A SCHUR NON-NEGATIVITY CONJECTURE RELATED TO DOUBLE-WIRING DIAGRAMS

CESAR CUENCA

ABSTRACT. We make an explicit combinatorial construction of the cluster algebra arising from a double wiring diagram. We also state a Schur non-negativity conjecture and prove it is true for small cases.

1. INTRODUCTION

Fomin and Zelevinski defined *double wiring diagrams* in [6]. A n-stringed double wiring diagram w consists of two families of n piecewise straight lines colored blue and red, such that each two lines of the same color intersect exactly once. The lines in each family are assigned numbers from $[n] = \{1, 2, \ldots, n\}$. The blue (resp. red) lines have increasing labels if we look at their left (resp. right) ends from top to bottom. Any such diagram w is divided into *chambers*. All chambers are assigned two subsets of $[n]$, each of which indicates which lines of the corresponding color pass below that chamber. A chamber with subsets I, J is associated the minor $\Delta_{I,J}$ of $X = (x_{ij})_{n\times n}$. These are called the *chamber minors* of w; the set of chamber minors of w is denoted as $\mathbf{z} = \mathbf{z}(w)$. For the example in Figure 1, the chamber minors are $1, x_{31}, x_{21}, x_{23}, x_{13}, \Delta_{23,12}, \Delta_{23,13}, \Delta_{12,13}, \Delta_{12,23}$ and $\Delta_{123,123} = \Delta$.

A double wiring diagram w can be associated a *quiver* $Q = Q(w)$. The associated *initial seed* $(Q(w), \mathbf{z}(w))$ then gives rise to a *cluster algebra* $\mathcal{A} = \mathcal{A}(w)$, a commutative ring with a special combinatorial structure (precise definitions are given in Section 2). The generators of the cluster algebra $\mathcal{A}$ are divided into clusters and are called cluster variables. A remarkable property of the cluster algebras is that each cluster variable of $\mathcal{A}$ can be expressed as a Laurent polynomial in the variables of the initial seed. Even more remarkable is that, for this special cluster algebra, each generator of $\mathcal{A}$ is in the (integer) polynomial ring on the variables of X, i.e., $\mathbb{Z}[\{x_{ij}\}_{n\times n}]$.

A *cluster monomial* is a finite product of cluster variables of the same cluster. It is conjectured that cluster monomials belong to the *dual canonical basis*. More than a conjecture, this was actually a motivating reason for the development of cluster algebras. In [7], Haiman showed, in a more general setting, that the dual canonical basis of $\mathcal{A}$ are Schur non-negative when evaluated in generalized Jacobi-Trudi matrices. More

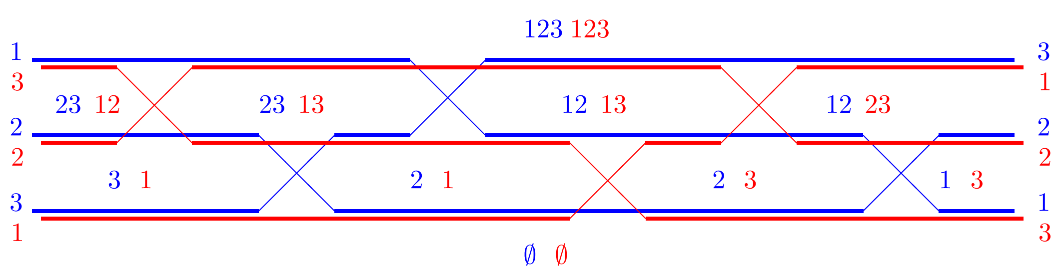


FIGURE 1. A double wiring diagram w for $n = 3$

recently, Lam and Rietsch explored similar ideas in [9], but with Stake basis instead of dual canonical basis and with k-Schur functions instead of Schur functions.

Observe that cluster variables are also cluster monomials. Therefore, if the conjecture above was true, Haiman's result would prove that all cluster variables in $\mathcal{A}$ are Schur non-negative when evaluated in generalized Jacobi-Trudi matrices. This is still an open problem and would be interesting to prove independently of Haiman's theorem.

Conjecture 1.1. *Let $\mathcal{A}$ be a cluster algebra generated by a double wiring diagram w. Any cluster variable of $\mathcal{A}$ is Schur non-negative when evaluated in generalized Jacobi-Trudi matrices.*

Instead of trying to prove Conjecture 1.1 in the general scenario, we restrict ourselves to a particular type of cluster variable. A cluster variable is said to be *Plucker-adjacent* if it is comes from one *seed mutation* to $(Q(w), \mathbf{z}(w))$ for some w. We propose the following specialization of Conjecture 1.1.

Conjecture 1.2. *Let $\mathcal{A}$ be a cluster algebra generated by a double wiring diagram w. Any Plucker-adjacent cluster variable of $\mathcal{A}$ is Schur non-negative when evaluated in generalized Jacobi-Trudi matrices.*

Proving this conjecture alone would be very interesting and provide us with general Schur positivity theorems. We prove the conjecture for small chambers.

Theorem 1.3. *Let c be a chamber in a double wiring diagram w with degree 6 or less in its associated quiver $Q = Q(w)$. Then $\Omega_w(c)$ is Schur non-negative when evaluated in generalized Jacobi-Trudi matrices.*

This report is organized as follows. The next section gives the necessary background on symmetric functions and cluster algebras. In section 3, we show some expressions for Plucker-adjacent cluster variables and prove a theorem for determinantal identities. In sections 4, we prove a general theorem for reducing Conjecture 1.2 to the case where w is as small as possible. In section 5, we introduce Temperley-Lieb immanants, a technique for proving Schur non-negativity. In section 6, we prove Theorem 1.3. In the last section, we show some examples of cluster variables that are not Plucker-adjacent.

2. Background on Symmetric functions and Cluster Algebras

We provide some background on symmetric functions and cluster algebras, as needed for this report. The interested reader can find more extensive expositions in [4], [3] and Ch. 7 of [10].

2.1. Symmetric functions.

The ring of symmetric polynomials has several important linear bases. We consider its basis of Schur polynomials $s_\lambda = \sum_T x^T$ and its basis of complete homogeneous symmetric polynomials $h_k = \sum_{1 \le i_1 \le \ldots \le i_k \le n} x_{i_1} \ldots x_{i_k}$. A symmetric polynomial f is said to be *Schur non-negative* if it can be expressed as a non-negative linear combination of Schur polynomials. We write $f \geq_s 0$.

For example, any skew-Schur polynomial $s_{\mu/\nu}$ is Schur non-negative and can be expressed as $s_{\mu/\nu} = \sum_\lambda c^{\mu}_{\nu,\lambda} s_\lambda$, where the coefficient $c^{\mu}_{\nu,\lambda}$ counts the number of *Littlewood-Richardson tableaux of shape μ/ν and weight λ*. In particular, they are non-negative integers. The product of two Schur non-negative polynomials is also Schur non-negative because of the Littlewood-Richardson rule $s_\lambda \cdot s_\mu = \sum_\nu c^{\nu}_{\lambda,\mu} s_\nu$. Moreover, any positive linear combination of Schur non-negative polynomials is also Schur non-negative. This gives the set of Schur non-negative polynomials the structure of a linear cone.

A *generalized Jacobi-Trudi matrix* is a $n \times n$ matrix of the form $H = (h_{\mu_i - \nu_j})$ for some integer partitions $\mu = (\mu_1 \geq \ldots \geq \mu_n \geq 0)$, $\nu = (\nu_1 \geq \ldots \geq \nu_n \geq 0)$. In the literature, the homogeneous polynomials h_k in

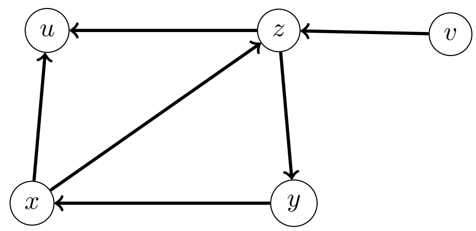


FIGURE 2. A quiver Q with 5 nodes

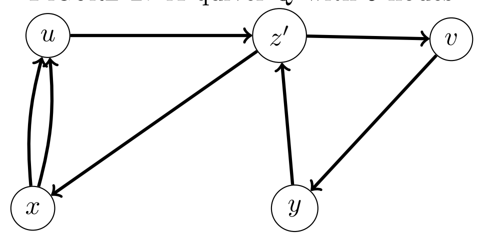


FIGURE 3. Example of quiver mutation $Q' = \mu_z(Q)$. Vertices u, v are frozen.

the matrix H are on the variables $x_1, \ldots, x_n$, but we consider that they can be on any number $m \geq n$ of variables $x_1, \ldots, x_m$. A polynomial F on the variables of a $n \times n$ matrix is *Schur non-negative when evaluated in generalized Jacobi-Trudi matrices* if $F(H)$ is Schur non-negative for any generalized Jacobi-Trudi matrix H. For simplicity, we simply say F is Schur non-negative and write $F \geq_s 0$. We also write $F \geq_s G$ if $F - G \geq_s 0$. The basic example of Schur non-negative polynomial in this context is any minor $\Delta_{I,J}$ of X. Indeed, the *generalized Jacobi-Trudi identity* implies $F(H) = det((h_{\mu_i - \nu_j})_{i \in I, j \in J}) = s_{\overline{\mu}/\overline{\nu}}$, where $\overline{\mu} = \{\mu_i\}_{i \in I}$ and $\overline{\nu} = \{\nu_j\}_{j \in J}$. From the previous remarks, any positive linear combination of products of minors is Schur non-negative as well.

