THE CYCLIC SIEVING PHENOMENON ON THE ALTERNATING SIGN MATRICES

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Abstract. We first present a previously unpublished result of Stanton [11] that the group of order four generated by rotation by 90° acting on alternating sign matrices exhibits the CSP with the obvious q-analogue of $|\mathrm{ASM}(n)|$.

In [12], Wieland introduced a much larger cyclic group that acts on the set of alternating sign matrices. Unfortunately, it has a very complex orbit structure that does not exhibit the CSP with that polynomial and is not suggestive of CSP with any simple polynomial. However, we found smaller groups that do exhibit the phenomenon, and in the process discovered an extremely large group of maps on the alternating sign matrices.

We finish by suggesting the existence of a group of order three on the ASMs that exhibits cyclic sieving and a class of subsets of the ASM(n) that exhibit cyclic sieving with gyration and a new set of polynomials.

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1. Introduction

Definition 1.1. An *alternating sign matrix* (ASM) of size n is an $n \times n$ matrix with all entries 0, 1, or -1 such that the non-zero entries in each row and column alternate sign and sum to one.

They arise naturally in the study of the lambda determinant (see [2]). We will call the set of ASMs of size n ASM(n). In [13], Zeilberger proved that

$$|\mathrm{ASM}(n)| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \tag{1.1}$$

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There is an easy bijection between ASMs and another class of objects called fully packed loop diagrams.

Definition 1.2. A *fully packed loop diagram* (FPL) of size n is an $n \times n$ grid with a boundary as in Figure 1.1 such that each vertex touches exactly two solid lines and two dotted lines.

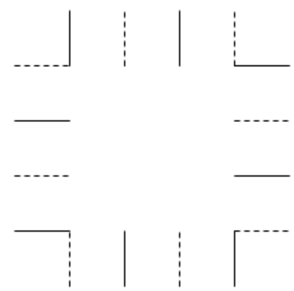

FIGURE 1.1. FPL Border

To convert an FPL to an ASM, place a 0 at any vertex where a solid line changes direction and a 1 or -1 at all other vertices such that the non-zero entries in each row alternate sign and sum to one. To convert back, draw the boundary as before and, starting at the bottom left corner, fill the squares of the ASM as in Figure 1.2 so that all vertices touch exactly two solid lines. Throughout the paper, we will use FPLs and ASMs interchangeably.

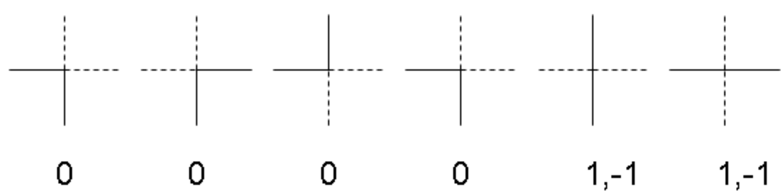

FIGURE 1.2. Pictures in FPL corresponding to numbers in ASM

We now consider a cyclic group of large order acting on the ASMs. First, label each vertex (i, j) of an FPL, as in Figure 1.3. If a square has bottom left vertex (i, j), we assign it the parity of $i+j$. This labelling allows us to define the following maps.

Definition 1.3. *Even (odd) gyration*, is an involution $G_{even} : \mathrm{ASM}(n) \to \mathrm{ASM}(n)$ ($G_{odd} : \mathrm{ASM}(n) \to \mathrm{ASM}(n)$) defined as follows. Visit every square labelled even (odd) on an FPL. If the square contains two parallel solid lines, change solid lines in the square to dotted lines and vice versa. Otherwise, do nothing.

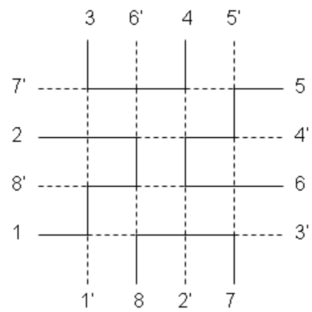

FIGURE 1.3. Labelled ASM

Definition 1.4. *Gyration* is the operation $G = G_{even} \circ G_{odd}$ first created by Wieland in [12]. It provides a bijection between ASMs with similar pairing patterns (see Theorem 5.3 for the details of this bijection and [12] for its proof).

We will also need two basic operations on ASMs.

Definition 1.5. *Rotation by* 90° is an operation, R_{90° : ASM(n) → ASM(n) such that it rotates an ASM 90° counterclockwise. We also define $(R_{90^\circ})^2 = R_{180^\circ}$ and $(R_{90^\circ})^3 = R_{270^\circ}$.

Definition 1.6. For a set X, a polynomial $X(q)$, and $H =< h >$ a cyclic group of order n that acts on X, we say $(X,\ X(q),\ H)$ exhibits the *cyclic sieving phenomenon* (CSP) if $\forall\, d$ such that $d \mid n$, one has $X(e^{2\pi\imath/d})=\mid \{x \in X \mid h^{n/d}(x) = x\} \mid$.

