Strongly connected graphs and polynomials

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Abstract

In this report, we give the exact solutions of the equation $P(A) = 0$ where P is a polynomial of degree 2 with integer coefficients, and A is the adjacency matrix of a strongly connected graph. Then we study the problem for P a polynomial of degree $n \geq 3$, and give a necessary condition on the trace of a rank one matrix. Finally, we give the solutions of maximal size in some particular cases.

1 Introduction

Definition. A *strongly connected graph* is a directed graph G such that for every pair (v_i, v_j) of vertices of G, there exists a path from v_i to v_j.

Example. The graph G_1 on figure 1 is strongly connected.

Definition. Given a directed graph G, we define the *adjacency matrix* $A(G)$ of G as follows: $A(G)$ is of size $n \times n$, where n is the number of vertices of G, and the entry (i, j) of $A(G)$ is equal to the number of edges from vertex v_i to vertex v_j for all $1 \leq i, j \leq n$.

Notice that the entry (i, j) of $A(G)^k$ gives the number of paths of length k from vertex v_i to vertex v_j for all $1 \leq i, j \leq n$.

Example. The adjacency matrix of the graph G_1 (figure 1) is $A(G_1) = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 3 & 1 \\ 3 & 2 & 0 \end{pmatrix}$.

Given a polynomial $P \in \mathbb{Z}[t]$, we want to find the strongly connected graphs G such that $P(A) = 0$, where $A = A(G)$ is the adjacency matrix of G.

Example. If P has a root a, where a is a positive integer, then the graph of figure 2 is a solution.

Definition. If $\lambda_1, ..., \lambda_n$ are the eigenvalues of a matrix A, then the *spectral radius* of A is equal to $\max_{1 \leq i \leq n} |\lambda_i|$.

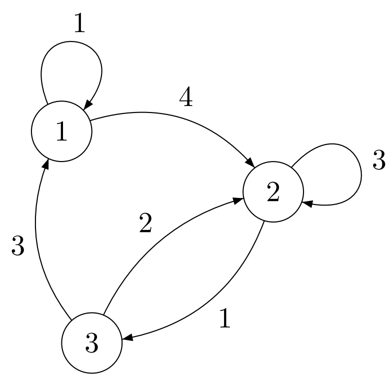

Figure 1: A strongly connected graph G_1.

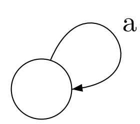

Figure 2: A graph with one vertex and a loops.

Let us start by giving an unpublished theorem by John Stembridge to show that the number of solutions is bounded.

Theorem 1.1. *For a fixed polynomial $P \in \mathbb{Z}[t]$, there are only finitely many strongly connected graphs G such that $P(A(G)) = 0$.*

Proof: Let $d = deg(P)$. Suppose that $P(A) = 0$, and $A = A(G)$ is a matrix of size $n \times n$. We can prove by induction that A^s is a linear combination of $I_n, A, ..., A^{d-1}$ for all $s \geq d$.

So there exists a path of length $\leq d - 1$ from any vertex to any other vertex in G. For purpose of contradiction, let us assume that the entry (i, j) is equal to 0 in all matrices $I_n, A, ..., A^{d-1}$. Then, for all $s \geq d$, the entry (i, j) of A^s is also equal to 0. Thus there exists no path from vertex v_i to vertex v_j, which means that G is not strongly connected and exhibits a contradiction.

Then all entries of $B := I_n + A + ... + A^{d-1}$ are positive, and therefore $\geq$ 1 because A has integer entries. So $tr(B^k) \geq n^k$ for every positive integer k. We also have $tr(B^k) = \sum_i \lambda_i^k$, where the λ_i's are the eigenvalues of B. Thus the spectral radius λ of B is at least as large as n. The eigenvalues of B are $\{1 + z + ... + z^{d-1} | \; z \; is \; eigenvalue \; of \; A\}$. All eigenvalues of A are roots of P. So

$n \leq \lambda \leq r$ where r is the largest root of P in module. The size of A is bounded.