2.2. **Cluster algebras.** A cluster algebra is a commutative ring with a special combinatorial structure. This combinatorial structure is given by a *quiver* Q, which is a directed multigraph with no loops and no 2-cycles. Some vertices of Q are denoted as *mutable* and the remaining ones are called *frozen*.

Definition 2.1. Let v be a mutable vertex of quiver Q. The *quiver mutation* of Q at v is an operation that produces another quiver $Q' = \mu_v(Q)$ via a sequence of three steps.

(1) Add a new edge $u \to w$ for each pair of edges $u \to v$, $v \to w$ in Q, except in the case when both u, w are frozen.
(2) Reverse the direction of all edges adjacent to v.
(3) Remove 2-cycles until none remains.

Definition 2.2. Let $\mathcal{F} \supset \mathbb{R}$ be any field. We say that pair $t = (Q, \mathbf{z})$ is a *seed* in $\mathcal{F}$ if Q is a quiver and $\mathbf{z}$, called the *extended cluster*, is a set consisting of algebraically independent elements of $\mathcal{F}$, as many as vertices of the quiver Q.

The elements of $\mathbf{z}$ corresponding to mutable vertices are called *cluster variables* and the ones corresponding to frozen vertices are called *frozen variables*.

A *seed mutation* at a cluster variable z transforms $t = (Q, z)$ into a new seed $t' = (Q', \mathbf{z}') = \mu_z(Q, \mathbf{z})$, where Q' is the cluster resulting after mutating Q at the vertex corresponding to z and $\mathbf{z}' = \mathbf{z} \cup \{z'\} \setminus \{z\}$, where the new variable z' is subject to the *mutation relation at z* $zz' = \prod_{z \leftarrow x} x + \prod_{z \rightarrow y} y$. We sometimes call it the *mutation relation at v* if v is the vertex of Q corresponding to z.

Starting from a seed $t = (Q, \mathbf{z})$, one can apply mutations ad infinitum to elements of the extended clusters. The elements in the union of all extended clusters are the generators of a subring of $\mathcal{F}$ that we call the cluster algebra $\mathcal{A}(Q, \mathbf{z})$ generated by seed $(Q, \mathbf{z})$. If $t' = (Q', \mathbf{z}')$ is a mutation of seed t, then $\mathcal{A}(Q, \mathbf{z})$ and $\mathcal{A}(Q', \mathbf{z}')$ are isomorphic. We bundle these facts into the next definition.

Definition 2.3. Two seeds $t = (Q, \mathbf{z})$ and $t' = (Q', \mathbf{z}')$ are said to be *mutation-equivalent* if one of them can be obtained from the other after a sequence of seed mutations. The *cluster algebra* $\mathcal{A}(Q, \mathbf{z})$ generated by an *initial seed* $t = (Q, \mathbf{z})$ is the subring of $\mathcal{F}$ generated by the elements of extended clusters that are mutation-equivalent to t.

2.2.1. *Quiver associated to a double-wiring diagram.* We describe how to associate a quiver $Q = Q(w)$ to any double wiring diagram w. This construction is described in a more general scenario in Subsection 2.2. of [1]. The construction provided here is purely combinatorial.

(1) The vertices of Q are the chambers of w.
(2) There is an edge between two chambers c and c' of Q in the following cases.
   (a) They are adjacent chambers in the same row. If the color of the crossing between them is blue, the edge is directed to the left. Otherwise, it is directed to the right.
   (b) If c' has ends of different color and lies completely above (or below) c. If the left end of c' is blue, the edge is directed from c to c'. Otherwise, it is directed from c' to c.
   (c) If the left end of c' is above c and the right end of c is below c' and both ends have the same color. If such common color is blue, the edge is directed from c to c'; otherwise, it is directed from c' to c.
   (d) If the right end of c' is above c and the left end of c is below c' and both ends have the same color. If such common color is blue, the edge is directed from c' to c; otherwise, it is directed from c to c'.

The mutable vertices of $Q(w)$ are the chambers in w that have two ends. Then the frozen vertices are the chambers with only one end, as well as the top and bottom chambers. Figure 2.2.1 shows the quiver that is associated to the double wiring diagram in Figure 1. It is important to observe that mutable chambers have at least 4 adjacent edges and they have equal indegree and outdegree. Moreover, incoming edges interlace with outgoing edges for each mutable vertex if we go around it.

Example 2.4. *Consider the chamber c in Figure 5. Its associated quiver has* 10 *vertices adjacent to it. Moreover, the incoming and outgoing edges interlace with each other as shown in Figure 2.4.*

2.2.2. *Cluster algebra associated to a double wiring diagram.* The cluster algebra $\mathcal{A} = \mathcal{A}(w)$ associated to the double wiring diagram w is the cluster algebra generated by the initial seed $t = (Q, \mathbf{z})$, where $Q = Q(w)$ is the quiver asssociated to w and the extended cluster $\mathbf{z}$ consists of the chamber minors of w. Naturally, the variable $\Delta_{I,J}$ is associated to chamber c if I and J are the sets of indices of blue and red wires below c, respectively.

At this point, it is worth to clarify the statement of Conjecture 1.2. Let us fix a double wiring diagram w with Q and $\mathcal{A}$ as its associated quiver and cluster algebra. A Plucker-adjacent cluster variable Ω is one that comes from mutating the initial seed at some cluster variable $\Delta_{I,J}$, corresponding to chamber c. We write

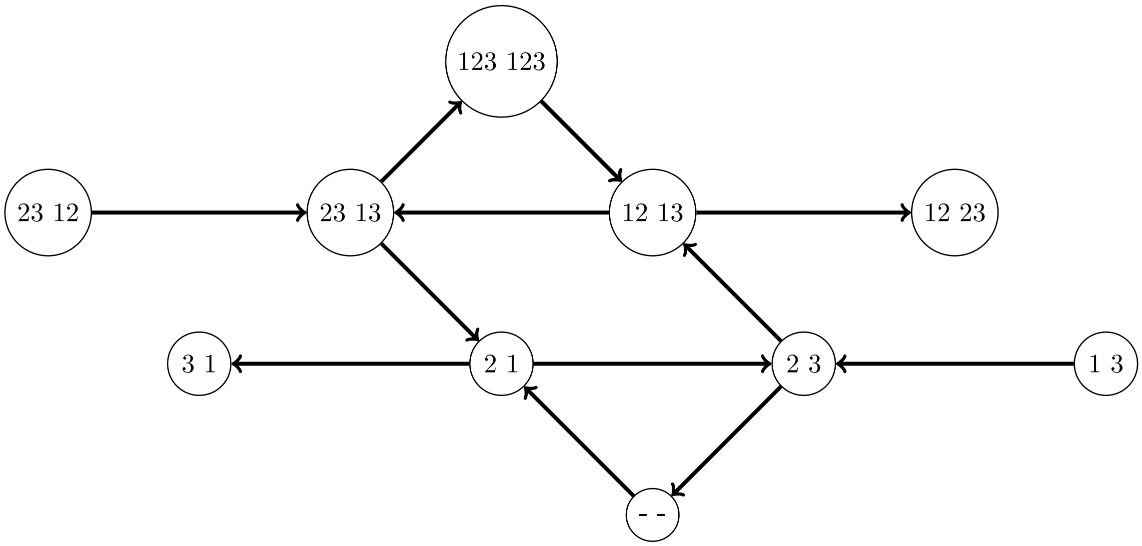


FIGURE 4. Quiver associated to double wiring diagram w in Figure 1. The mutable vertices are the ones associated to $\Delta_{23,13}$, $\Delta_{12,13}$, x_{21} and x_{23}.