We will often drop this formality, and say that a specific bijection ϕ : ASM(n) → ASM(n) exhibits the CSP, by which we mean that (ASM(n), ASM(n,q), $< \phi >$) exhibits the CSP, where ASM(n,q) is defined below.

$$ASM(n,q) = \prod_{i=0}^{n-1} \frac{(3i+1)!_q}{(n+i)!_q}. \tag{1.2}$$

Note that this is not an arbitrary choice. The following theorem was conjectured by Andrews in [1] and proven by Mills et. al. in [5] .

Theorem 1.7. *$ASM(n,q) = \sum_m M(n,m)q^m$ where $M(n,m)$ is the number of descending plane partitions of n with the number of rows, the number of columns, and all entries less than m.*

Proof. See [5]. □

Corollary 1.8. *$ASM(n,q) \in \mathbb{N}[q]$.*

Proof. Theorem 1.7 □

2. Cyclic Sieving Phenomenon with Rotation

We first give a previously unpublished theorem due to Stanton [11].

Definition 2.1. *The set of half-turn (quarter-turn) symmetric matrices* is
$\text{ASM}(n)_{HT} = \{A \in \text{ASM}(n) : R_{180^\circ}(A) = A\}$
$\text{ASM}(n)_{QT} = \{A \in \text{ASM}(n) : R_{90^\circ}(A) = A\}$

Theorem 2.2.
(1) $ASM(n, -1) = |ASM(n)_{HT}|$
(2) $ASM(n, \imath) = |ASM(n)_{QT}|$

Proof. We will prove (1) for the case $n = 2m$ and (2) for $n = 4m + 2$ and omit the rest of the proof because it is similar and tedious.

(1) Note the following basic facts,

$$lim_{q\to -1}[n]_q = \begin{cases} lim_{q\to -1}(1+q)(n/2) & \text{even } n \\ 1 & \text{odd } n \end{cases} \tag{2.1}$$

$$lim_{q\to -1} n!_q = lim_{q\to -1}\lfloor n/2\rfloor ! * (1+q)^{\lfloor n/2\rfloor} \tag{2.2}$$

Plugging (2.2) into (1.2),

$$\text{ASM}(n, -1) = lim_{q\to -1}\prod_{i=0}^{n-1}\frac{\lfloor\frac{3i+1}{2}\rfloor!(1+q)^{\lfloor\frac{3i+1}{2}\rfloor}}{\lfloor\frac{i+n}{2}\rfloor!(1+q)^{\lfloor\frac{i+n}{2}\rfloor}} \tag{2.3}$$

First we need to get rid of the limit.

Lemma 2.3. *$ASM(2m, -1)$ is non-zero and finite.*

Proof. It follows from 1.8 that $\text{ASM}(2m, -1)$ is finite.

Now, it suffices to show that $\sum_{i=0}^{2m-1}(\lfloor\frac{3i+1}{2}\rfloor - \lfloor\frac{i+2m}{2}\rfloor) = 0$.

$$\sum_{i=0}^{2m-1}(\lfloor\frac{3i+1}{2}\rfloor - \lfloor\frac{i+2m}{2}\rfloor) = \sum_{\substack{i=0\\ i \text{ even}}}^{2m-2}(i) + \sum_{\substack{i=1\\ i \text{ odd}}}^{2m-1}(i+1) - \sum_{i=0}^{2m-1}(m)$$
$$= \sum_{i=0}^{2m-1}(i-m) + m = 0$$

In fact, this gives an alternative proof that $\text{ASM}(2m, -1)$ is finite. □

We can now simplify (2.7) for $n = 2m$ [1].

$$\text{ASM}(2m, -1) = \prod_{i=0}^{2m-1}\frac{\lfloor\frac{3i+1}{2}\rfloor!}{\lfloor\frac{i+2m}{2}\rfloor!} \tag{2.4}$$

[1] In fact, if we replace $2m$ by n, the formula holds for all n.

Then,

$$\mathrm{ASM}(2m,-1)=\left(\prod_{\substack{i=0\\ i \text{ even}}}^{2m-2}\frac{\frac{3i}{2}!}{\frac{2m+i}{2}!}\right)\left(\prod_{\substack{i=1\\ i \text{ odd}}}^{2m-1}\frac{\frac{3i+1}{2}!}{\frac{2m+i-1}{2}!}\right)=\prod_{i=0}^{m-1}\frac{(3i)!(3i+2)!}{(m+i)!^2}$$

Kuperberg ([4]) proved that

$$|\mathrm{ASM}(n)_{HT}|=\prod_{i=0}^{m-1}\frac{(3i)!(3i+2)!}{(m+i)!^2} \tag{2.5}$$

So, we have $\mathrm{ASM}(2m,-1)=|\mathrm{ASM}(2m)_{HT}|$.

For the case $n=2m+1$, see [7].

(2) First, we present a simple lemma due to Robbins [10].