The entry (i, j) of A is at most $tr(A^k)$, where k is the length of the smallest cycle containing (v_i, v_j). This estimation is very rough, but we only want to show that the number of solutions is finite. Then $k \leq d$, so $tr(A^k) \leq tr(A^d)$ and $tr(A^d)$ is bounded because all eigenvalues of A are roots of P. So the size and the value of the entries of A are bounded. There are only finitely many strongly connected graphs G such that $P(A(G)) = 0$. □

Now that we know that there are only finitely many solutions, let us try to find them. We will start by giving a lemma which will help us in the two following sections. To do this, we need the Perron-Frobenius theorem [3]:

Theorem 1.2. *If A is the adjacency matrix of a strongly connected graph with spectral radius $\rho(A)$, then:*
a) A has a positive eigenvector v with eigenvalue $\rho(A)$,
b) the $\rho(A)$-eigenspace is one-dimensional,
c) v is the unique non-negative eigenvector of A (up to scalar multiples),
d) $\rho(A)$ is a simple root of the characteristic polynomial of A.

Now let us use the Perron-Frobenius theorem in our lemma.

Lemma 1.3. *Let $P(t) = (t - a)q(t)$, where all roots of q are smaller than a in module. Let $P(A) = 0$ and assume that a is eigenvalue of A. Then*

$$q(A) = \frac{q(a)}{{}^tVU} U {}^tV,$$

where U (resp. ${}^tV$) is a right (resp. left) eigenvector of A corresponding to the eigenvalue a. In other terms, $q(A)$ has to be a rank one matrix.

Proof: We have $(A - a)q(A) = 0$, so $q(A)$ is a rank one matrix because all its columns are eigenvectors of A with respect to the eigenvalue a and are therefore proportional by Perron-Frobenius theorem. Let U (resp. ${}^tV$) be a right (resp. left) eigenvector of A corresponding to the eigenvalue a. Then all rows are proportional to ${}^tV$ and all columns are proportional to V. Thus we deduce that $q(A) = cU{}^tV$, where c is some constant. But we have $q(A)U = q(a)U = cU({}^tVU)$, and ${}^tVU$ is a positive integer, so $c = \frac{q(a)}{{}^tVU}$, which completes our proof. □

This preliminary result established, we can try to solve the equation.

2 The case of polynomials of degree 2

In this section we give the exact solutions of the equation $P(A) = 0$, where P is a polynomial of degree 2 with integer coefficients.

Let P be a polynomial of degree 2 in $\mathbb{Z}[t]$. Then P can be written in the form $P(t) = c(t - a)(t - b)$, where $a, b \in \mathbb{C}$ and $c \in \mathbb{Z}$. But $P(t) = 0$ if and only if $(t - a)(t - b) = 0$, so we can omit the coefficient c and only consider polynomials of the form $(t - a)(t - b)$ without loss of generality.

In a first part we are going to solve the equation $(A - a)(A - b) = 0$, with a and b integers, $a > 0$ and $|b| < a$, and in a second part we will study the other cases.

2.1 The equation $(A - a)(A - b) = 0$, with a and b integers, $a > 0$ and $|b| < a$

If A is a matrix of size 1, then the solutions are a and b if b is a positive integer, and only a otherwise..

For matrices of size $n \geq 2$, we establish the following result:

Theorem 2.1. *The strongly connected solutions of size ≥ 2 of the equation*

$$(A-a)(A-b)=0,$$

with a and b integers, $a>0$ and $|b|<a$, are exactly the matrices $bI_n + X^tC$, where X^tC is a rank one matrix with positive integer entries such that $tr(X^tC) = a-b$.