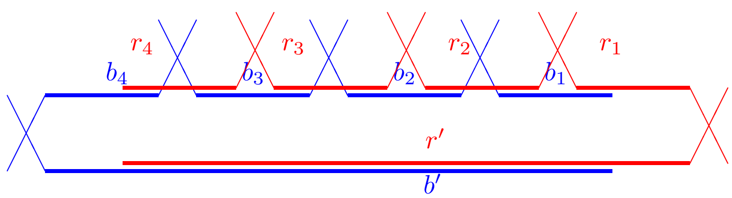


FIGURE 5. Chamber c with blue left end and red right end

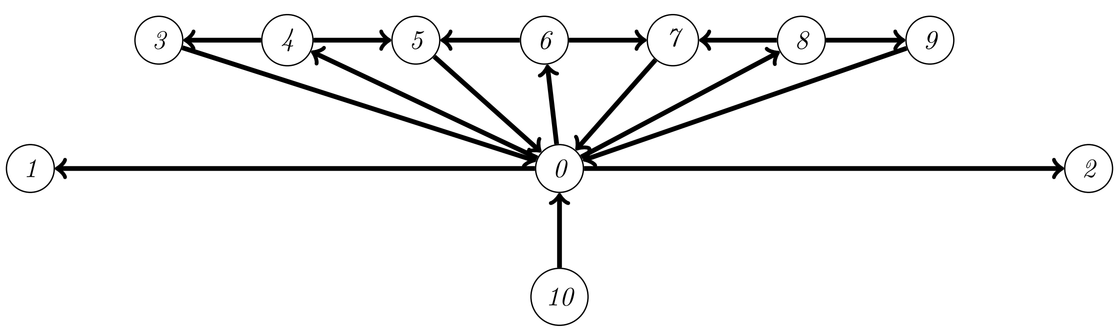


FIGURE 6. Part of the quiver of double wiring diagram containing c in Figure 5

the respective Plucker-adjacent cluster variable as $\Omega_w(c)$. We can safely assume always that c does not have chambers with both ends of the same color completely above or below c. This follows immediately from the construction of the quiver Q because none of these chambers shares an edge with c in Q. Moreover, from the symmetry of the construction of Q, it suffices to analyze the cases where both ends of c are blue and where the left end of c is blue and its right end is red.

Let $w(I', J')$ denote the chamber with cluster variable $\Delta_{I',J'}$ in w if it exists. Write $c \leftarrow w(I', J')$ if there is an edge from $w(I', J')$ to c in Q and define $c \to w(I', J')$ likewise. From the mutation relation at c, we have $\Omega_w(c) = \dfrac{\prod_{c \leftarrow w(I',J')} \Delta_{I',J'} + \prod_{c \to w(I',J')} \Delta_{I',J'}}{\Delta_{I,J}}$. As remarked in the introduction and proved in Theorem 1.12 of [5], $\Omega_w(c)$ belongs to the polynomial ring $\mathbb{Z}[\{x_{ij}\}_{n\times n}]$. Conjecture 1.2 would imply that such expression is Schur non-negative when evaluated in any generalized Jacobi-Trudi matrix $H = (h_{\mu_i - \nu_j})_{n\times n}$. In the next section, we find some recurrences for computing $\Omega_w(c)$.

3. Determinantal identities and formulas for some $\Omega_w(c)$

In this section, we disgress in two directions that will be useful later. We first prove Theorem 3.2 that shows how to obtain general determinantal formulas from ones in small dimensions. Then we give recurrence relations and explicit formulas for some Plucker-adjacent variables $\Omega_w(c)$.

3.1. Determinantal Identities.

The main tool for proving Theorem 3.2 will be Jacobi's identity, which is stated as it was in [2].

Theorem 3.1. *(Jacobi's identity) For a subset $S \subset [n]$, let $s(S)$ be the sum of numbers in S and $\bar{S} = [n] \setminus S$. If $I, J \subset [n]$ are subsets of the same cardinality, then*

$$\Delta_{\bar{I},\bar{J}}(X^{-1}) = \frac{(-1)^{s(I)+s(J)}}{\det(X)} \cdot \Delta_{I,J}(X)$$

Theorem 3.2. *Let $F(z_1, \ldots, z_k)$ be an homogeneous polynomial of degree $d > 0$. Let $I_1, \ldots, I_k,\ J_1, \ldots, J_k \subset [n]$ be subsets of $[n]$ with $|I_s| = |J_s|$ for all s. Let $\bar{I} := [n]\backslash I$ for any subset $I \subset [n]$. If $F(\Delta_{I_1,J_1}, \Delta_{I_2,J_2}, \ldots, \Delta_{I_k,J_k}) = 0$, then*

1. *$F(\Delta_{\bar{I}_1,\bar{J}_1}, \ldots, \Delta_{\bar{I}_k,\bar{J}_k}) = 0$.*
2. *If $I, J \subset [n]$ are subsets of the same cardinality and such that I (resp. J) is disjoint from all sets I_s (resp. J_s), then $F(\Delta_{I\cup I_1, J\cup J_1}, \Delta_{I\cup I_2, J\cup J_2}, \ldots, \Delta_{I\cup I_k, J\cup J_k}) = 0$.*

Proof. To prove the first item, rewrite Jacobi's identity in 3.1 as

$$\Delta_{\bar{I},\bar{J}}(X) = \frac{(-1)^{s(I)+s(J)}}{\Delta(X^{-1})} \cdot \Delta_{I,J}(X^{-1}). \tag{3.1}$$

Since F is homogeneous of degree d, i.e., $r^d F(x_1, \ldots, x_k) = F(r^d x_1, \ldots, r^d x_k)$ for any $r \in \mathbb{C}$, it is easy to see that the desired result comes from multiplying $F(\Delta_{I_1,J_1}(X^{-1}), \ldots, \Delta_{I_k,J_k}(X^{-1})) = 0$ by $\left(\dfrac{(-1)^{s(I)+s(J)}}{\Delta(X^{-1})}\right)^d$ and using Equation 3.1.

For the second item, consider Y to be the submatrix of X consisting of rows $[n] \setminus I$ and columns $[n] \setminus J$. Since I is disjoint from all I_s and J is disjoint from all J_s, we have that $\Delta_{I_s,J_s}(X)$ is a minor of Y for all s. The complementary minor of $\Delta_{I_s,J_s}(X)$ in Y is $\Delta_{([n]\backslash I)\backslash I_s, ([n]\backslash J)\backslash J_s} = \Delta_{[n]\backslash(I\cup I_s), [n]\backslash(J\cup J_s)}$. From the first item, we have $F(\Delta_{[n]\backslash(I\cup I_1), [n]\backslash(J\cup J_1)}, \ldots, \Delta_{[n]\backslash(I\cup I_n), [n]\backslash(J\cup J_n)}) = 0$. We can apply the first item again to the last equation (the complementary minor in X of $\Delta_{[n]\backslash(I\cup I_s), [n]\backslash(J\cup J_s)}$ is $\Delta_{I\cup I_s, J\cup J_s}$) and we are done. □

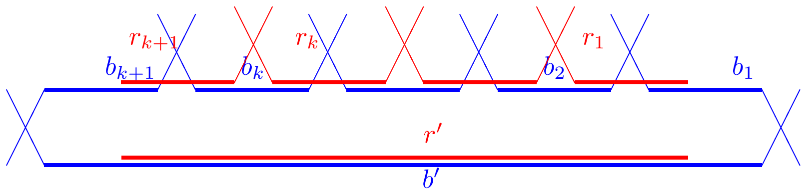


Figure 7. Chamber c with both blue ends

3.2. **Recurrences and formulas for Plucker-adjacent variables.** As remarked previously, c has the same number $d = d(c)$ of incoming and outgoing edges. After this subsection, we will see that $\Omega_w(c)$ can be written as sums and differences of products of $d-1$ minors of X (e.g. formula 3.6 below). We should remark that this subsection is completely elementary and is mostly given as an illustration of how to find explicit formulas for $\Omega_w(c)$. We only analyze the case where c has both blue ends, hoping that it becomes obvious how to do the analogous computations in the case where c has blue left end and red right end. Moreover, we assume that chamber c is in the smallest possible double wiring diagram w (, i.e., w has the smallest possible number of wires) that may contain such a chamber c. Next section shows that the latter restriction is not important.

3.2.1. *c has no crossings below it.* Let us begin by considering a chamber c as the one in Figure 7. Let b', r' be the only wires below c and $B = \{b_1 < b_2 < \ldots < b_{k+1}\}$, $R = \{r_1 > r_2 > \ldots > r_k\}$ be the sets of labels of blue and red wires right above c; this implies $b_1 < b' < b_{k+1}$. We write $\Omega_w(c)$ as $\Omega^{B,R}_{b',r'}$. The mutation relation at c is

$$x_{b',r'} \cdot \Omega^{B,R}_{b',r'} = x_{b_1,r'} \cdot \prod_{i=1}^{k} \Delta_{b'b_{i+1},r'r_i} + x_{b_{k+1},r'} \cdot \prod_{i=1}^{k} \Delta_{b'b_i,r'r_i} \tag{3.2}$$

We find a recurrence relation if $b' < b_k$. Consider a chamber c_1 which is similar to c, but with wires b_{k+1}, r_k removed. This chamber exists since $b_k > b'$. Let $B_1 = B \setminus \{b_{k+1}\}$, $R_1 = R \setminus \{r_k\}$, so that the Plucker-adjacent variable to c_1 is $\Omega^{B_1,R_1}_{b',r'}$. The mutation relation at c_1 is

$$x_{b',r'} \cdot \Omega^{B_1,R_1}_{b',r'} = x_{b_1,r'} \cdot \prod_{i=1}^{k-1} \Delta_{b'b_{i+1},r'r_i} + x_{b_k,r'} \cdot \prod_{i=1}^{k-1} \Delta_{b'b_i,r'r_i} \tag{3.3}$$