Lemma 2.4. *For $n=4m+2$, $|ASM(n)_{QT}|=0$.*

Proof. An ASM of size $4m+2\times 4m+2$ must have all of its entries sum to $4m+2$. If it is invariant under R_{90°, all four quadrants must have the same number of 1s and -1s. Therefore, the sums of the entries in the four quadrants must all be $m+1/4$. Obviously, this is not possible, so no such matrices exist. □

As in (1), we first need a basic fact about the q-factorial.

$$lim_{q\to i}n!_q=lim_{q\to i}\lfloor n/4\rfloor!(1+q+q^2+q^3)^{\lfloor n/4\rfloor} \tag{2.6}$$

Plugging in (2.6) into (1.2),

$$\mathrm{ASM}(n,\imath)=lim_{q\to i}\prod_{j=0}^{n-1}\frac{\lfloor\frac{3j+1}{4}\rfloor!(1+q+q^2+q^3)^{\lfloor\frac{3j+1}{4}\rfloor}}{\lfloor\frac{j+n}{4}\rfloor!(1+q+q+q^3)^{\lfloor\frac{j+n}{4}\rfloor}} \tag{2.7}$$

For $n=2m+2$, the exponent on $(1+q+q^2+q^3)$ is $\sum_{j=0}^{4m+1}\left(\lfloor\frac{3j+1}{4}\rfloor-\lfloor\frac{4m+j+2}{4}\rfloor\right)$

$$=\sum_{j=0}^{4m+1}(-m)+\sum_{\substack{j=0\\ j\equiv 0(4)}}^{4m}(\frac{j}{2})+\sum_{\substack{j=1\\ j\equiv 1(4)}}^{4m+1}(\frac{j+1}{2})+\sum_{\substack{j=2\\ j\equiv 2(4)}}^{4m-2}(\frac{j-2}{2})+\sum_{\substack{j=3\\ j\equiv 3(4)}}^{4m-1}(\frac{j-1}{2})$$

$$=\sum_{j=0}^{4m+1}(\frac{j}{2}-m)-m+\frac{3}{2}$$

$$=2$$

So, $\mathrm{ASM}(2m+2,\imath)=0$, which agrees with Lemma 2.4 .

For the case $n=2m+1$ see [8]. For $n=4m$ see [4].

□

Table 3.1. Gyration Orbits

n	Orbit Size	Number of Orbits
1	1	1
2	2	1
3	2	2
	3	1
4	2	3
	4	1
	8	4
5	2	4
	4	4
	5	5
	10	38
6	2	6
	4	16
	6	24
	10	4
	12	532
	24	20
	36	4
	84	2
7	2	12
	4	28
	6	20
	7	98
	8	4
	10	14
	12	12
	14	11,464
	21	10
	22	6
	28	476
	42	360
	56	56
	70	106
	98	80
	126	20
	154	20
	210	8
	266	8

3. Cyclic Sieving Phenomenon with Gyration

Our original hope was to find CSP with gyration itself and the obvious q-analogue of (1.1) .

Unfortunately, gyration does not exhibit the CSP as we'd hoped. See Table 3.1 for the orbits of gyration. Note that they get rather large, so they're not very suggestive of CSP with a simple polynomial. For example, when $n = 5$, $\mathrm{ASM}(5, -1) = 25$, but there are actually 413 ASMs fixed by $G^{|G|/2} = G^{10}$.

However, we did find CSP with smaller groups related to gyration.

Theorem 3.1. *The following maps exhibit the CSP with Equation* (1.2) *with orders two, two, and four respectively:*

(1) $G_{odd} \circ R_{180^\circ}$ *for all* n.
(2) $G_{even} \circ R_{180^\circ}$ *for odd* n.
(3) $G_{odd} \circ R_{90^\circ}$ *for even* n.

Proof. Let $G_i^{TL}, G_i^{TR}, G_i^{BL}$, and G_i^{BR}, be G_i applied only to the top left, top right, bottom right, bottom left and bottom right corners respectively (Note that we have defined these not to act on their boundary diagonals). Similarly, define G_i^M, G_i^A to be G_i applied only to the main diagonal and the antidiagonal respectively (see Figure 3.1).

We note the following basic relations on the G_i^j:

$$(G_i^j)^2 = 1 \tag{3.1}$$

$$G_i^j \circ R_{180^\circ} = R_{180^\circ} \circ G_i^j \tag{3.2}$$

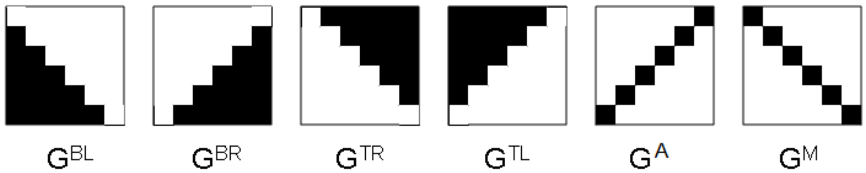

FIGURE 3.1. Operation Diagrams

$$G_i \circ R_{90^\circ} = R_{90^\circ} \circ G_i (n \text{ even}) \tag{3.3}$$

$$G_{even} \circ R_{90^\circ} = R_{90^\circ} \circ G_{odd} (n \text{ odd}) \tag{3.4}$$