Proof: If a is not eigenvalue of A, then $A = bI_n$, which does not correspond to a strongly connected graph if $n \geq 2$. Let us assume that $(A-a)(A-b) = 0$ and that a is eigenvalue of A. We assume that A is of size $n \times n$ with $n \geq 2$. Then, by Lemma 1.3, $A - b$ has to be a rank one matrix. So let us search for A of the form $A = bI_n + X^tC$, where $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ are two vectors (we can assume that $x_1, ..., x_n$ and $c_1, ..., c_n$ are non-negative integers).

Then

$$\begin{aligned} A^2 =& (bI_n + X^tC)^2 \\ =& b^2 I_n + 2bX^tC + X(^tCX)^tC \\ =& b^2 I_n + (2b + tr(X^tC))X^tC, \end{aligned} \tag{2.1}$$

because $^tCX = tr(X^tC)$.

We can prove by induction that for every positive integer k,

$$A^k = d_k I_n + e_k X^tC,$$

where d_k and e_k are non-negative integers. So we deduce that for every $i \in \{1, ..., n\}$, $c_i > 0$ and $x_i > 0$. Let us indeed assume for purpose of contradiction that for example $x_i = 0$ for some i. Then, for every positive integer k, the entry $(2, i)$ of A^k will be 0, so in the corresponding graph, it is impossible to reach the vertex v_i from the vertex v_2, which means that the graph is not strongly connected and gives us a contradiction.

By equation 2.1, we have:

$$\begin{aligned} (A-a)(A-b) =& A^2 - (a+b)A + abI_n \\ =& b^2 I_n + (2b + tr(X^tC))X^tC - (a+b)(bI_n + X^tC) + abI_n \\ =& (b - a + tr(X^tC))X^tC. \end{aligned}$$

All c_i's and x_i's are positive so X^tC cannot be the zero matrix. So, in order to have $(A-a)(A-b) = 0$, we must have $tr(X^tC) = a - b$.

We conclude that all solutions of the equation have to be of the form $A = bI_n + X^tC$, where X^tC is a rank one matrix, with c_i and x_i positive integers for every $i \in \{1, ..., n\}$ and

$$tr(X^tC) = \sum_{i=1}^{n} c_i x_i = a - b.$$

Conversely, all such A's are solutions of the equation, so we found exactly all the solutions. □

Example. Let us find the solutions of $(A-2)(A-5) = 0$. The solutions of size 1 are 2 and 5. By Theorem 2.1, the solutions of size $n \geq 2$ are of the form $2I_n + X^tC$, where X^tC is a rank one matrix with positive integer entries of trace $5 - 2 = 3$. So the solutions of size 2 are: $2I_2 + \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $2I_2 + \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, $2I_2 + \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ and $2I_2 + \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$. The solution of size 3 is $3I_3 + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. There are no solutions of size $n \geq 4$. The strongly connected graphs corresponding to these solutions are represented on figure 3.

We deduce from Theorem 2.1 that the solutions cannot be of size bigger than $a - b$. Indeed the entries of the rank one matrix are all ≥ 1, so if the size of the matrix is bigger than $a - b$, its trace is also greater than $a - b$.

In the next paragraph, we will study all other possible cases.

2.2 The other cases

Let $P(t) = (t-a)(t-b)$ with $a, b \in \mathbb{C}$ be a polynomial of degree 2 in $\mathbb{Z}[t]$. Different cases can happen.

2.2.1 a and b are two complex conjugate roots of P

Let $P(t) = (t-a)(t-b)$ with $a, b \in \mathbb{C} - \mathbb{R}$. Then the equation $(A-a)(A-b) = 0$ has no strongly connected solutions. Indeed the eigenvalues of A are in the set $\{a, b\}$. But the Perron-Frobenius theorem says that if A is the adjacency matrix of a strongly connected graph, it has a real positive eigenvalue, which cannot be the case here.

2.2.2 $a = b$ is a double root of P

If $a = b$, then a has to be integer, otherwise P would not be in $\mathbb{Z}[t]$.