Combining these two equations yields

$$\begin{aligned} x_{b',r'} \cdot \Omega^{B,R}_{b',r'} &= \Delta_{b'b_{k+1},r'r_k}\big(x_{b',r'} \cdot \Omega^{B_1,R_1}_{b',r'} - x_{b_k,r'} \cdot \prod_{i=1}^{k-1} \Delta_{b'b_i,r'r_i}\big) + x_{b_{k+1},r'} \cdot \prod_{i=1}^{k} \Delta_{b'b_i,r'r_i} \\ &= \Delta_{b'b_{k+1},r'r_k} x_{b',r'} \cdot \Omega^{B_1,R_1}_{b',r'} + (x_{b_{k+1},r'}\Delta_{b'b_k,r'r_k} - x_{b_k,r'}\Delta_{b'b_{k+1},r'r_k}) \cdot \prod_{i=1}^{k-1} \Delta_{b'b_i,r'r_i} \\ &= \Delta_{b'b_{k+1},r'r_k} x_{b',r'} \cdot \Omega^{B_1,R_1}_{b',r'} - (x_{b',r'}\Delta_{b_k b_{k+1},r'r_k}) \cdot \prod_{i=1}^{k-1} \Delta_{b'b_i,r'r_i} \\ &= x_{b',r'}\big(\Delta_{b'b_{k+1},r'r_k} \cdot \Omega^{B_1,R_1}_{b',r'} - \Delta_{b_k b_{k+1},r'r_k} \cdot \prod_{i=1}^{k-1} \Delta_{b'b_i,r'r_i}\big), \end{aligned}$$

where we used the trivial identity $x_{b_{k+1},r'}\Delta_{b'b_k,r'r_k} - x_{b_k,r'}\Delta_{b'b_{k+1},r'r_k} = -x_{b',r'}\Delta_{b_k b_{k+1},r'r_k}$ (which holds whenever $b' < b_k < b_{k+1}$). We can thus cancel $x_{b',r'}$ and obtain the recurrence relation

$$\Omega^{B,R}_{b',r'} = \Delta_{b'b_{k+1},r'r_k} \cdot \Omega^{B_1,R_1}_{b',r'} - \Delta_{b_k b_{k+1},r'r_k} \cdot \prod_{i=1}^{k-1} \Delta_{b'b_i,r'r_i} \tag{3.4}$$

We can proceed likewise if $b_2 < b'$. In this case, if we let $B_1 = B \setminus \{b_1\}$ and $R_1 = R \setminus \{r_1\}$, we obtain the similar-looking recurrence

$$\Omega^{B,R}_{b',r'} = \Delta_{b'b_1,r'r_1} \cdot \Omega^{B_1,R_1}_{b',r'} - \Delta_{b_1 b_2,r'r_1} \cdot \prod_{i=2}^{k} \Delta_{b'b_{i+1},r'r_i} \tag{3.5}$$

Both recurrence relations above are enough to find an explicit formula for $\Omega^{B,R}_{b',r'}$ when $b_i < b' < b_{i+1}$, for any $1 \le i \le k$. For instance, when $b_1 < b' < b_2 < \ldots < b_{k+1}$, the recurrence relation 3.4 suffices to obtain

$$\Omega^{B,R}_{b',r'} = \Delta_{b_1 b_2,r'r_1} \cdot \prod_{i=2}^{k} \Delta_{b'b_{i+1},r'r_i} - \sum_{j=2}^{k}\big(\Delta_{b_j b_{j+1},r'r_j} \cdot \prod_{i=1}^{j-1} \Delta_{b'b_i,rr_i} \cdot \prod_{i=j+2}^{k+1} \Delta_{b'b_i,r'r_{i-1}}\big) \tag{3.6}$$

3.2.2. *c has no crossings above it.* Let us find recurrences similar to 3.4 and 3.5 in this case. For a chamber c, let b, r the only blue and red wire right above c and $B' = \{b'_1 < \ldots < b'_{l+1}\}$, $R'' = \{r'_1 > \ldots > r'_l\}$ be the sets of labels of blue and red wires right below c. Notice there is one more red wire below c that is not right below it. Let us call it r' and let $R' = R'' + r'$.

Denote by $\Omega^{b,r}_{B',R'}$ to the Plucker-adjacent variable coming from c. For any subsets $I, J \subset [n]$, we call $\Delta_{\overline{I},\overline{J}}$ the *complementary minor* of $\Delta_{I,J}$ in X. The mutation relation at c is

$$\Delta_{\overline{b},\overline{r}} \cdot \Omega^{b,r}_{B',R'} = \Delta_{\overline{b'_1},\overline{r}} \cdot \prod_{i=1}^{l} \Delta_{\overline{bb'_{i+1}},\overline{rr'_i}} + \Delta_{\overline{b'_{l+1}},\overline{r}} \cdot \prod_{i=1}^{l} \Delta_{\overline{bb'_i},\overline{rr'_i}} \tag{3.7}$$

which is in a very similar form as equation 3.3, except that minors are replaced by complementary minors. To obtain recurrence 3.4, we used identity $x_{b_{k+1},r'}\Delta_{b'b_k,r'r_k} - x_{b_k,r'}\Delta_{b'b_{k+1},r'r_k} = -x_{b',r'}\Delta_{b_k b_{k+1},r'r_k}$. We can proceed analogously, but using instead the identity $\Delta_{\overline{b'_{k+1}},\overline{r}}\Delta_{\overline{bb'_k},\overline{rr'_k}} - \Delta_{\overline{b'_k},\overline{r}}\Delta_{\overline{bb'_{k+1}},\overline{rr'_k}} = -\Delta_{\overline{b},\overline{r}}\Delta_{\overline{b'_k b'_{k+1}},\overline{rr'_k}}$, which

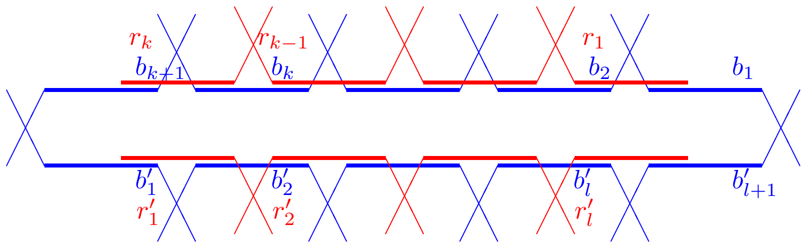


FIGURE 8. General arrangement for a chamber with both blue ends

follows from the first identity and (1) of Theorem 3.2. If $b'_l > b$, let $B'_1 = \{b'_1, \ldots, b'_l\}$, $R'_1 = \{r_1, \ldots, r'_{l-1}\}$; the analogous to recurrence 3.4 is:

$$\Omega^{b,r}_{B',R'} = \Delta_{\overline{bb'_{k+1}},\overline{rr'_k}} \cdot \Omega^{b,r}_{B'_1,R'_1} - \Delta_{\overline{b'_k b'_{k+1}},\overline{rr'_k}} \cdot \prod_{i=1}^{k-1} \Delta_{\overline{bb'_i},\overline{rr'_i}} \tag{3.8}$$

We can proceed in the same manner when $b'_2 < b$. If we let $B'_1 = \{b'_2, \ldots, b'_{l+1}\}$, $R'_1 = \{r'_2, \ldots, r'_l\}$, the analogous recurrence to 3.5 is:

$$\Omega^{b,r}_{B',R'} = \Delta_{\overline{bb'_1},\overline{rr'_1}} \cdot \Omega^{b,r}_{B'_1,R'_1} - \Delta_{\overline{b'_1 b'_2},\overline{rr'_1}} \cdot \prod_{i=2}^{k} \Delta_{\overline{bb'_{i+1}},\overline{rr'_i}} \tag{3.9}$$

3.2.3. *General chambers c.* We now consider chambers with crossings right above and right below it (see Figure 8). We let $B = \{b_1 < \ldots < b_{k+1}\}$, $R = \{r_1 > \ldots > r_k\}$, $B' = \{b'_1 < b'_2 < \ldots < b'_{l+1}\}$ and $R'' = \{r'_1 > r'_2 > \ldots > r'_l\}$ be the sets of labels of blue and red wires as it is shown in Figure 8. As before, these is a red wire r' below c and not right below it; then write $R' = R'' + r'$. It then follows that $b'_1 < b_{k+1}$ and $b_1 < b'_{l+1}$; we write $\Omega_w(c)$ as $\Omega^{B,R}_{B',R'}$.

It should be no surprise at this point that, depending on the relative ordering of the b_i, b'_i, one can find recurrences like (3.4) and (3.5) by considering chambers c' that come from removing pairs of a blue and red wire. The point of all these recurrences will be to prove that $\Omega_w(c)$ can be expressed as sums and differences of products of $d-1$ minors of X, if chamber c has $2d$ edges adjacent to it.