$$G_i^j \circ G_i^k = G_i^k \circ G_i^j \tag{3.5}$$

$$G_i = G_i^{BL} \circ G_i^{TR} \circ G_i^M = G_i^{BR} \circ G_i^{TL} \circ G_i^A \tag{3.6}$$

$$G_{odd}^A = 1 \tag{3.7}$$

$$G_{odd}^M = 1 \ (n \text{ even}) \tag{3.8}$$

$$G_{even}^M = 1 \ (n \text{ odd}) \tag{3.9}$$

$$G_i^{TR} \circ R_{180^\circ} = R_{180^\circ} \circ G_i^{BL}, \quad G_i^{TL} \circ R_{180^\circ} = R_{180^\circ} \circ G_i^{BR} \tag{3.10}$$

$$R_{90^\circ} \circ G_i^{BL} \circ G_i^{BR} = G_i^{TR} \circ G_i^{BR} \circ R_{90^\circ} (n \ odd) \tag{3.11}$$

The orders of the maps in question follow trivially from (3.2) and (3.4). We can now prove Theorem 3.1.

(1) Suppose A is an ASM invariant under R_{180°. Then consider $A' = G_{odd}^{TL}(A)$.

$$\begin{aligned}
G_{odd} \circ R_{180^\circ}(A') &= G_{odd}^{BR} \circ G_{odd}^{TL} \circ G_{odd}^A \circ R_{180^\circ}(A') && (3.6)\\
&= G_{odd}^{BR} \circ G_{odd}^{TL} \circ R_{180^\circ}(A') && (3.7)\\
&= G_{odd}^{BR} \circ G_{odd}^{TL} \circ R_{180^\circ} \circ G_{odd}^{TL}(A) &&\\
&= (G_{odd}^{BR})^2 \circ G_{odd}^{TL} \circ R_{180^\circ}(A) && (3.10)\\
&= G_{odd}^{TL}(A) && (3.1)\\
&= A'
\end{aligned}$$

Since G_{odd}^{TL} is an involution, it must define a bijection between the fixed points of R_{180° and $G_{odd} \circ R_{180^\circ}$, and since $G_{odd} \circ R_{180^\circ}$ has order two, we are done.

(2) Suppose A is an $n \times n$ ASM invariant under R_{180° and n is odd. Then consider $A' = G^{TR}_{even}(A)$.

$$G_{even} \circ R_{180^\circ}(A') = G^{TR}_{even} \circ G^{BL}_{even} \circ G^{M}_{even} \circ R_{180^\circ}(A') \quad (3.6)$$

$$= G^{TR}_{even} \circ G^{BL}_{even} \circ R_{180^\circ}(A') \quad (3.9)$$

$$= G^{TR}_{even} \circ G^{BL}_{even} \circ R_{180^\circ} \circ G^{TR}_{even}(A)$$

$$= (G^{BL}_{even})^2 \circ G^{TR}_{even} \circ R_{180^\circ}(A) \quad (3.10)$$

$$= G^{TR}_{even}(A) \quad (3.1)$$

$$= A'$$

Since G^{TR}_{even} is an involution, it must define a bijection between the fixed points of R_{180° and $G_{even} \circ R_{180^\circ}$, and since $G_{even} \circ R_{180^\circ}$ has order two, we are done.

(3) Suppose A is an $n \times n$ ASM invariant under R_{90° and n is even. Then consider $A' = G^{BL}_{odd} \circ G^{BR}_{odd}(A)$.

$$G_{odd} \circ R_{90^\circ}(A') = G^{TR}_{odd} \circ G^{BL}_{odd} \circ G^{M}_{odd} \circ R_{90^\circ}(A') \quad (3.6)$$

$$= G^{TR}_{odd} \circ G^{BL}_{odd} \circ R_{90^\circ}(A') \quad (3.8)$$

$$= G^{TR}_{odd} \circ G^{BL}_{odd} \circ R_{90^\circ} \circ G^{BL}_{odd} \circ G^{BR}_{odd}(A)$$

$$= (G^{TR}_{odd})^2 \circ G^{BL}_{odd} \circ G^{BR}_{odd} \circ R_{90^\circ}(A) \quad (3.11)$$

$$= G^{BL}_{odd} \circ G^{BR}_{odd}(A) \quad (3.1)$$

$$= A'$$

Since $G^{BL}_{odd} \circ G^{BR}_{odd}$ is an involution, it must define a bijection between the fixed points of R_{90° and $G_{odd} \circ R_{90^\circ}$. Also, note that for n even, $(G_{odd} \circ R_{90^\circ})^2 = R_{180^\circ}$. So $(G_{odd} \circ R_{90^\circ})^2$ trivially has the same fixed points as $(R_{90^\circ})^2 = R_{180^\circ}$. Since $G_{odd} \circ R_{90^\circ}$ has order four for n even, we are done.