In that case, the equation is $(A-a)^2 = 0$. So the only possible eigenvalue of A is a. But by the Perron-Frobenius theorem, a is a simple root of the characteristic

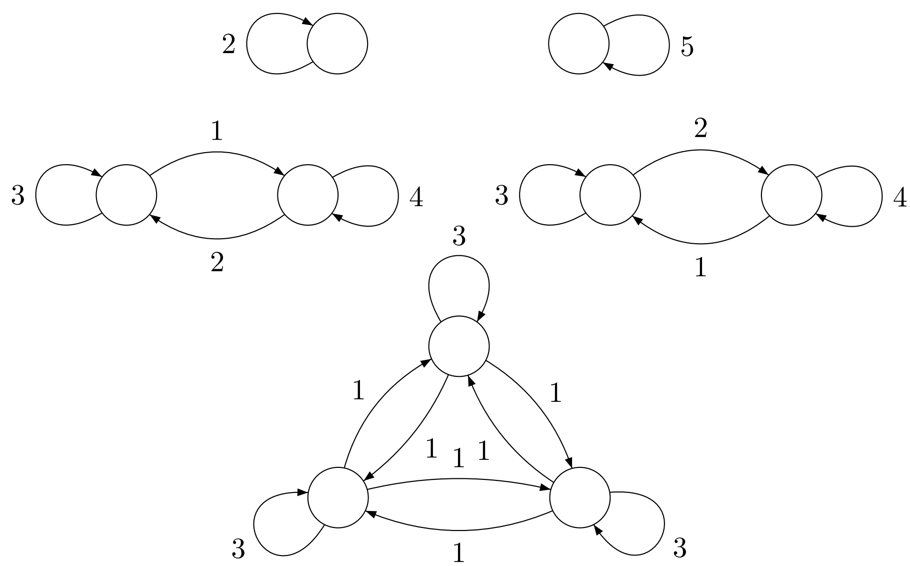

Figure 3: The solutions of $(A-5)(A-2)=0$.

polynomial χ_A. So $\chi_A(A) = A - a = 0$. If A is of size one, we have the solution a. Otherwise we have no solution because the graph corresponding to aI_n is not strongly connected.

2.2.3 a and $b = -a$ are the roots of P

In that case, the equation is $(A-a)(A+a) = A^2 - a^2 = 0$, ie. $A^2 = a^2 I_n$. By induction, for every $k \in \mathbb{N}$, $A^{2k} = a^{2k} I_n$ and $A^{2k+1} = a^{2k}A$. So, if we want the corresponding graph to be strongly connected, for every couple (i,j), $1 \le i,j \le n$, either the entry (i,j) of A or the entry (i,j) of I_n has to be positive. Thus for every (i,j) such that $i \ne j$, a_{ij} the entry (i,j) of A is positive. But then for $n \ge 3$, the entry $(1,2)$ of A^2 is equal to $\sum_{i=1}^n a_{1i}a_{i2}$ and therefore is ≥ 1, which is impossible. Therefore there are no solutions of size $n \ge 3$. For $n = 1$, $|a|$ is the only solution if a is an integer, and there is no solution otherwise. For $n = 2$, the solutions are the matrices $A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ such that b and c are positive integers whose product is equal to a^2.

2.2.4 a and b are irrational, $|a| > |b|$

First notice that since $P(t) = t^2 - (a+b)t + ab$, if a is rational, then b has to rational, and if they are rational, they have to be integers (otherwise P would not have integer coefficients).