As a last calculation in this subsection, let us find a recurrence for $\Omega_w(c)$ if $b'_1 < b_k$; it will be a more general version of (3.4). Let $X = \prod_{i=1}^{l} \Delta_{B' \setminus \{b_i\}, R' \setminus \{r_i\}}$ be the product of variables of chambers with edges directed to c and below it. Also let $Y = \prod_{i=1}^{l} \Delta_{B' \setminus \{b_{i+1}\}, R' \setminus \{r_i\}}$ be the product of variables of chambers with edges directed from c to them and below c. The mutation relation at c is the following, more general version of 3.3:

$$\Delta_{B',R'} \cdot \Omega^{B,R}_{B',R'} = \Delta_{B'+b_1-b'_{l+1},R'} \cdot \prod_{i=1}^{k} \Delta_{B'+b_{i+1},R'+r_i} \cdot X + \Delta_{B'+b_{k+1}-b'_1,R'} \cdot \prod_{i=1}^{k} \Delta_{B'+b_i,R'+r_i} \cdot Y$$

We will use the identity $\Delta_{B'+b_{k+1}-b'_1,R'} \Delta_{B'+b_k,R'+r_k} - \Delta_{B'+b_k-b'_1,R'} \Delta_{B'+b_{k+1},R'+r_k} = -\Delta_{B',R'} \Delta_{B'+b_k+b_{k+1}-b'_1,R'+r_k}$, which follows from the trivial $x_{b_{k+1},r'} \Delta_{b'_1 b_k, r' r_k} - x_{b_k,r'} \Delta_{b'_1 b_{k+1}, r' r_k} = -x_{b'_1,r'} \Delta_{b_k b_{k+1}, r' r_k}$ and (2) of Theorem 3.2. If $b'_1 < b_k$, let $B'_1 = \{b'_1, \ldots, b'_l\}$, $R'_1 = \{r_1, \ldots, r'_{l-1}\}$; the analogous to recurrence 3.4 is:

$$\Omega^{B,R}_{B',R'} = \Delta_{B'+b_{k+1},R'+r_k} \cdot \Omega^{B_1,R_1}_{B',R'} - \Delta_{B'+b_k+b_{k+1}-b,R'+r_k} \cdot \prod_{i=1}^{k-1} \Delta_{B'+b_i,R'+r_i} \cdot Y. \tag{3.10}$$

As remarked previously, the goal of this section is to convince the reader of the truth of the following proposition.

Proposition 3.3. *A product of m minors of X, of arbitrary dimensions, will be called a (m, X) - product, or simply m-product if X is implicit. If c has $2d$ neighboring edges, then $\Omega_w(c)$ can be written as the difference between a $(d-1)$-product and the sum of $d-2$ $(d-1)$-products (see, e.g., formula (3.6)).*

In particular, there exists an homogeneous polynomial F on the minors of X such that $\Omega_w(c) = F(\{\Delta_{I,J}\}_{|I|=|J|})$ and with degree $(d-1)$.

This is not hard to prove by induction if one had recurrences for general chambers like (3.10), and also for chambers with only crossings on one side (3.4), (3.5), (3.8) and (3.9). Of course, one needs several of these, depending on the order relation of some wires and also in the case where c has blue left end and red right end. The calculation of the previous recurrences should however be enough to convince the reader doing all these cases is only a tedious task.

4. Simplifying the double wiring diagram

In this section, we prove the following proposition:

Proposition 4.1. *If $\Omega_{w'}(c)$ is Schur non-negative, then $\Omega_w(c)$ is Schur non-negative for any double wiring diagram w that contains the wires of w' (and possibly additional ones) and such that it contains c as a chamber.*

This will allow us to prove Schur non-negativity of $\Omega_w(c)$ by proving it for the simplest double wiring diagram w that contains c. The first trivial observation is that wires above c that do not make crossings right above c are unimportant. This is because the variables involved in the mutation rule at c do not depend on those wires. What is a little more surprising is that, for Schur non-negativity, the wires below c that do not make crossings right below it are also unimportant. Let w, w' be two doule wiring diagrams that contain a chamber c, but they differ in that w has additional sets B_0, R_0 (of the same cardinality) of labels of blue and red wires below it.

Denote by B', R' the sets of labels of blue and red wires in w', so that $B' \cup B_0 = B$, $R' \cup R_0 = R$ are the sets of labels of blue and red wires of w. We show that if $\Omega_{w'}(c')$ is Schur non-negative when evaluated in generalized Jacobi-Trudi matrices, then so is $\Omega_w(c)$. Denote by X' to the submatrix $X(B', R')$. Let $w'(I, J)$ denote the chamber in w' with cluster variable $\Delta_{I,J}$ if it exists. From Proposition 3.3, $\Omega_{w'}(c')$ is an homogeneous polynomial F on the minors of $X' = X(B', R')$. The mutation relation of $Q(w')$ at c' gives

$$\Omega_{w'}(c') = \frac{\prod_{c' \leftarrow w'(I,J)} \Delta_{I,J} + \prod_{c' \rightarrow w'(I,J)} \Delta_{I,J}}{\Delta_{I,J}} = F(\{\Delta_{I,J}\}_{|I|=|J|, I \subset B', J \subset R'})$$

Since B_0 and R_0 are disjoint from the sets of blue and red wires of w', we can apply part (2) of Theorem 3.2 to the expression above and obtain

$$\frac{\prod_{c' \leftarrow w'(I,J)} \Delta_{B_0 \cup I, R_0 \cup J} + \prod_{c' \rightarrow w'(I,J)} \Delta_{B_0 \cup I, R_0 \cup J}}{\Delta_{B_0 \cup I, R_0 \cup J}} = F(\{\Delta_{B_0 \cup I, R_0 \cup J}\}_{|I|=|J|, I \subset B', J \subset R'})$$

The left hand side of the above expression corresponds to $\Omega_w(c)$ because of the mutation rule of $Q(w)$ at c and because $c' \leftarrow w'(I,J) \Longleftrightarrow c \leftarrow w(I\cup B_0, J\cup R_0)$, $c' \rightarrow w'(I,J) \Longleftrightarrow c \rightarrow w(I\cup B_0, J\cup R_0)$. Therefore $\Omega_w(c) = F(\{\Delta_{B_0\cup I, R_0\cup J}\}_{|I|=|J|, I\subset B', J\subset R'})$.

Let H be any $n\times n$ generalized Jacobi-Trudi matrix and $H' = H(B', R')$. From assumption, we know that $\Omega_{w'}(c')|_{H'} = F(\{\Delta_{I,J}\}_{I\subset B', J\subset R'})|_{H'}$ is Schur positive, i.e., can be written as $\sum_\nu a_\nu s_\nu$, for $a_\nu \geq 0$ and Schur polynomials s_ν on the variables of X'. From the Jacobi-Trudi identity, we have that $s_\nu = \det(h_{\nu_i+j-i})$, therefore

$$F(\{\Delta_{I,J}\}_{|I|=|J|, I\subset B', J\subset R'})|_{H'} = \sum_\nu a_\nu \det(h_{\nu_i+j-i}). \tag{4.1}$$

We can assume that all entries in $H(I,J)$, $I\subset B'$, $J\subset R'$, are nonzero (otherwise, we use the minors expansion formula for $\Delta_{I,J}$ and get rid of the zero entries). Consider the (generalized Jacobi-Trudi) matrix $\widetilde{H}$ that is of the form below and is such that all $H(I,J)$, $I\subset B'$, $J\subset R'$, and (h_{ν_i+j-i}) are submatrices of it.

$$\widetilde{H} = \begin{pmatrix} h_{r+s} & \cdots & h_{r+s} & h_{r+s+1} & \cdots & h_{r+s+k} & \cdots & h_{r+s+k+l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_r & \cdots & h_r & h_{r+1} & \cdots & h_{r+k} & \cdots & h_{r+k+l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_1 & \cdots & h_1 & h_2 & \cdots & h_{k+1} & \cdots & h_{k+l+1} \\ h_0 & \cdots & h_0 & h_1 & \cdots & h_k & \cdots & h_{k+l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ h_0 & \cdots & h_0 & h_1 & \cdots & h_k & \cdots & h_{k+l} \end{pmatrix}$$

We also want $\widetilde{H}$ to be large enough so that all $H(I\cup B_0, J\cup R_0)$ are submatrices of it. For each $I\subset B'$, $J\subset R'$, denote by $\tilde{I}, \tilde{J}$ to the rows ans columns of $\tilde{H}$ such that $\widetilde{H}(\widetilde{I}, \widetilde{J}) = H(I,J)$ and also denote by I_ν, J_ν to the rows and columns of $\widetilde{H}$ such that $\widetilde{H}(I_\nu, J_\nu) = (h_{\nu_i+j-i})$. Finally, denote by $\widetilde{\Delta}_{I,J}$ to the repective minor of $\tilde{H}$. Then equation (4.1) can be rewritten as

$$F(\{\widetilde{\Delta}_{\widetilde{I},\widetilde{J}}\}_{|I|=|J|, I\subset B, J\subset R}) = \sum_\nu a_\nu \widetilde{\Delta}_{I_\nu, J_\nu}. \tag{4.2}$$

