two composable elements of $\overline{\{s_1, \cdots, s_k\}}$ is in $\overline{\{s_1, \cdots, s_k\}}$. The composition of $n \in B^C$ and $s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_m}$ can be rewritten as $(n \circ s_{i_1}) \circ (s_{i_2} \circ \ldots \circ s_{i_m})$, but $B^C \cup \{s_{i_1}\}$ is biclosed, so $n_1 = n \circ s_{i_1} \in B^C$. Continuing in this manner, utilizing that each $B^C \cup \{s_i\}$ is biclosed, we see that the composition of composable elements from B^C and from $\overline{\{s_1, \cdots, s_k\}}$ is itself in B^C and thus in $B^C \cup \overline{\{s_1, \cdots, s_k\}}$. So simply, $\bigwedge_{i=1}^{k} B_i = (B^C \cup \overline{\{s_1, \cdots, s_k\}})^C = B - \overline{\{s_1, \cdots, s_k\}}$. □

Lemma 6.10. If $s = \lambda(B_i, B)$ for some B_i covered by the biclosed set B, then $R \cup \{s\}$ is biclosed.

Proof. A composition of two composable elements in $R \cup \{s\}$ is either a composition of two elements in R which is therefore in R or the composition of s with an element, r in R. A segment $s \circ r$ is in B so it is either in R or in $\overline{S}$. If $s \circ r$ is in $\overline{S}$ then we can write it as $s_{i_1} \circ s_{i_2} \circ \ldots s_{i_m}$, where s_{i_1} shares an endpoint with r and s_{i_m} shares an endpoint with s. However s is not a split of s_{i_m} or vice versa, so $s = s_{i_m}$. If $m = 1$, this would mean $r \circ s = s$, so $m > 1$. Thus $r = s_{i_1} \circ s_{i_2} \circ \cdots s_{i_{m-1}}$, a contradiction since $r \in R = B - \overline{S}$. Thus $s \circ r \in R \subset R \cup \{s\}$. So $R\{s\}$ is closed. A composition of two composable elements in $(R\{s\})^C = R^C - \{s\}$ is a composition of elements in R^C so the only way it would not be in $(R\{s\})^C$ is if it equals s. Suppose $n_1 \circ n_2 = s$ for $n_1, n_2 \in R^C$. If $n_1, n_2 \in R^C - \overline{S}$, then $n_1 \circ n_2 \in R - \overline{S}$, a contradiction, so either n_1 or n_2 is in $\overline{S}$. Without loss of generality, say $n_1 = s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_m}$, where s_{i_1} shares an endpoint with s. This means s_{i_1} is a split of s, contradicting 6.8. So $(R \cup \{s\})^C$ is closed, and thus $R \cup \{s\}$ is biclosed. □

Consider the set of segments in $\overline{S}$. We can consider a subset, A of $\overline{S}$ to be biclosed if for every two composable segments $a, b \in A$, we have $a \circ b \in A$ and for every break of a segment $s \in A$ where the splits of the break are in $\overline{S}$, at least one of those splits is in A.

Lemma 6.11. The biclosed subsets in the interval $[R, B]$ are precisely those of the form $R \cup A$ where A is a biclosed subset of $\overline{S}$.

Proof. Let A be a subset of $\overline{S}$. $R \subset R \cup A$ and $A \subset \overline{S} \subset (B)$, so $R \cup A \subset B$. Thus, if $R \cup A$ is biclosed, it is in the interval $[R, B]$. (If A is not a subset of $\overline{S}$, then $R \cup A$ would either contain some element not in B or would be equivalent to $R \cup A'$ for some $A' \in \overline{S}$)

If A is a biclosed subset of $\overline{S}$, consider the composition of two composable segments in $R \cup A$. If they are both in R, their composition is in R and if they are both in A, their composition is in A. If one is some $r \in R$ and the other is some $s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_m} \in A$, then their composition is $r \circ s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_m}$. However $r \circ s_{i_1}$ is in the biclosed set $R \cup \{s_{i_1}\}$, but it is not equal to s_{i_1}, so it must be some other $r_1 \in R$. In general for $r_j \in R$, $r_j \circ s_{i_{j+1}} \in R \cup \{s_{i_{j+1}}\}$ but is not equal to $s_{i_{j+1}}$, so it is equal to some other $r_{j+1} \in R$. Thus

$$\begin{aligned}r \circ s_{i_1} \circ s_{i_2} \circ \cdots \circ s_{i_m} &= r_1 \circ s_{i_2} \circ \cdots \circ s_{i_m} \\ &= r_2 \circ s_{i_3} \circ \cdots \circ s_{i_m} \\ &= \ldots \\ &= r_{m-1} \circ s_{i_m} = r_m \in R \subset (R \cup A)\end{aligned}$$

so $R \cup A$ is closed.