Assume that a is irrational. Then for the same reason b is also irrational. Now we can apply the same method as in the case where a and b are integers, and search the solutions of the form $A = bI_n + M$ where M is a rank one matrix. Let

$$M = \begin{pmatrix} y_1 & c_2 y_1 & \dots & c_n y_1 \\ y_2 & c_2 y_2 & \dots & c_n y_2 \\ \vdots & \vdots & \dots & \vdots \\ y_n & c_2 y_n & \dots & c_n y_n \end{pmatrix}.$$

A has integer entries so $y_2, ..., y_n \in \mathbb{Z}$. But b is irrational so $c_2 y_2$ has to be irrational, which means that c_2 is irrational. But then $c_2 y_n$ would not be integer if $n \geq 3$, which is impossible. There are no solutions of size $n \geq 3$.

If $n = 1$, the only possible solutions are a and b, but they are not integers, so there is no solution.

If $n = 2$, we use the same techniques as for a and b integers, and we find that the solutions are matrices of the form $bI_n + \begin{pmatrix} m_1 - b & c(m1 - b) \\ \frac{1}{c}(m_2 - b) & m_2 - b \end{pmatrix}$ such that m_1, m_2, $c(m1 - b)$ and $\frac{1}{c}(m_2 - b)$ are integers, and $m_1 - b + m_2 - b = a - b$, ie. $m_1 + m_2 = a + b$.

Partial conclusion and remarks

We studied all possible cases and we found the strongly connected solutions of the equation $P(A) = 0$ for every polynomial P of degree 2 with integer coefficients.

We did not specify it in every case, but if a is not eigenvalue of A, then $A = bI_n$ which never corresponds to a strongly connected graph if $n \geq 2$ and b is a solution of size 1 if b is a positive integer.

We can also solve the equations of the type $(A - a)^k (A - b) = 0$, $a > |b|$. Indeed, by the Perron-Frobenius theorem, if a is eigenvalue of A, then it is a simple root of the characteristic polynomial of A. So it is a simple root in the minimal polynomial of A too, and we deduce that we also have $(A-a)(A-b) = 0$, which we just solved.

We will now try to solve this type of equations with polynomials of degree greater than 2.

3 On some particular cases of the matrix equation $P(A) = 0$ with $deg(P) \geq 3$

3.1 A necessary condition on the trace of the rank one matrix

In the case of the equation $(A-a)(A-b)=0$, assuming that a is eigenvalue of A we had the necessary condition that $A-b$ is a rank one matrix of trace $a-b$. We can find a similar necessary condition for polynomials of any degree.

Let P be a polynomial and A the matrix of a strongly connected graph such that $P(A)=0$. By the Perron-Frobenius theorem, A has a real non-negative eigenvalue, which must be a root of P. So P can always be written $P(t) = (t-a)q(t)$, with a non-negative real number.

We have the following necessary condition:

Theorem 3.1. *Let us assume that $(A-a)q(A)=0$, a is eigenvalue of A and a is bigger than every root of q (in module). Then $q(A)$ has to be a rank one matrix of trace $q(a)$.*

Proof: By lemma 1.3, we know that

$$q(A) = \frac{q(a)}{{}^tVU} U {}^tV,$$

where U (resp. ${}^tV$) is a right (resp. left) eigenvector of A corresponding to the eigenvalue a. We deduce that:

$$\begin{aligned} tr(q(A)) =& \frac{q(a)}{{}^tVU} tr(U {}^tV) \\ =& \frac{q(a)}{{}^tVU} tr({}^tVU) \\ =& q(a). \end{aligned}$$

So the trace of the rank one matrix $q(A)$ has to be $q(a)$. □

We can use this condition to try to find the solutions of maximal size of $P(A) = 0$ in some particular cases, which we will do in the next section.

3.2 The solutions of maximal size of $(A-a)(A^l + A^{l+1}) = 0$

For every non-negative integer l, the polynomial $(t-a)(t^l + t^{l+1})$ is in $\mathbb{Z}[t]$ if and only if a is an integer. By Theorem 3.1, if a is eigenvalue of A, $A^l + A^{l+1}$ has to be a rank one matrix of trace $a^l + a^{l+1} = (a+1)a^l$. Let us try to find the solutions of maximal size, ie. of size $n = a^l + a^{l+1}$. In that case, we have $A^l + A^{l+1} = J_n$, where J_n is the matrix of size $n \times n$ with all entries equal to 1.