There exist sets of rows and columns $\widetilde{B}, \widetilde{R}$ so that $\widetilde{H}(\widetilde{I}\cup\widetilde{B}, \widetilde{J}\cup\widetilde{R}) = H(I\cup B_0, J\cup R_0)$ for all $|I| = |J|$. Therefore, one can apply part (2) of Theorem 3.2 to (4.2) and obtain

$$F(\{\widetilde{\Delta}_{\widetilde{I}\cup\widetilde{B}, \widetilde{J}\cup\widetilde{R}}\}_{|I|=|J|, I\subset B, J\subset R}) = \sum_\nu a_\nu \widetilde{\Delta}_{I_\nu\cup\widetilde{B}, J_\nu\cup\widetilde{R}}. \tag{4.3}$$

By the condition imposed, we have that the left-hand side of (4.3) is $F(\{\widetilde{\Delta}_{\widetilde{I}\cup\widetilde{B}, \widetilde{J}\cup\widetilde{R}}\}_{|I|=|J|, I\subset B, J\subset R}) = F(\{\Delta_{I\cup B_0, J\cup R_0}\}_{|I|=|J|, I\subset B', J\subset R'}) = \Omega_w(c)$. The generalized Jacobi-Trudi identity says that $\widetilde{\Delta}_{I_\nu, J_\nu}$ is a skew Schur function, which is Schur non-negative. Therefore, the positive linear combination (which is the right-hand side of (4.3)) $\sum_\nu a_\nu \widetilde{\Delta}_{I_\nu\cup\widetilde{B}, J_\nu\cup\widetilde{R}}$ is also Schur non-negative, as desired.

In the subsequent sections, especially in section 6, when calculating $\Omega_w(c)$, we assume that the double wiring diagram w that contains c is the smallest one in terms of number of wires. For example, if c has 4 blue crossings next to it and 2 red crossings next to it, then w contains only 4 wires of each color.

5. Temperley-Lieb Immanants

In this section, we develop a technique for proving Conjecture 1.2 when chamber c has 6 or less adjacent chambers in Q. We give a rough overview of some results we found in [8], but which are originally due to Haiman, and Rhoades-Skandera.

5.1. Background.

For a $n \times n$ matrix X, $w \in S_n$, the *Kazhdan-Lusztig immanant* $Imm_w(X)$ is defined in terms of the Kazhdan-Lusztig polynomials indexed by elements of the symmetric group S_n. The details of the definition are not important for our purposes. We only remark that the Kazhdan-Lusztig immanants are essentially the dual canonical basis and thus Haiman's result shows:

Theorem 5.1. *If M is a generalized Jacobi-Trudi matrix, then $Imm_w(M)$ is Schur non-negative.*

The Temperley-Lieb algebra $TL(\xi)$ is the $\mathbb{C}[\xi]$-algebra generated by $t_1, t_2, \ldots, t_{n-1}$, subject to the relations $t_i^2 = \xi t_i$, $t_i t_j t_i = t_i$ if $|i-j| = 1$ and $t_i t_j = t_j t_i$ if $|i-j| \geq 2$ (see [8] for details). The dimension of $TL_n(\xi)$ is the n-th Catalan number. For any $w \in S_n$, define $t_w := t_{i_1} \cdots t_{i_k}$ for a reduced decomposition $w = s_{i_1} \cdots s_{i_k}$. A natural basis for $TL(\xi)$ is the set of all t_w for 321-avoiding permutations $w \in S_n$.

The Temperley-Lieb immanant of an $n \times n$ matrix X is defined as

$$Imm_w^{TL}(X) := \sum_{v \in_n} f_w(v) x_{1,v(1)} \cdots x_{n,v(n)},$$

where $f_w(v)$ is defined as the coefficient of the basis element $t_w \in TL(2)$ in the expansion of $(t_{i_1} - 1) \cdots (t_{i_k} - 1) \in TL(2)$ for a reduced decomposition $v = s_{i_1} \cdots s_{i_k}$.

Theorem 5.2. *For a 321-avoiding permutation $w \in S_n$, we have $Imm_w(X) = Imm_w^{TL}(X)$. Thus, from Theorem 5.1, $Imm_w^{TL}(X)$ is Schur non-negative when evaluated in any generalized Jacobi-Trudi matrix.*

A product of generators $t_{i_1} \cdots t_{i_k}$ in TL_n can be visualized using the *Temperley-Lieb diagrams*. These are diagrams with n non-crossing strands connecting the vertices $1, 2, \ldots, 2n$, with possibly some internal loops (see [8]). The set of pairs of vertices connected by a strand form a non-crossing matching on the set of vertices $[2n]$, i.e., for edges (a, b) and (c, d) of the matching, we never have $a < c < b < d$. This association is a bijection between the basis $\{t_w : w \text{ is a 321-avoiding permutation of } S_n\}$ of TL_n and the set of non-crossing matchings on the vertex set $[2n]$.

For a subset $S \subset [2n]$, say that $w \in S_n$ is S-compatible if each strand of its Temperley-Lieb diagram has exactly one endpoint in S. A basis element t_w is S-compatible if and only if each edge of the associated matching has one endpoint in S and the other in $[2n] \setminus S$. Denote by $\Theta(S)$ to all the 321-avoiding permutations $w \in S_n$ for which t_w is S-compatible. For a subset $I \subset [n]$, let $I' := \{2n + 1 - i : i \in I\}$.

Theorem 5.3. *For two subsets $I, J \subset [n]$ of the same cardinality and $S = S(I, J) = J \cup (\overline{I})'$, we have*

$$\Delta_{I,J}(X) \cdot \Delta_{\overline{I},\overline{J}}(X) = \sum_{w \in \Theta(S)} Imm_w^{TL}(X).$$

Combining Theorems 5.2 and 5.3 provide a wonderful tool for proving Schur non-negativity for expressions involving products of complementary minors as we shall see soon. For a 321-avoiding permutation $w \in S_n$, denote by M_w to its associated Temperley-Lieb diagram, which is a non-crossing matching and write $Imm_w^{TL}(X)$ as $Imm(M_w)$ for simplicity. For two subsets $I, J \subset [n]$ of the same cardinality, let $S(I, J) = J \cup (\overline{I})'$ and $M(I, J)$ be the set of matchings M_w such that $w \in \Theta(S(I, J))$. Equivalently, $M(I, J)$ is the set of matchings on vertex set $[2n]$ that are $S(I, J)$-compatible. Theorem 5.3 can then be rewritten as $\Delta_{I,J}(X) \cdot \Delta_{\overline{I},\overline{J}}(X) = \sum_{M_w \in M(I,J)} Imm(M_w)$. We show a few proofs that some expressions are Schur non-negative by expressing them as positive linear combinations of immanants.

Example 5.4. *The expression* $\Omega = x_{11} \cdot \Delta_{23,23} - \Delta_{123,123}$ *is Schur non-negative.*

We have that $M(\{1\},\{1\})$ *consists of matchings* $\{16, 25, 34\}$ *and* $\{12, 34, 56\}$ *since they are the only ones that are* $\{1, 4, 5\}$*-compatible on vertex set* [6]*. Likewise,* $M(\{1, 2, 3\}, \emptyset)$ *consists of the single matching* $\{16, 25, 34\}$*. Therefore, Theorem 5.3 tells us that* $\Omega = x_{11} \cdot \Delta_{23,23} - \Delta_{123,123} = (Imm(\{16, 25, 34\}) + Imm(\{12, 34, 56\})) - Imm(\{16, 25, 34\}) = Imm(\{16, 25, 34\})$*. Theorem 5.2 then implies* Ω *is Schur non-negative.*

Proposition 5.5. *The following hold for minors of a* 4×4 *matrix* X*.*

(1) $\Delta_{13,24}\Delta_{24,13} \geq_s \Delta_{34,13}\Delta_{12,24}$.

(2) $\Delta_{13,14}\Delta_{24,23} \geq_s \Delta_{12,14}\Delta_{34,23}$.

Proof. The proofs simply involve counting S-compatible matchings for some sets S. For the first item, observe that the left-hand side of the inequality is $\Delta_{I_1,J_1}\Delta_{\overline{I_1},\overline{J_1}}$ for $I_1 = \{1, 3\}$ and $J_1 = \{2, 4\}$; then let $S_1 = S(I_1, J_1) = S(\{1, 3, 6, 8\})$. For the right-hand side, the set of interest is $S_2 = S(I_2, J_2) = \{3, 4, 5, 7\}$. Next, let M_1, M_2 be the sets of S_1 - and S_2 - compatible matchings respectively, then

(1) M_1 has elements $\{12, 34, 56, 78\}$, $\{12, 34, 58, 67\}$, $\{14, 23, 56, 78\}$ and $\{14, 23, 58, 67\}$.

(2) M_2 has elements $\{14, 23, 56, 78\}$ and $\{14, 23, 58, 67\}$.

