Rowmotion Orbits of Trapezoid Posets

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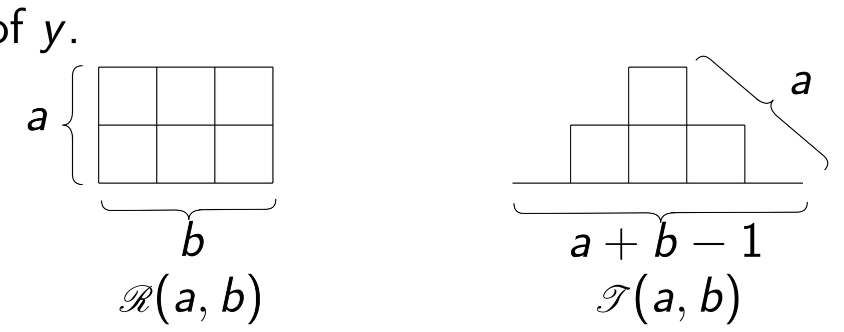
Preliminaries

Posets, Order Ideals

A **poset** is a set $\mathcal{P}$ with a binary relation $\leq$ that is reflexive, anti-symmetric, and transitive. An **order ideal** I of $\mathcal{P}$ is a sub-poset of $\mathcal{P}$ that is downward closed. The set of order ideals of $\mathcal{P}$ is denoted $J(\mathcal{P})$.

The posets of interest to us are the trapezoid posets (denoted $\mathscr{T}(a,b)$) and rectangle posets (denoted $\mathscr{R}(a,b)$) which can be represented by the graphs below where the vertices are the elements of our set and the relation $\leq$ is $x \leq y$ if x is southwest of y.



Diagrams like these are called *Hasse diagrams*.

Rowmotion

Let $\mathcal{P}$ be a poset, and $I \in J(\mathcal{P})$ an order ideal of $\mathcal{P}$. Then the **rowmotion** of I, denoted Row(I) is the order ideal generated by the minimal elements of that are not in I, i.e.

$$\text{Row}(I) = \langle a \in \mathcal{P} : a \in \min\{\mathcal{P} \setminus I\}\rangle.$$

Motivation

The action of rowmotion on the rectangle poset $\mathscr{R}(a,b)$ is known to exhibit various 'nice' properties such as cyclic sieving property [RS13] and acting the same way as cyclic rotation on necklaces with a black beads and b white beads. The trapezoid poset $\mathscr{T}(a,b)$ is closely related to the rectangle poset $\mathscr{R}(a,b)$ by virtue of being its minuscule dopplegänger partner [HPPW18]; in particular, they have the same number of order ideals. However, the action of rowmotion on the trapezoid poset remains poorly understood.

Main Question

Does the action of rowmotion on order ideals of the trapezoid poset $\mathscr{T}(a,b)$ have the same orbit structure as rowmotion on order ideals of the rectangle poset $\mathscr{R}(a,b)$?

The trapezoid poset has an important place on the edge of our current knowledge of posets. Learning about rowmotion on trapezoid poset brings us closer to a conjecture of Reiner, Tenner and Yong that the distributive lattice of order ideals of the trapezoid poset has the so-called coincidental-down-degree expectation property.

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Bijection φ of Hamaker, Patrias, Pechenik and Williams [HPPW18]

Both the trapezoid and rectangle posets can be realized as shifted skew shapes λ/μ. In both the trapezoid and rectangle posets we also have a rank function

$$rank(s) = \text{ minimum taxicab distance from } s \text{ to a minimal element of } \lambda/\mu.$$

Increasing Tableaux

An **increasing tableaux** of shape λ/μ is a function $T : \lambda/\mu \to \mathbb{Z}_{\geq 0}$ such that whenever $x < y$ in λ/μ, $T(x) < T(y)$. We say an increasing tableaux is an **almost-minimal increasing tableaux** if for all elements $s \in \lambda/\mu$

$$T(s) - rank(s) \in \{0, 1\}.$$

There is a bijection between almost-minimal increasing tablueax and order ideals obtained by subtracting rank from the increasing tableaux. We now define φ, which relies on the following action on increasing tableaux:

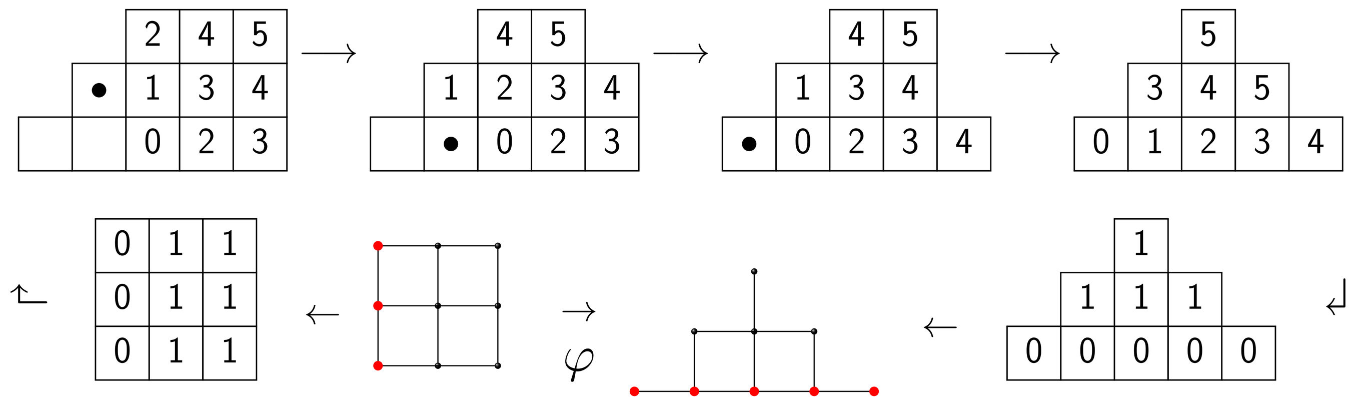
$$\text{swap}_{a,b}(T)(x) = \begin{cases} a & \text{if } T(x) = b \text{ and } a \text{ is adjacent to } x, \\ b & \text{if } T(x) = a \text{ and } b \text{ is adjacent to } x, . \\ T(x) & \text{otherwise.} \end{cases}$$

K-jeu-de-taquin (K-jdt) and φ

The **K-jdt forward slide** of a tableaux T and subset C of maximal elements of μ, is the result of first adding $\bullet$'s to C (denote this new tableaux $T \cup C$) and then

$$\text{jdt}_C(T) = \left(\left(\prod_{b=1}^{\infty} \text{swap}_{\bullet,b}\right)(T \cup C)\right) \text{ with the } \bullet\text{'s removed}$$

The bijection φ is the product of K-jdt forward slides with C = all maximal elements of μ.



The map φ on the almost-minimal tableaux corresponding to order ideals below where red nodes are elements in the order ideal. Each arrow in the above diagram is a K-jdt forward slide.

Main Theorem

Theorem 1: Commuting of rowmotion and φ

For any order ideal $\mathcal{I} \in J(\mathscr{R}(a,b))$, we have

$$\varphi \circ \text{Row}(\mathcal{I}) = \text{Row} \circ \varphi(\mathcal{I})$$

References

[BS16] Anders Skovsted Buch and Matthew J Samuel. K-theory of minuscule varieties. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2016(719):133–171, 2016.

[HPPW18] Zachary Hamaker, Rebecca Patrias, Oliver Pechenik, and Nathan Williams. Doppelgängers: Bijections of Plane Partitions. *International Mathematics Research Notices*, 03 2018.

[RS13] David B Rush and XiaoLin Shi. On orbits of order ideals of minuscule posets. *Journal of Algebraic Combinatorics*, 37(3):545–569, 2013.

K-jdt Equivalence

The K-jdt forward slides and their inverses (called *K-jdt backwards slides*) define an equivalence relation on shifted tableaux, where $T \sim S$ if and only if T can be transformed into S by a series of K-jdt slides. Similarly, K-jdt slides define a different equivalence relation on standard tableaux $T \sim S$ when T can be transformed into S by a series of K-jdt slides.

Theorem 2: Necessary Equivalence Condition

If non-skew standard tableaux T, S are K-jdt equivalent, then the set of boxes x where $T(x) = rank(x)$ is the same as the set of boxes where $S(x) = rank(x)$.