For two composable elements of $(R \cup A)^C = R^C - A$, their composition is in R^C. $(R \cup A)^C$ is closed unless the composition of some $n_1, n_2 \in R^C - A$ is in A. If $n_1, n_2 \in R^C - \overline{S}$, their composition is in $R^C - \overline{S}$, so without loss of generality, $n_1 \in \overline{S}$. Given $n_1 \circ n_2 = (s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_m}) \circ n_2 = s_{j_1} \circ s_{j_2} \circ \ldots \circ s_{j_d} = a \in A$, where s_{i_1} and s_{j_1} share an endpoint, we have that one of s_{i_1} and s_{j_1} is a split of the other unless $s_{i_1} = s_{j_1}$. Thus $(s_{i_2} \circ \ldots \circ s_{i_m}) \circ n_2 = s_{j_2} \circ \ldots \circ s_{j_d}$ where s_{i_2} and s_{j_2} share and endpoint, so one is a split of the other unless they are the same. Thus $(s_{i_3} \circ \ldots \circ s_{i_m}) \circ n_2 = s_{j_3} \circ \ldots \circ s_{j_d}$. Continuing in this manner, we see that $n_2 = s_{j_{m+1}} \circ \ldots \circ s_{j_d}$. This means that both $n_1, n_2 \in \overline{S}$. Since A is a biclosed subset of $\overline{S}$ and $a \in A$, either n_1 or n_2 must be in A.

So the composition of $n_1, n_2 \in R^C - A$ cannot be in A, and we get that $R^C - A$ is closed, and hence $R \cup A$ is biclosed.

If A is not a biclosed subset of $\overline{S}$, it either contains (i) some a, b without $a \circ b$ or (ii) there exist some $a, b \in \overline{S}$ for which $a, b \notin A$, but $a \circ b \in A$. In case (i), $R \cup A$ contains a, b but neither R nor A contains $a \circ b$, so $R \cup A$ is not closed. In case (ii), $R \cup A$ contains $a \circ b$ but since $a, b \notin R, A$, it does not contains a or b. Thus $(R \cup A)^C$ is not closed. So $R \cup A$ is not biclosed in either case.

□

For a segment $t = [e, f] = s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_m}$ in $\overline{S} = B - R$, it has $m - 1$ breaks. To show that t with any of the 2^{m-1} choices of one split per break occurs as a label in $[R, B]$ consider the following:

We will build a set D with the property that D is a biclosed subset of $\overline{S}$ which contains an arbitrarily chosen split for each faultline break of t.

First, choose one split for each of the $m - 1$ breaks of t and add them to D. For each split p included in D:

(1) Consider a faultline break of p into a, b where a shares an endpoint with t. If a was one of the original splits chosen to be in D, do nothing. If $a \notin D$, add b to D.
(2) Now add compositions of segments in D to D.
(3) Repeat this process for the next break of p.

We never add something that is a split of t, except perhaps in the phase where we add compositions of segments already in D. Since none of the original splits $p \in D$ are composable, such a composition will include at least one $b = [x, y]$ (with x closer to e than y is) that was added later. This would require the presence of either $[e, x]$ or $[y, f]$ in D in order for a split of t to result from a composition. Initially, there is no break of t for which both splits are contained in D. By induction, this is true before we add b to D. In the case that $[x, y]$ was added to D as a split of $p = [e, y]$: If $[e, x]$ was already in D, then it was one of the original elements of D and we would already have the split $a = [e, x]$ of the split $p = [e, y]$ and not have added $[x, y]$ to D per the rules outlined earlier. If $[y, f]$ was already in D, then $[e, y]$ was not, a contradiction.