□

Note that these proofs fail for the cases not listed in the theorem. For example, $G_{even} \circ R_{180^\circ}$ does not exhibit the CSP for even n due to the fact that squares on both the main and antidiagonals are even.

Table 4.1. Structure of GS_n

n	order	structure
1	1	C_1
2	2	C_2
3	6	S_3
4	24	S_4
5	$8640 = 2^6 * 3^3 * 5$	$C_2 \times C_2 \times ((C_3 \times (GL_2(4) \rtimes C_2)) \rtimes C_2)$
6	$16639583300553277440 = 2^{33} * 3^{18} * 5$	Not Available

4. The Generalized Symmetric Group

The $n \times n$ ASMs are a generalization of the $n \times n$ permutation matrices, so it seems natural to define a group of operations on the ASMs that generalizes the symmetric group.

Recall that the symmetric group is generated by elements $(i, i+1)$ which swap the ith and $(i+1)$th rows of a matrix.

We define the generalized symmetric group of size n (GS_n) to be the group generated by the maps $[i, i+1]$ $1 \leq i < n$, defined as follows on an ASM A:

$$[i,\ i+1](A) = \begin{cases} (i,\ i+1)(A) & \text{if } (i, i+1)(A) \text{ is an ASM} \\ A & \text{otherwise} \end{cases}$$

This group is significantly more complicated than the symmetric group. First note that if we define $[1,\ 3]$ in the obvious way, $[1,\ 2][2,\ 3][1,\ 2] \neq [1,\ 3]$. In fact, the element $[1,\ 3]$ is not in GS_n for $n > 4$, so one can consider an even larger generalization that contains all

$$[x_1,\ x_2,\ x_3, \dots\ x_m] = \begin{cases} (x_1,\ x_2,\ x_3, \dots\ x_m)(A) & \text{if } (x_1,\ x_2,\ x_3, \dots\ x_m)(A) \text{ is an ASM} \\ A & \text{otherwise} \end{cases}$$

Question 4.1. Does there exist an element $X = [x_1,\ x_2,\ x_3,\ \dots\ x_m]$ such that $X \in \mathrm{GS}_n$ for all sufficiently large n?

Worse still, $[1,\ 2][2,\ 3][1,\ 2] \neq [2,\ 3][1,\ 2][2,\ 3]$ for $n > 4$. Because of facts like this, the group is much much larger than the symmetric group (see table 4.1). However, the group's order is divisible by very small primes, suggesting the following questions.

Question 4.2. What is the order of the GS_n for all n? Does there exist a simple product formula that gives this order for any n?

Question 4.3. What is the structure of GS_n for all n?

$\mathrm{GS}(n)$ also suggests a natural partition on $\mathrm{ASM}(n)$.

Definition 4.4. For $v = (v_1, v_2, \dots v_n) \in \mathbb{Z}^n$, let $\mathrm{ASM}(n)_v = \{A \in \mathrm{ASM}(n) : A$ has $v_i\ -1$s in its ith column $\}$.

Question 4.5. What is $|\mathrm{ASM}(n)_v|$ for each v.

Question 4.6. What is the order and structure of GS_n restricted to $\mathrm{ASM}(n)_v$ for each v.

We also discovered a subset of GS_n that exhibits a very nice property.

Definition 4.7. For $\sigma \in \mathrm{GS}_n$, we say that σ is a *palindrome* iff

$$\sigma \circ R_{180^\circ} = R_{180^\circ} \circ \sigma^{-1} \tag{4.1}$$

In some sense, these maps act on the top of the matrix as they do on the bottom. I.e., they can be written as

$[x_1,\ x_1{+}1][x_2,\ x_2{+}1]\dots[x_m,\ x_m{+}1][n{-}x_m,\ n{-}x_m{+}1]\dots[n{-}x_2,\ n{-}x_2{+}1][n{-}x_1,\ n{-}x_1{+}1]$.

To better understand palindromes, we introduce an obvious group isomorphic to GS_n also acting on $\mathrm{ASM}(n)$.

Definition 4.8. *The column generalized symmetric group of size* n $(\mathrm{GS}_{n,C})$ is the group generated by elements $[i, i{+}1]_C$ that permute the columns of $\mathrm{ASM}(n)$ exactly as the generators of GS_n permute the rows.