First we need a theorem about matrix equations of the type $P(A) = J_n$, due to A. J. Hoffman and M. H. McAndrew [2].

Definition. Let G be a directed graph. For every vertex v_i, let d_i be the number of edges with terminal vertex v_i and e_i the number of edges with initial vertex v_i. G is said to be *strongly regular* if $d_i = e_i = d$ for all $1 \leq i \leq n$.

Theorem 3.2. *Let A be the adjacency matrix of a directed graph G. There exists a polynomial P such that $P(A) = J$ if and only if A is the adjacency matrix of a strongly connected and strongly regular graph.*

Now we are going to use the results of Y. Wu and Q. Li about the matrix equation $A^l + A^{l+1} = J_n$ [5] to prove the following theorem about the solutions of maximal size of our equation.

Let us first define what a Kautz digraph is.

Definition. The *Kautz digraph* $K(d, k+1)$ is a directed graph with $(a+1)a^k$ vertices labelled by all possible strings $s_0...s_k$ of length $k+1$ on an alphabet Σ of $d+1$ letters, such that $s_i \neq s_{i+1}$ for all $0 \leq i \leq k-1$. It has $(a+1)a^{k+1}$ edges $\{(s_0s_1...s_k, s_1s_2...s_{k+1})|s_i \in \Sigma, s_i \neq s_{i+1}\}$.

Example. Figure 4 shows the Kautz digraph $K(2,2)$ on the alphabet a, b, c.

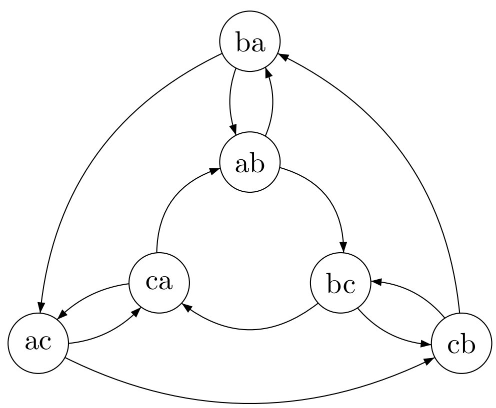

Figure 4: The Kautz digraph $K(2,2)$

Theorem 3.3. *The solution of maximal size of the equation $(A-a)(A^l+A^{l+1}) = 0$ is the Kautz digraph $K(a, l+1)$.*

Proof: If a is not eigenvalue of A, then A is solution of $A^l + A^{l+1} = 0$, which is impossible because A is the matrix of a strongly connected graph.

From now let us assume that a is eigenvalue of A. According to Wu and Li [5], a $(0,1)$-matrix A of size $n \times n$ satisfies $A^l + A^{l+1} = J_n$ if and only if $n = d^l + d^{l+1}$ and A is the adjacency matrix of the Kautz digraph $K(d, l+1)$.

In our case, $n = a^l + a^{l+1}$ so the Kautz digraph $K(a, l+1)$ is the only $(0,1)$-solution of $A^l + A^{l+1} = J_n$. But A has non-negative integer entries and is the adjacency matrix of a strongly connected graph, so the only possible solutions are $(0,1)$-solutions. So $K(a, l+1)$ is the only possible solution of maximal size of our equation.

Conversely, let us show that A the adjacency matrix of $K(a, l+1)$ satisfies $(A-a)(A^l + A^{l+1}) = 0$. By Theorem 3.2, $K(a, l+1)$ is strongly regular. It has $(a+1)a^l$ vertices and $(a+1)a^{l+1}$ edges, so every vertex v_i has in-degree d_i and out-degree e_i equal to a. Thus

$$\begin{aligned}(A-a)(A^l + A^{l+1}) =& (A-a)J_n \\ =& \begin{pmatrix} e_1 - a & \dots & e_1 - a \\ \vdots & & \vdots \\ e_n - a & \dots & e_n - a \end{pmatrix} \\ =& 0,\end{aligned}$$

which completes our proof. □

Example. The Kautz digraph $K(2,2)$ of figure 4 is the only solution of maximal size of the equation $(A-2)(A+A^2) = 0$.