Therefore Theorem 5.3 says that the difference between the left - and right - hand side of (1) is $Imm(\{12, 34, 58, 67\}) + Imm(\{14, 23, 58, 67\})$. Theorem 5.2 then implies this expression is Schur non-negative.

For the second item, we obtain $S_1' = \{1, 3, 6, 7\}$, $S_2' = \{1, 3, 6, 7\}$. Then, if M_1', M_2' are the sets of S_1' - and S_2' - compatible matchings respectively, we have that

(1) M_1' has elements $\{12, 34, 56, 78\}$, $\{12, 38, 47, 56\}$, $\{14, 23, 56, 78\}$, $\{18, 23, 47, 56\}$ and $\{18, 27, 34, 56\}$.

(2) M_2' has elements $\{14, 23, 56, 78\}$ and $\{18, 23, 47, 56\}$.

As above, Theorem 5.3 shows that the difference between the left - and right - hand side of (2) is $Imm(\{12, 34, 56, 78\}) + Imm(\{12, 38, 47, 56\}) + Imm(\{18, 27, 34, 56\})$. Theorem 5.2 then implies this expression is Schur non-negative. □

The following proposition proves the Schur non-negativity of some expressions using Temperley-Lieb immanants and the technique of *copying rows and columns of generalized Jacobi-Trudi matrices.*

Proposition 5.6. *The following expressions hold for minors of a* 4×3 *matrix* X*.*

(1) $\Delta_{13,13}\Delta_{24,12} \geq_s \Delta_{34,12}\Delta_{12,13}$.

(2) $\Delta_{13,23}\Delta_{24,12} \geq_s \Delta_{34,12}\Delta_{12,23}$.

(3) $\Delta_{13,23}\Delta_{24,13} \geq_s \Delta_{34,13}\Delta_{12,23}$.

(4) $\Delta_{13,13}\Delta_{24,12} \geq_s \Delta_{12,13}\Delta_{34,12}$.

(5) $\Delta_{13,23}\Delta_{24,12} \geq_s \Delta_{12,23}\Delta_{34,12}$.

(6) $\Delta_{13,23}\Delta_{24,13} \geq_s \Delta_{12,23}\Delta_{34,13}$

Proof. Let us prove (1), i.e., $\Delta_{13,13}\Delta_{24,12} - \Delta_{34,12}\Delta_{12,13} \geq_s 0$. We need to show this is Schur non-negative for any 4×3 generalized Jacobi-Trudi matrix $M = (H_{\mu_i - \nu_j})_{i,j}$. Let M' be the 4×4 generalized Jacobi-Trudi matrix that comes from duplicating the first column of M. Let $1'$ the duplicated column and let the columns of X be labeled $1, 1', 2, 3$ in that order. Let $\Delta_{I,J}|_{M'}$ be the (I, J)- minor of X' evaluated in M'. From (1) in Proposition 5.5, we have $\Delta_{13,1'3}|_{M'}\Delta_{24,12}|_{M'} \geq_s \Delta_{34,12}|_{M'}\Delta_{12,1'3M'}$. Since columns 1 and $1'$ are equal in M', the identity becomes $\Delta_{13,13}|_M\Delta_{24,12}|_M \geq_s \Delta_{34,12}|_M\Delta_{12,13}|_M$. Since this holds for any generalized Jaoobi-Trudi matrix M, we are done.

Items (2) and (3) also follow from (1) of Proposition 5.5, but duplicating the second and third column, respectively.

Items (4), (5) and (6) above follow from (2) Proposition 5.5, via the same technique and duplicating the first, second and third column, respectively. □

6. Proof of Theorem 1.3

We prove conjecture 1.2 for Plucker-adjacent cluster variables $\Omega_w(c)$ that come from chambers c in w, whose corresponding vertex in $Q = Q(w)$ has 6 or less adjacent vertices. From the construction of Q, any chamber c has an even number of adjacent vertices that is at least 4.

If chamber c has degree 4 in Q, then it was proven in Lemma 18 of [6] that its mutated variable $\Omega_w(c)$ is a chamber minor Δ_{I_1,J_1} of a double wiring diagram w' that comes from making a *local or braid move* to w at c. In particular, the generalized Jacobi-Trudi identity implies that when we evaluate $\Omega_w(c)$ at a generalized Jacobi-Trudi matrix, we obtain some skew-Schur function s_{μ_1/ν_1}, which is Schur non-negative.

From section 4, we can assume the double wiring diagram w of c is as small as possible. For chambers of degree 6, then the simplest w has at most 4 wires. In each case below, we first calculate $\Omega_w(c)$ with general labels for the wires and then let them be numbers from $1, 2, 3, 4$ according to the relative ordering of them. When c has degree 6 in Q, then there will be 3 general chamber arrangements when c has both blue ends and 2 general chamber arrangements when c has blue left end and red right end.

Case 1: Both ends of c are blue

Subcase 1: c has no crossings below it (see Figure 7).

Depending on whether $b_1 < b < b_2$ or $b_2 < b < b_3$, we can use recurrence 3.4 or 3.5 to obtain a formula for $\Omega_w(c)$. In the first case, we have $\Omega_w(c) = \Delta_{b_1b_2,rr_1}\Delta_{bb_3,rr_2} - \Delta_{b_2b_3,rr_2}\Delta_{bb_1,rr_1}$ and in the second case, we have $\Omega_w(c) = \Delta_{bb_1,rr_1}\Delta_{b_2b_3,rr_2} - \Delta_{b_1b_2,rr_1}\Delta_{bb_3,rr_2}$.

Proposition 6.1. *All of these expressions for $\Omega_w(c)$ above are Schur non-negative.*

Proof. In the first case, we can let $b_1 = 1, b = 2, b_2 = 3$ and $b_3 = 4$. Three subcases arise depending on the relative order of r with respect to r_1, r_2. The three formulas for $\Omega_w(c)$ are, when $r < r_2 < r_1$, $r_2 < r < r_1$ and $r_2 < r_1 < r$ respectively,

(1) $\Delta_{13,13}\Delta_{24,12} - \Delta_{34,12}\Delta_{12,13}$

(2) $\Delta_{13,23}\Delta_{24,12} - \Delta_{34,12}\Delta_{12,23}$

(3) $\Delta_{13,23}\Delta_{24,13} - \Delta_{34,13}\Delta_{12,23}$

In the second case, we let $b_1 = 1, b_2 = 2, b = 3$ and $b_3 = 4$. As above, the three formulas for $\Omega_w(c)$ are

(1) $\Delta_{13,13}\Delta_{24,12} - \Delta_{12,13}\Delta_{34,12}$

(2) $\Delta_{13,23}\Delta_{24,12} - \Delta_{12,23}\Delta_{34,12}$

(3) $\Delta_{13,23}\Delta_{24,13} - \Delta_{12,23}\Delta_{34,13}$

It was proven in Proposition 5.6 that all 6 of these expressions are Schur non-negative, as desired. □

Subcase 2: c has no crossings above it.

In this case, we can use recurrences 3.8 and 3.9 to obtain all the formulas for $\Omega_w(c)$. It is not surprising that they turn out to be of the same form as the formulas in the first subcase, but each minor is replaced by its complementary minor in X. For example, The analogous to (1) is $\Delta_{\overline{13},\overline{13}}\Delta_{\overline{24},\overline{12}} - \Delta_{\overline{34},\overline{12}}\Delta_{\overline{12},\overline{13}} = \Delta_{24,24}\Delta_{13,34} - \Delta_{12,34}\Delta_{34,24}$. We could prove these 6 expressions are Schur non-negative using immanants and copying rows and columns, as it was done in the previous subcase. However, this will not be necessary.

Let H be any 4×4 generalized Jacobi-Trudi matrix. Let H' be the reflection of H over the secondary diagonal. It is clear that H' is also a generalized Jacobi-Trudi matrix and $\big(\Delta_{24,24}\Delta_{13,34} - \Delta_{12,34}\Delta_{34,24}\big)|_H = \big(\Delta_{13,13}\Delta_{24,12} - \Delta_{34,12}\Delta_{12,13}\big)|_{H'}$. The latter expression is Schur non-negative from (1) of the previous subcase. Since H was arbitrary, this shows $\Delta_{24,24}\Delta_{13,34} - \Delta_{12,34}\Delta_{34,24} \geq_s 0$. A similar reasoning applies to all other formulas for $\Omega_w(c)$.

Subcase 3: c has one crossing above it and one crossing below it (see Figure 8 with $k = l = 1$).

In this case, if $b_1' < b_1$, we can use recurrence 3.10 to obtain $\Omega_w(c) = \Delta_{b_1'b_2'b_2,r'r_1'r_1} \cdot x_{b_1,r'} - \Delta_{b_2'b_1b_2,r'r_1'r_1} \cdot x_{b_1',r'}$. Letting $b_1' = 1, b_1 = 2$, we have three formulas for $\Omega_w(c)$ which are

(1) $x_{2,1} \cdot \Delta_{134,123} - x_{1,1} \cdot \Delta_{234,123}$

(2) $x_{2,2} \cdot \Delta_{134,123} - x_{1,2} \cdot \Delta_{234,123}$

(3) $x_{2,3} \cdot \Delta_{134,123} - x_{1,3} \cdot \Delta_{234,123}$

Proposition 6.2. *All of these expressions for $\Omega_w(c)$ are Schur non-negative.*

Proof. The proof for all of these is similar to that of Proposition 5.6.