In the case that $[x, y]$ was added to D as a split of $p = [x, f]$: If $[e, x]$ was already in D, then $[x, f]$ could not have been, a contradiction. If $[y, f]$ was already in D then the split $a = [y, f]$ of $[x, f]$ would have already been in D and we would not have added $[x, y]$ as per the rules outlined earlier.

The last step of this process involves taking compositions of all the elements already in D and adding them to D so the resulting D is closed. D will be biclosed if for every element, $q = [x, y]$ of D and every break of q, some split corresponding to that break is also in D.

By virtue of how the b's were added to D any split, p, of t that was used as a seed for D has a split in D corresponding to each of its breaks.

Consider any other $[x, y] \in D$ where x is closer to e than y is. Since this does not have e or f as an endpoint, it is the composition of some set of b's added during phase (1), including some $[g, y]$. Assume for the sake of contradiction that there is some vertex $c \in [x, y]$ where neither $[x, c]$ nor $[c, y]$ is in D. Either $[e, c]$ or $[c, f]$ was one of the splits used to seed D. Without loss of generality, say $[c, f]$ is in D. Since the split $[c, y]$ of $[c, f]$ is not in D, we must have $[y, f] \in D$. However, this means $[y, f]$ was one of the original splits used to seed D. Since $[g, y], [y, f] \in D$, then $[g, f] \in D$. Thus $[g, f]$ was one of the splits used to seed D. But since the split $[y, f]$ of $[g, f]$ was already in D, the split $[g, y]$ would not have been added in phase (1) except perhaps as a split of $[e, y]$. However, the split $[y, f] \in D$ so $[e, y] \notin D$ and thus there is no way $[g, y]$ could have been added, a contradiction.

So D is a biclosed subset of $\overline{S}$. The only way it could include t would be if t had been added as a composition of two splits corresponding to the same break of t. But that situation never occurs so the biclosed set $R \cup D$ does not include t, but B does. Some covering relation on a chain from $R \cup D$ to B will have a label t_μ where μ includes the splits we originally chose as seeds for D.

For label $t_\delta \in \Psi(B)$ and each non-faultline break, $([e, h], [h, f])$ of t, either $[e, h]$ or $[f, h]$ is in δ. h is a vertex inside some segment $s_{i_j} = [x, y]$ within t, where x is the endpoint of s_{i_j} closer to e. If λ_{i_j} contains $[x, h]$, then B (and R)contain $[x, h]$. R contains either $[e, h]$ or $[h, f]$. If R contains $[h, f]$, then R contains $[x, f]$ since R is biclosed. This contradicts $[x, f] \in \overline{S}$, so R contains $[e, h]$. Thus, δ contains $[e, h]$.

Similarly, if λ_{i_j} contains $[h, y]$, then B (and R) contains $[h, y]$. If R contains $[e, h]$, then R contains $[e, y] \in \overline{S}$, a contradiction. So R, and thus δ, contains $[h, f]$. □

For biclosed sets B and B' of $Seg(T)$, we can consider the structure of $\psi(B) \cap \psi(B')$. Suppose $\lambda(B_i, B) = s_i$ while $\lambda(B'_j, B') = t_j$. Then any label in $\psi(B) \cap \psi(B')$ consists of a segment s which can be expressed as a composition of both s_i's and t_i's along with some subscripts. If this composition of s_i's and the corresponding composition of t_i's share m *faultlines*, then we can consider this label as corresponding to both a composition of $m+1$ $s_i's$ and a combination of $m+1$ $t_i's$ which do not share any faultlines. We call labels in $\psi(B)\cap\psi(B')$ corresponding to compositions of s_i's and t_i's that share no faultlines *pseudominimal*. An element of $\psi(B)\cap\psi(B')$ is of the form s_δ where s is a composition of pseudominimal elements of $\psi(B)\cap\psi(B')$ and δ consists of one split per break of s. There are independent choices for which faultline splits are included in δ but which non-faultline splits are included are predetermined by the splits in the subscripts of the pseudominimal elements, in much the same way the non-faultline splits of an element of $\psi(B)$ depend on the subscripts of the labels on the covering relations (B', B).

As a result, if the pseudominimal elements of $\psi(B)\cap\psi(B')$ form a canonical join representation, then we can take B'' equal to their join to obtain $\psi(B'')=\psi(B)\cap\psi(B')$, which would prove Conjecture 6.3.