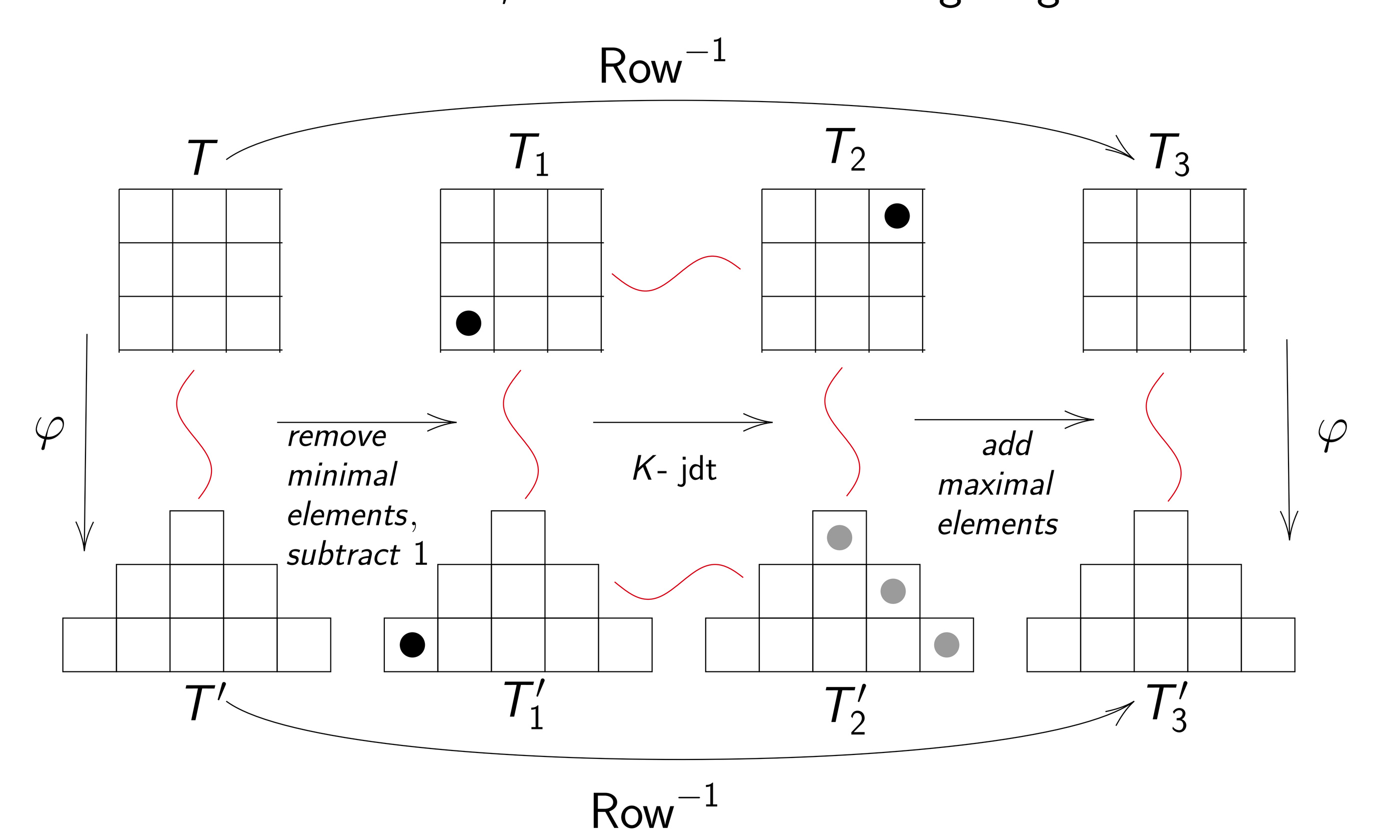
The above theorem extends to the K-jdt equivalence relation on shifted tableaux using a proposition of Buch and Samuel [BS16, Proposition 7.1], leading to the corollary

Corollary 3: Unique Representatives

All almost-minimal (shifted or standard) tableaux of shape λ are in separate K-jdt equivalence classes.

Commuting of Rowmotion and φ

Using our unique representatives corollary, the bijection φ can be rephrased as sending an almost-minimal tableaux of rectangle shape to the unique almost-minimal tableaux of trapezoid shape in the same K-jdt equivalance class. Additionally, the inverse of rowmotion on order ideals can be phrased as the following action on almost-minimal increasing tableaux - replace any minimal entries by a bullet point, perform K-jdt, decrement the new entries by 1, and replace the remaining bullet points with the maximum entry. Using this K-jdt description of rowmotion and Theorem 2, we show the following diagram commutes:



The red squiggles indicate K-jdt equivalence. Note that where the large dot(s) ends up in the second to rightmost trapezoid will depend of the order ideal ideals hence the grey dots.

Future Work

- Find a bijection between plane partitions of the rectangle and trapezoid which commutes with piecewise-linear rowmotion.
- Find a more explicit combinatorial description of rowmotion on the trapezoid poset.
- Show the down-degree statistic is homomesic with respect to rowmotion for order ideals of the trapezoid.