3.3 The solutions of maximal size of $(A-a)(A-b)^2 = 0$, $a > |b|$

In this section, we use a paper by F. Curtis, J. Drew, C. Li and D. Pragel about the $(0,1)$-solutions of $A^2 = J$ [1] to find solutions of maximal size of the equation $(A-a)(A-b)^2 = 0$, where $a > |b|$. Notice that if $b > |a|$, we already found all the solutions in the conclusion of Section 2.

Theorem 3.4. *The matrices of the form $A = bI_n + P_1 + ... + P_{a-b}$, where $P_1, ..., P_{a-b}$ are permutation matrices such that P_iP_j and P_rP_s have no common non-zero entries if $(i,j) \neq (r,s)$ are solutions of maximal size $n = a - b$ of the equation $(A-a)(A-b)^2 = 0$, $a > |b|$. If $b = 0$ they are the only solutions of size n.*

Proof: In [1], we learn that the equation $A^2 = J_n$ has solutions if and only if $n = k^2$ for some integer k, and in that case the $(0,1)$-solutions are matrices of the form $A = P_1 + ... + P_{a-b}$, where $P_1, ..., P_{a-b}$ are permutation matrices such that P_iP_j and P_rP_s have no common non-zero entries if $(i,j) \neq (r,s)$.

In our case, $n = (a-b)^2$ so we obtain the solutions $A - bI_n = P_1 + ... + P_{a-b}$, where $P_1, ..., P_{a-b}$ are permutation matrices such that P_iP_j and P_rP_s have no common non-zero entries if $(i,j) \neq (r,s)$.

We can notice that by definition of the set of solutions, all matrices A that are solutions have their row sums equal to $a - b + b = a$. We can use the same argument as in the previous section to prove that conversely, those solutions are also solutions of the equation $(A-a)(A-b)^2 = 0$.

If $b = 0$, the equation is $(A - a)A^2 = 0$. In that case we obtain all the $(0, 1)$-matrices solutions of $A^2 = J$. But as A is the matrix of a strongly connected graph, so if it has an entry bigger than 1, then we can find two vertices v_i and v_j such that there is more than one path of length 2 from v_i to v_j, and therefore A^2 could not be equal to J. So in that case all the solutions are $(0, 1)$-matrices. $\square$

Example. All solutions of maximal size of the equation $(A - 2)A^2 = 0$ are isomorphic to the directed graph $L(G)$ of figure 6.

We give a more intuitive justification of this fact in the next section.

3.4 An algorithm to compute the solutions of maximal size of $(A-a)A^k = 0$

If a is not integer, then the polynomial doesn't have integer coefficients, so a has to be integer. If a is a non-positive integer, then there can be no positive eigenvalue, so by the Perron-Frobenius theorem, no strongly connected graph can be solution of this equation.

From now, let us assume that a is a positive integer. To study this case, let us introduce the notion of line digraph.

Definition. Given a directed graph G, the *line digraph* $L(G)$ of G is a directed digraph such that each vertex uv of $L(G)$ represents an edge (u, v) of G, and there is an edge (uv, wz) in $L(G)$ if and only if $v = w$.

Example. Figure 5 shows an example of the line digraph $L(G)$ of a directed graph G.

Now we need a lemma due to J. Gimbert and Y. Wu [4].

Lemma 3.5. *Let G be a strongly regular digraph of degree d with n vertices. Let A and A_L be the adjacency matrices of G and $L(G)$ respectively. Then*

$$P(A) = J_n \iff A_L P(A_L) = J_{nd},$$

where P is a polynomial.