Remark 6.12. $\Psi(T)$ is not always graded. For example, $\Psi(S_4)$ and $\Psi(S_6)$ are not graded.

Proof. Label the degree 4 vertex of S_4 as 0 and label the others clockwise as 1,2,3,4. For convenience, let $a=[0,1], b=[0,2], c=[0,3], d=[0,4], e=[1,2], f=[2,3], g=[3,4], h=[1,4]$.

We demonstrate that $\Psi(S_4)$ is not graded by producing two maximal chains of differing lengths:

$\psi(\{\})=\{\}\lessdot\psi(\{b,e\})=\{e_{\{b\}}\}\lessdot\psi(\{b,e,f\})=\{e_{\{b\}},f_{\{b\}}\}\lessdot\psi(\{b,d,e,f,g\})=\{e_{\{b\}},f_{\{b\}},g_{\{d\}}\}\lessdot\psi(\{b,d,e,f,g,h\})=\{e_{\{b\}},f_{\{b\}},g_{\{d\}},h_{\{d\}}\}\lessdot\psi(\{a,b,c,d,e,f,g,h\})=\{a_{\{\}},b_{\{\}},c_{\{\}},d_{\{\}},e_{\{a\}},e_{\{b\}},f_{\{b\}},f_{\{c\}},g_{\{c\}},g_{\{d\}},h_{\{d\}},h_{\{a\}}\}$

and

$\psi(\{\})=\{\}\lessdot\psi(\{b,e\})=\{e_{\{b\}}\}\lessdot\psi(\{b,c,e,f\})=\{c_{\{\}},e_{\{b\}}\}\lessdot\psi(\{a,b,c,e,f\})=\{a_{\{\}},b_{\{\}},c_{\{\}},e_{\{a\}},e_{\{b\}},f_{\{b\}},f_{\{c\}}\}\lessdot\psi(\{a,b,c,d,e,f,g,h\})=\{a_{\{\}},b_{\{\}},c_{\{\}},d_{\{\}},e_{\{a\}},e_{\{b\}},f_{\{b\}},f_{\{c\}},g_{\{c\}},g_{\{d\}},h_{\{d\}},h_{\{a\}}\}$

□

Conjecture 6.13. $\Psi(Bic(S_k))$ for $n\geqslant 3$ is graded if and only if n is odd.

7. Enumerative Results

The results developed in the Section 3 can be used to make calculations concerning the noncrossing tree partition lattice. In particular, there are nice methods to calculate the numbers of maximal chains for the noncrossing tree partition lattices corresponding to one-parameter families of trees. For a lattice L, we let $\mathrm{mc}(L)$ denote the number of maximal chains in L. We focus on a family of trees called **stars**.

Definition 7.1. Define the star S_k to be a tree with $k+1$ interior vertices, one of which is adjacent to the other k, all of which have degree 3.

Proposition 7.2. The number of maximal chains, $\mathrm{mc}(\mathrm{NCP}(S_k))$ is $\frac{k!F_{k+1}}{2}$ where F_k is the kth Fibonacci number with $F_0=1, F_1=1$.

Proof. Let $\{a_i\}_{i=1}^m$ be the set of coatoms of S_k.

First note that $\mathrm{mc}(\mathrm{NCP}(S_k))=\sum_{i=1}^m \mathrm{mc}([\hat{0},a_i])$. Also, each a_i consists of two blocks. There are two possibilities: either there is a block of size 1 (not the vertex of degree k) and a block of size k or there is a block of size $k-1$ involving the vertex of degree k along with a block of size 2.

For $k\geqslant 3$, there are k of each type. So if a coatom a_i consists of blocks A and B, we have $\mathrm{mc}([\hat{0},a_i])=\mathrm{mc}(\mathrm{NCP}(A)\times\mathrm{NCP}(B))$.

Thus, $\mathrm{mc}(\mathrm{NCP}(S_k))=k(\mathrm{mc}(\mathrm{NCP}(S_1)\times\mathrm{NCP}(S_{k-2}))+\mathrm{mc}(\mathrm{NCP}(S_0)\times\mathrm{NCP}(S_{k-1}))=k(\mathrm{mc}(\mathrm{NCP}(S_1)\times\mathrm{NCP}(S_{k-2}))+\mathrm{mc}(\mathrm{NCP}(S_{k-1})))$.