Theorem 4.9. *If $\sigma \in GS_n$ is a palindrome and $m, k \in \mathbb{Z}$, then*

1. *For $n > 1$, $\sigma^k \circ \sigma_C^{2m-k} \circ R_{180^\circ}$ has order 2 and exhibits the CSP with ASM(n,q)*
2. *For $n > 2$, $\sigma^k \circ \sigma_C^{2m-k} \circ R_{90^\circ}$ has order 4 and exhibits the CSP with ASM(n,q).*

First note some basic facts. For a $\sigma \in \mathrm{GS}_n$,

$$\sigma_C = R_{90^\circ} \circ \sigma \circ R_{270^\circ} \tag{4.2}$$

$$\sigma \circ \sigma_C = \sigma_C \circ \sigma \tag{4.3}$$

$$R_{90^\circ} \circ \sigma = \sigma_C \circ R_{90^\circ},\ R_{90^\circ} \circ \sigma_C = \sigma^{-1} \circ R_{90^\circ} \text{ (iff } \sigma \text{ a palindrome)} \tag{4.4}$$

Next we prove a basic lemma.

Lemma 4.10. *If $\sigma \in GS_n$ is a palindrome and $m, k \in \mathbb{Z}$, then*

1. *$\sigma^k \circ \sigma_C^{2m-k} \circ R_{180^\circ}$ is an involution.*
2. *$\sigma^k \circ \sigma_C^{2m-k} \circ R_{90^\circ}$ has order dividing four.*

Proof.

(1)

$$\begin{aligned}
&\left(\sigma^k \circ \sigma_C^{2m-k} \circ R_{180^\circ}\right) \circ \left(\sigma^k \circ \sigma_C^{2m-k} \circ R_{180^\circ}\right) \\
&= \sigma^k \circ \sigma_C^{2m-k} \circ \sigma^{-k} \circ \sigma_C^{k-2m} && (4.1) \\
&= 1 && (4.3)
\end{aligned}$$

(2)

$$\begin{aligned}&\left(\sigma^k \circ \sigma_C^{2m-k} \circ R_{90^\circ}\right) \circ \left(\sigma^k \circ \sigma_C^{2m-k} \circ R_{90^\circ}\right)\\ &= \sigma^k \circ \sigma_C^{2m-k} \circ \sigma_C^{k} \circ \sigma^{k-2m} \circ R_{180^\circ} && (4.2)\\ &= \sigma^{2m} \circ \sigma_C^{2m} \circ R_{180^\circ} && (4.3)\end{aligned}$$

But, by (1), this is an involution. Therefore the original map has order dividing four.

□

Now, the theorem will follow from the next lemma.

Lemma 4.11. *If $\sigma \in GS_n$ is a palindrome and $m \in \mathbb{Z}$, then*

(1) *For $n > 1$ $\sigma^{2m} \circ R_{180^\circ}$ has order 2 and exhibits the CSP with ASM(n,q)*

(2) *For $n > 2$ $\sigma^{2m} \circ R_{90^\circ}$ has order four and exhibits the CSP with ASM(n,q).*

Proof.

(1) Suppose A is an ASM invariant under R_{180°. Consider $A' = \sigma^m(A)$.

$$\begin{aligned}&\sigma^{2m} \circ R_{180^\circ}(A') = \sigma^{2m} \circ R_{180^\circ} \circ \sigma^m(A)\\ &= \sigma^m(A) && (4.1)\\ &= A'\end{aligned}$$

So, A' is invariant under $\sigma^{2m} \circ R_{180^\circ}$. But σ^m has an inverse, so it defines a bijection. So, $\sigma^{2m} \circ R_{180^\circ}$ has the same number of fixed points as R_{180°, and by Lemma 4.10 it is an involution and therefore has CSP with ASM(n, q).

(2) Suppose A is an ASM invariant under R_{90°. Then consider $A' = \sigma^m \sigma_C^m(A)$.

$$\begin{aligned}&\sigma^{2m} \circ R_{90^\circ}(A') = \sigma^{2m} \circ R_{90^\circ} \circ \sigma^m \sigma_C^m(A)\\ &= \sigma^{2m} \circ \sigma_C^m \circ \sigma^{-m}(A) && (4.4)\\ &= \sigma^m \circ \sigma_C^m(A) && (4.3)\\ &= A'\end{aligned}$$

So, A' is invariant under $\sigma^{2m} \circ R_{90^\circ}$. Now, suppose B is an ASM invariant under R_{180°. Then consider $B' = \sigma^m \sigma_C^m(B)$

$$\begin{aligned}&(\sigma^{2m} \circ R_{90^\circ})^2(B') = \sigma^{2m} \circ R_{90^\circ} \circ \sigma^{2m} \circ R_{90^\circ}(B')\\ &= \sigma^{2m} \circ \sigma_C^{2m} \circ R_{180^\circ}(B') && (4.4)\\ &= \sigma^{2m} \circ \sigma_C^{2m} \circ R_{180^\circ} \sigma^m \sigma_C^m(B)\\ &= \sigma^{2m} \circ \sigma_C^{2m} \circ \sigma^{-m} \sigma_C^{-m}(B) && (4.1)\\ &= \sigma^m \sigma_C^m(B) && (4.3)\\ &= B'\end{aligned}$$

So, B' is invariant under $\left(\sigma^{2m} \circ R_{90^\circ}\right)^2$. Since the map $\sigma^m \sigma_C^m$ has an inverse, $\sigma_C^{-m} \sigma^{-m}$, it forms a bijection from the fixed points of R_{90° and R_{180° to those of $\sigma^{2m} \circ R_{90^\circ}$ and $\left(\sigma^{2m} \circ R_{90^\circ}\right)^2$ respectively.