For the first two, we prove $x_{2,2}\Delta_{134,134} - x_{1,2}\Delta_{234,134} \geq_s 0$. The first item then follows from copying the first column of that expression and the second item follows from copying the second column.

It is not hard to check that $x_{2,3}\Delta_{13,12} = Imm(\{12, 36, 45, 78\}) + Imm(\{12, 38, 45, 67\}) + Imm(\{18, 23, 45, 67\}) + Imm(\{18, 27, 36, 45\})$ and $x_{1,3}\Delta_{23,12} = Imm(\{12, 36, 45, 78\}) + Imm(\{12, 38, 45, 67\})$. Thus the statement follows from Theorem 5.1.

For the last expression , we prove $x_{2,3}\Delta_{134,124} - x_{1,4}\Delta_{234,123} \geq_s 0$; the statement follows from copying the third column.

Again, this simply follows from the equalities $x_{2,3}\Delta_{134,124} = Imm(\{12, 38, 45, 67\}) + Imm(\{12, 38, 47, 56\}) + Imm(\{18, 23, 45, 67\}) + Imm(\{18, 23, 47, 56\})$ and $x_{1,4}\Delta_{234,123} = Imm(\{12, 38, 47, 56\})$. □

In the case $b_1 < b_1'$, we obtain the formulas where each minor above is replaced by its complementary minor in X, e.g., $\Delta_{\overline{2},\overline{1}} \cdot \Delta_{\overline{134},\overline{123}} - \Delta_{\overline{1},\overline{1}} \cdot \Delta_{\overline{234},\overline{123}} = \Delta_{134,234} \cdot x_{2,4} - \Delta_{234,234} \cdot x_{1,4}$. The Schur non-negativity of these expressions follows from the analysis above and the same trick applied to subcase 2.

Case 2: c has both ends of different color

Subcase 1: c has no crossings below it (see Figure 5).

In this case, the minimal double wiring diagram for c has 3 wires. Moreover, we must have $b_2 > b_1, b$ and $r_1 > r_2, r$. The mutation relation at c is

$$x_{b,r} \cdot \Omega_w(c) = x_{b,r_1} \cdot x_{b_2,r} \cdot \Delta_{bb_1,rr_2} + 1 \cdot \Delta_{bb_1} \cdot \Delta_{bb_2,rr_2}.$$

If $b_1 > b$, we have $\Omega_w(c) = x_{b_1,r_1} \cdot \Delta_{bb_2,rr_2} - x_{b,r_1}\Delta_{b_1b_2,rr_2}$. By renaming the wires $b = 1, b_1 = 2, b_2 = 3, r_1 = 3$ and $\{r, r_2\} = \{1, 2\}$, we have the expression $\Omega_w(c) = x_{2,3}\Delta_{13,12} - x_{1,3}\Delta_{23,12}$.

If $b > b_1$, we have $\Omega_w(c) = x_{b,r_1}\Delta_{b_1b_2,rr_2} - x_{b_1,r_1} \cdot \Delta_{bb_2,rr_2}$. In this case, renaming the variables $b = 2, b_1 = 1, b_2 = 3, r_1 = 3$ and $\{r, r_2\} = \{1, 2\}$, we have the same expression as the one above. Thus the following proposition suffices for this subcase.

Proposition 6.3. $x_{2,3}\Delta_{13,12} - x_{1,3}\Delta_{23,12} \geq_s 0$

Proof. $x_{2,3}\Delta_{13,12} = Imm(\{12, 36, 45\}) + Imm(\{16, 23, 45\})$.

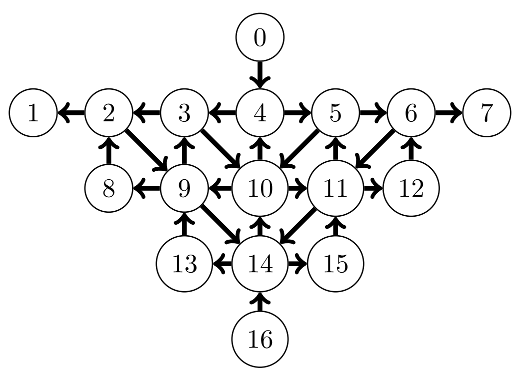
$x_{1,3}\Delta_{23,12} = Imm(\{12, 36, 45\})$. □

Subcase 2: c has no crossings above it.

This case follows from the previous subcase, as it was seen in the subcase 2 of case 1.

7. Final remarks

7.1. **Examples of non Plucker-adjacent cluster variables.** A single wiring diagram can be associated with a reduced word for the permutation $(1, 2, \ldots, n) \to (n, \ldots, 2, 1)$ of S_n. A transposition $s_i = (i, i+1)$ simply corresponds to a crossing between the wires at the levels i and $i+1$. We consider a double wiring diagram w with $n = 4$, for which all blue crossings are to the left of the red crossings, the blue crossings correspond to the reduced word $s_1s_2s_3s_1s_2s_3$ in S_4 and the red crossings correspond to the reduced word $t_1t_2t_1t_3t_2t_1$ ($t_i = (i, i+1) = s_i$, but we stick to the notation in [8]). The associated quiver is the one in Figure 7.1, where for example, chambers 4, 9 and 15 correspond to cluster variables $\Delta_{123,123}, \Delta_{14,12}$ and $x_{1,4}$. The vertices $0, 1, 7, 8, 12, 13, 15, 16$ are frozen ones.



We obtained some cluster variables that are not Plucker-adjacent. For all the ones we found, but one, one can prove Schur non-negativity via the method of immanants and copying rows and columns, as in Proposition 5.6. That special cluster variable is $z_{11} = \Delta_{14,34}\Delta_{123,123} - x_{13}\Delta_{1234,1234}$ and comes from mutating the initial seed at 10 and then at 11.

Other examples of cluster variables that are not Plucker-adjacent are given below and they come from the seed after mutating at $3, 5, 10, 9, 11, 10, 14, 3, 5$ in that order. The motivation to mutate at these vertices is that the quiver associated to the final seed has a double edge from 14 to 4. Then by mutating at 14 and 4 repeatedly, one obtains infinitely many cluster variables, but the size of these grows very rapidly.

(1) $z_3 = \Delta_{34,23}\Delta_{12,13} - \Delta_{12,23}\Delta_{34,13}$.

(2) $z_5 = x_{33}\Delta_{123,124} - x_{34}\Delta_{123,123}$.

(3) $z_{14} = x_{32}\Delta_{12,13} - x_{31}\Delta_{12,23}$.

8. Acknowledgements

This research was conducted at the 2012 summer REU (Research Experience for Undergraduates) program at the University of Minnesota, Twin Cities, and was supported by NSF grants DMS-1067183 and DMS-1148634. The author would like to thank Profs. Gregg Musiker, Pavlo Pylyavskyy, Joel Lewis and Dennis Stanton, who directed the program, for their support, and express particular gratitude to Prof. Pylyavskyy for introducing this problem and for his guidance throughout the research process. The author is also indebted to Thomas McConville for his assistance in editing this report.

References

[1] A. Berenstein, S. Fomin and A. Zelevinsky, Cluster Algebras III: Upper Bounds and Double Bruhat Cells, *Duke Math J.* **Vol. 126, Number 1** (2005), 1 - 52.

[2] R. Brualdi and H. Schneider, Determinantal Identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley, *Linear Algebra and its Applications* **52/53** (1983), 769 - 791.

[3] S. Fomin and P. Pylyavskyy, Tensor diagrams and cluster algebras, preprint, 2012; **arXiv:1210.1888**.

[4] S. Fomin and A. Zelevinski, Cluster algebras I: Foundations, *J. Amer. Math. Soc.* **15** (2002), 497 - 529.

[5] S. Fomin and A. Zelevinski, Double Bruhat cells and total positivity, *J. Amer. Math. Soc.* **12** (1999), 335-380.

[6] S. Fomin and A. Zelevinski, Total Positivity: tests and parametrizations, *Math. Intelligencer* **22** (1), 23 - 33.

[7] M. Haiman, Hecke algebra characters and immanant conjectures, *J. Amer. Math. Soc.* **6** (1993), 569 - 595.

[8] T. Lam, A. Postnikov and P. Pylyavskyy, Schur Positivity and Schur Log-Concavity, *American Journal of Mathematics* **129** (2007), 1611 - 1622.

[9] T. Lam and K. Rietsch, Total positivity, Schubert positivity, and Geometric Satake preprint, 2012; **arXiv:1203.1682**.

[10] R. P. Stanley, Enumerative Combinatorics, Vol. 2. Cambridge University Press. ISBN 0-521-55309-1, 0-521-56069-1.