The minimal element in $\mathrm{NCP}(S_{k-2})$ has $k-1$ blocks and the maximal element has 1, so a maximal chain has $k-2$ covering relations. Thus there are $(k-2)+1=k-1$ ways to insert the single covering relation of $\mathrm{NCP}(S_1)$ into a maximal chain of $\mathrm{NCP}(S_{k-2})$ to form a maximal chain of $\mathrm{NCP}(S_1)\times\mathrm{NCP}(S_{k-2})$. So $\mathrm{mc}(\mathrm{NCP}(S_k))=k[(k-1)*\mathrm{mc}(\mathrm{NCP}(S_{k-2}))+\mathrm{mc}(\mathrm{NCP}(S_{k-1}))]$

Note that $\mathrm{mc}(\mathrm{NCP}(S_1))=1$ and $\mathrm{mc}(\mathrm{NCP}(S_2))=3$. These don't meet the specifications for T but it is still possible to count their maximal chains. Now for $l\geqslant 3$, assume we have $\mathrm{mc}(\mathrm{NCP}(S_k))=\frac{k!F_{k+1}}{2}$ for $1\leqslant k<l$.

Then $\mathrm{mc}(\mathrm{NCP}(S_l))=l[(-1)*\mathrm{mc}(\mathrm{NCP}(S_{l-2}))+\mathrm{mc}(\mathrm{NCP}(S_{l-1}))]=l[(l-1)\frac{(l-2)!F_{l-1}}{2}+\frac{(l-1)!*F_l}{2}]=l[(l-1)!\frac{F_{l-1}+F_l}{2}]=l!\frac{F_{l+1}}{2}$ and the induction is complete.

□

Definition 7.3. Two vertices of degree 3 in S_k, which are each adjacent to 2 leaves, are called *outer neighbors* if there is a segment with those endpoints. Such a segment would necessarily consist of two edges.

Proposition 7.4. $|NCP(S_3)| = 14, |NCP(S_4)| = 34$. For $k \geqslant 5$, $|NCP(S_k)| = 2|NCP(S_{k-1})| + |NCP(S_{k-2})|$. These are known as the Pell-Lucas numbers.

Proof. One can verify $|NCP(S_3)| = 14, |NCP(S_4)| = 34$.

For $n \geqslant 5$, consider some interior vertex, A of S_k which has degree 3. A is either (i) in a block of size 1, (ii) in a block of size 2 along with some other interior vertex of degree 3, or (iii) in the same block as the vertex of degree k.

Let Sub_k denote the number of noncrossing partitions of S_k such that a given pair of vertices of degree 3 (in particular vertices of degree 3 adjacent to two leaves) form a block of size 2.

Case (i): The remaining vertices other than A form a S_{k-1} but the two vertices which had been outer neighbors of A cannot be in the same block of size 2. Thus there are $|NCP(S_{k-1})| - Sub_{k-1}$ possibilities.

Case(ii): There are two choices for which outer neighbor, B, of A is included in the same block as A. The remaining vertices form a S_{k-2} where the vertices which had been outer neighbors of A and B respectively cannot constitute a block of size 2. Thus there are $2 * (|NCP(S_{k-2})| - Sub_{k-2})$ possibilities.

Case(iii): We can contract the edge between A and the vertex of degree k in order to form a S_{k-1}. Choosing a noncrossing tree partition of the S_{k-1} which does include a block of size 2 constituted by what were the outer neighbors of A and then uncontracting the edge we contracted is equivalent to choosing a noncrossing tree partition of S_k in which A is in the same block as the vertex of degree k. Thus there are $|NCP(S_{k-1})| - Sub_{k-1}$ possibilities.

In total:

$$\begin{aligned}|NCP(S_k)| &= |NCP(S_{k-1})| - Sub_{k-1} + 2 * (|NCP(S_{k-2})| - Sub_{k-2}) + |NCP(S_{k-1})| - Sub_{k-1} \\ &= 2|NCP(S_{k-1})| + 2|NCP(S_{k-2})| - 2(Sub_{k-1} + Sub_{k-2})\end{aligned}$$

Note that $Sub_3 = 2, Sub_4 = 5$, and $Sub_5 = 12$. We claim that for $k \geqslant 3$: $|NCP(S_k)| = 2Sub_{k+1} + 2Sub_k$. We will prove this by induction.