We still need to prove that $\alpha = \sigma^{2m} \circ R_{90^\circ}$ has order four. Since the fixed points of α^2 are in bijection with those of R_{180° and for $n > 2$, R_{180° is not the identity. Then by Lemma 4.10, $|\alpha| = 4$.

□

We can now prove Theorem 4.9.

Proof.

(1) Suppose k is even, then 2m-k is even, and

$$\sigma^k \circ \sigma_C^{2m-k} \circ R_{180^\circ} = \sigma^k \circ \sigma_C^{(2m-k)/2} \circ R_{180^\circ} \circ \sigma_C^{(k-2m)/2} \quad (4.1)$$
$$= \sigma_C^{(2m-k)/2} \circ \sigma^k \circ R_{180^\circ} \circ \sigma_C^{(k-2m)/2} \quad (4.3)$$

Therefore $\sigma^k \circ \sigma_C^{2m-k} \circ R_{180^\circ}$ is just a conjugate of $\sigma^k \circ R_{180^\circ}$, so it has the correct order and exhibits CSP with ASM(n,q) by Lemma 4.11.

Now suppose k is odd. Then clearly $\sigma^{-1} \circ \sigma_C^{-1}$ defines a bijection from the fixed points of $\alpha = \sigma^k \circ \sigma_C^{2m-k} \circ R_{180^\circ}$ and those of $\sigma^{k+1} \circ \sigma_C^{2m-k-1} \circ R_{180^\circ}$. By Lemma 4.10, α has the correct order. Therefore, $\forall k \in \mathbb{Z}$, $\sigma^k \circ \sigma_C^{2m-k} \circ R_{180^\circ}$ has order two and exhibits the CSP with ASM(n, q).

(2) Similarly, $\forall k \in \mathbb{Z}$,

$$\sigma^k \circ \sigma_C^{2m-k} \circ R_{90^\circ} = \sigma^{-2m} \circ \sigma_C^{2m-k} \circ \sigma^{k-2k} \circ R_{90^\circ}$$
$$= \sigma^{-2m} \circ \sigma_C^{2m-k} \circ R_{90^\circ} \circ \sigma_C^{k-2k} \quad (4.4)$$

So $\sigma^k \circ \sigma_C^{2m-k} \circ R_{90^\circ}$ is a conjugate of σ^{-2m}, and therefore has the correct order and exhibits the CSP with ASM(n,q) by Lemma 4.11.

□

5. A Set of ASMs on which Gyration is Well-Behaved

Recall that in Section 3 we showed that gyration does not exhibit CSP with ASM(n, q) and we noted that the orbit structure of gyration seems much too complex to exhibit CSP with a simple polynomial.

Definition 5.1. A *solid (dotted) pair* in an FPL is an ordered pair (i, j) such that a solid (dotted) line connects the boundary line i to boundary line j (labelled as in Figure 1.3). π_S (π_D) is the set of all solid (dotted) line pairings on some FPL.

Definition 5.2. An *adjacent pair* is a pair in an FPL linking i to $i+1$ or 1 to $2n$.

For reasons best explained in [3], the ASMs with more adjacent loops tend to be in higher orbits of gyration. So, we choose to look at a smaller set of ASMs. To motivate this, we first present a result of Wieland [12] .

Theorem 5.3. *Let $A_n(\pi_S, \pi_D)$ be the set of ASMs of order n in which the solid line subgraph induces pairing π_S, the dotted line subgraph induces pairing π_D. If π'_S is π_S rotated clockwise, and π'_D is π_D rotated counterclockwise, then the sets $A_n(\pi_S, \pi_D)$ and $A_n(\pi'_S, \pi'_D)$ are in bijection.*

Proof. Gyration defines the bijection between these two sets. See Wieland [12] for a proof. □

In particular, FPLs with the same number of solid adjacent pairs are closed under gyration by Theorem 5.3. Also, FPLs with fewer adjacent loops tend to be in smaller orbits of gyration (see [3] for more detail). This motivates the next definition.

Definition 5.4. $\mathrm{ASM}(n)_k = \{A \in \mathrm{ASM}(n) : A$ has at most $n-k$ solid adjacent pairs$\}$.

Propp and Wilson conjectured that $|\mathrm{ASM}(n)_1| = |\mathrm{ASM}(n)| - 2|\mathrm{ASM}(n-1)|$ [6]. So, we define $\mathrm{ASM}(n,q)_1 = \mathrm{ASM}(n,q) - (1+q^n)\mathrm{ASM}(n-1,q)$

We found this surprising result.

Theorem 5.5. *For $n \leq 5$, gyration restricted to $ASM(n)_1$ exhibits the CSP with the polynomial $ASM(n,q)_1$.*

Proof. Computation. (See Table 5.1) □

Unfortunately, Theorem 5.5 does not generalize simply for n = 6 or 7. However, the orbit structures of gyration on $\mathrm{ASM}(6)_2$, $\mathrm{ASM}(6)_3$, and $\mathrm{ASM}(7)_3$ are very suggestive of CSP (see Table 5.2). This leads to a natural question.

Question 5.6. Does there exist a simple function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ and set of polynomials $\mathrm{ASM}(n,q)_k$ such that gyration restricted to $\mathrm{ASM}(n)_{f(n)}$ exhibits the CSP with $\mathrm{ASM}(n,q)_{f(n)}$? Is the order of the gyration restricted to $\mathrm{ASM}(n)_{f(n)}$ always $2n$ for $n > 3$?

If such a set of functions and polynomials were found, it would prove (or disprove) many of the open conjectures in [14].

TABLE 5.1. Gyration Orbits on $\mathrm{ASM}(n)_1$

n	Orbit Size	Number of Orbits
3	3	1
4	4	1
	8	3
5	5	5
	10	32

TABLE 5.2. Gyration Orbits on $\mathrm{ASM}(6)_2$ and $\mathrm{ASM}(7)_3$

Set	Orbit Size	Number of Orbits
$\mathrm{ASM}(6)_2$	4	5
	6	15
	12	242
$\mathrm{ASM}(6)_3$	4	5
	6	1
	12	30
$\mathrm{ASM}(7)_3$	7	35
	14	1936

6. ASM(n,q) at a Third Root of Unity

Theorem 6.1. *$|ASM(n, e^{2\pi\imath/3})| \in \mathbb{Z}$ for all n.*

Proof. Let $\omega = e^{2\pi\imath/3}$ and $\omega_6 = e^{\pi\imath/3}$. Then note the following relations.

$$lim_{q\to\omega}[n]_q = \begin{cases} lim_{q\to\omega}(1+q+q^2)(n/3)! & \text{if } n \equiv 0\ (3) \\ 1 & \text{if } n \equiv 1\ (3) \\ 1+\omega = \omega_6 & \text{if } n \equiv 2\ (3) \end{cases} \tag{6.1}$$

$$lim_{q\to\omega} n!_q = lim_{q\to\omega}(1+q+q^2)^{\lfloor \frac{n}{3} \rfloor} \lfloor \frac{n}{3} \rfloor ! (\omega_6)^{\lfloor \frac{n+1}{3} \rfloor} \tag{6.2}$$

Lemma 6.2. *ASM(n,ω) is non-zero and finite.*

Proof. Since ASM$(n,q) \in \mathbb{N}[q]$ by Corollary 1.8, ASM(n,ω) is finite.

The proof that it is non-zero is straight-forward but tedious. It is essentially the same as the proof of Lemma 2.3. We don't require this fact for our main theorem, so we omit the proof. □

Plugging in (6.2) to (1.2) and noting that Lemma 6.2 allows us to ignore the $lim_{q\to\omega}(1+q+q^2)^i$ term, we have a nice formula for ASM(n,ω).

$$\mathrm{ASM}(n,\omega) = \prod_{i=0}^{n-1} \frac{i!(\omega_6)^i}{\lfloor \frac{n+i}{3} \rfloor ! (\omega_6)^{\lfloor \frac{n+1+i}{3} \rfloor}} \tag{6.3}$$

Lemma 6.3. *$|ASM(n,\omega)|$ is either integral or irrational.*

Proof. Since ASM(n,q) is a polynomial in $\mathbb{N}[q]$ by Corollary 1.8, ASM$(n,\omega) \in \mathbb{Z}[\omega]$. Note that $|a + b\omega| = \sqrt{a^2 - ab + b^2}$. If $a, b \in \mathbb{Z}$, then $|a+b\omega| \in \mathbb{Z} \bigcup (\mathbb{R} - \mathbb{Q})$. □

By taking the absolute value of (6.3), we get

$$|\mathrm{ASM}(n,\omega)| = \prod_{i=0}^{n-1} \frac{i!}{\lfloor \frac{n+i}{3} \rfloor !} \tag{6.4}$$

This is clearly rational, so by Lemma 6.3, it is an integer. □

See Table 6.1 for the values for small n. Note that they are typically very small compared to ASM(n). This suggests the following question.

Question 6.4. Is there a map of order three on ASM(n) that exhibits CSP with $|\mathrm{ASM}(n,q)|$?

TABLE 6.1. Values of $|\mathrm{ASM}(n,q)|$ at a Third Root of Unity

n	$\lvert\mathrm{ASM}(n,\omega)\rvert$	$\lvert\mathrm{ASM}(n)\rvert$
1	1	1
2	1	2
3	2	7
4	3	42
5	6	429
6	20	7436
7	50	218,348
8	175	10,850,216

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