Base cases: $|NCP(S_3)| = 14 = 2*5+2*2 = 2Sub_4+2Sub_3$ and $|NCP(S_4)| = 34 = 2*12+2*5 = 2Sub_5+2Sub_4$ For some $n \geqslant 4$, we have $NCP(S_k) = 2Sub_{k+1} + 2Sub_k$ for all $k \leqslant n$. Now consider

$$\begin{aligned}|NCP(S_{n+1})| &= 2|NCP(S_n)| + 2|NCP(S_{n-1})| - 2(Sub_n + Sub_{n-1}) \\ &= 2(2Sub_{n+1} + 2Sub_n) + 2(2Sub_n + 2Sub_{n-1}) - 2(Sub_n + Sub_{n-1}) \\ &= 4Sub_{n+1} + 6Sub_n + 2Sub_{n-1}\end{aligned}$$

Now consider Sub_{k+1}. It is the number of noncrossing partitions of S_{k+1} which include a given block of size 2 constituted by 2 outer neighbors. This is equivalent to

$$|NCP(S_{k-1})| - Sub_{k-1} = 2Sub_k + Sub_{k-1}$$

Thus $|NCP(S_{n+1})| = 2Sub_{n+2} + 2Sub_{n+1}$, completing the induction. So

$$\begin{aligned}|NCP(S_k)| &= 2|NCP(S_{k-1})| + 2|NCP(S_{k-2})| - 2(Sub_{k-1} + Sub_{k-2}) \\ &= 2|NCP(S_{k-1})| + |NCP(S_{k-2})|\end{aligned}$$

□

In the case of a star, it is also possible to explicitly compute $|Bic(T)|$.

Proposition 7.5. For $k \geqslant 3, |\operatorname{Bic}(S_k)| = 3^k + (-1)^k$

Proof. Each member of $\operatorname{Bic}(S_k)$ is analogous to choosing some vertices and edges of an k-gon such that if two adjacent edges are chosen, the vertex between them must be chosen and if two adjacent edges are both not chosen, then the vertex between them cannot be chosen either.

The number of vertices for which one neighboring edge is chosen and the other is not is necessarily even. For each choice of of such vertices, there are two ways to pick which of the sets of intervening edges are chosen and which are not. There are then two ways to pick whether each of these vertices is chosen.

Thus

$$|\operatorname{Bic}(S_k)| = \sum_{\substack{i=0 \\ i \text{ even}}}^{i=k} \binom{k}{i} * 2^{i+1}$$

Note that

$$3^k = (2+1)^k = \sum_{i=0}^{i=k} \binom{k}{i} * 2^i$$

and

$$1 = (2-1)^k = \sum_{i=0}^{i=k} (-1)^{k-i} * \binom{k}{i} * 2^i$$

Adding these expressions together gives us:

$$3^k + 1 = 2 * \sum_{\substack{i=0 \\ i \text{ even}}}^{i=k} \binom{k}{i} * 2^i$$

for even k and

$$3^k - 1 = 2 * \sum_{\substack{i=0 \\ i \text{ even}}}^{i=k} \binom{k}{i} * 2^i$$

for odd k.

So in either case, $|\operatorname{Bic}(S_k)| = 3^k + (-1)^k$

□

Acknowledgements

This research was carried out as part of the 2016 summer REU program at the School of Mathematics, University of Minnesota, Twin Cities, and was supported by NSF RTG grant DMS-1148634. The authors would like to thank Craig Corsi, Thomas McConville, and Vic Reiner for their valuable advice and conversations. The authors especially thank Al Garver for his mentorship.

References

[1] A. Björner. Shellable and Cohen-Macaulay partially ordered sets. *Transactions of the American Mathematical Society*, 1980.
[2] A. Björner and M. Wachs. On lexicographically shellable posets. *Transactions of the American Mathematical Society*, 1983.
[3] A. Garver and T. McConville. Oriented flip graphs and noncrossing tree partitions. *arXiv: 1604.06009*, 2016.
[4] N. Reading. Lattice theory of the poset of regions. In *Lattice Theory: Selected Topics and Applications. Volume 2.* Birkhäuser/Springer Basel AG, 2016.
[5] R. Simion and D. Ullman. On the structure of the lattice of noncrossing partitions. *Discrete Mathematics*, 1989.