We can extend this lemma to the k-th iterated line digraphs.

Definition. The k-th iterated line digraph $L^k(G)$ of a directed graph G is defined as follows: If $k = 0$, it is G. If $k \geq 1$, $L^k(G)$ is the line digraph of $L^{k-1}(G)$.

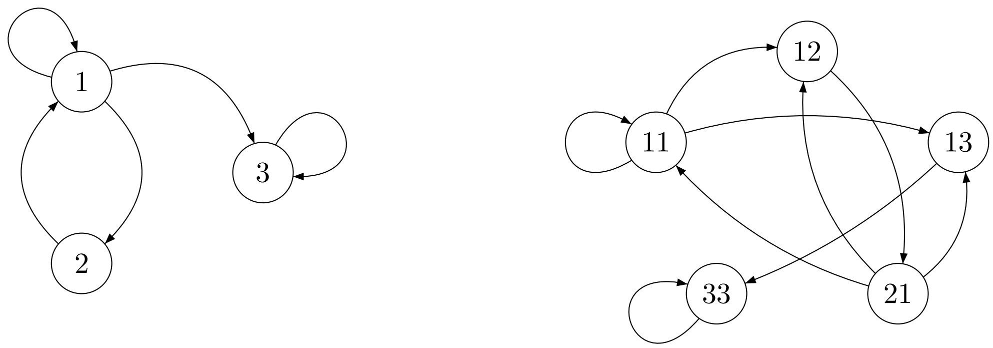

Figure 5: On the left, a directed graph G. On the right, its line digraph $L(G)$.

Lemma 3.6. *Let G be a strongly regular digraph of degree d with n vertices. Let A and A_{L^k} be the adjacency matrices of G and $L^k(G)$ respectively. Then*

$$P(A) = J_n \Longleftrightarrow A_{L^k}^k P(A_{L^k}) = J_{nd^k},$$

where P is a polynomial.

Proof: Let us prove this lemma by induction. If $k = 1$, it is Lemma 3.5. Let us assume that the lemma is true for $k \geq 1$ and let us show it for $k + 1$. We have:

$$P(A) = J_n \Longleftrightarrow A_{L^k}^k P(A_{L^k}) = J_{nd^k}.$$

And by Lemma 3.5, we also have:

$$A_{L^k}^k P(A_{L^k}) = J_{nd^k} \Longleftrightarrow A_{L^{k+1}}^{k+1} P(A_{L^{k+1}}) = J_{nd^{k+1}}.$$

We deduce that

$$P(A) = J_n \Longleftrightarrow A_{L^{k+1}}^{k+1} P(A_{L^{k+1}}) = J_{nd^{k+1}},$$

and the lemma is also true for $k + 1$. □

From this lemma, we can deduce a theorem and an algorithm to find the solutions of maximal size of the equation $(A - a)A^k = 0$.

Theorem 3.7. *The adjacency matrix $A(G)$ of a strongly connected graph G is a solution of maximal size of $(A - a)A^k = 0$ if and only if G is the $(k - 1)$-th iterated line digraph of the graph G' such that $A(G') = J_a$.*

Proof: $A(G)$ is solution of maximal size of $(A-a)A^k = 0$ if and only if $A(G)^k = J_{a^k}$. By Lemma 3.6,

$$A = J_a \iff A^k_{L^{k-1}} = J_{a^k}.$$

So the solutions are isomorphic to the $(k-1)$-th iterated line digraph of the graph G' such that $A(G') = J_a$. □

Example. Let us consider the equation $(A-2)A^2 = 0$. By Theorem 3.7, to find the solution of maximal size, we have to compute the line digraph $L(G)$ of G such that $A(G) = J_2$.

The graph $L(G)$ of figure 6 is the solution of maximal size of our equation.

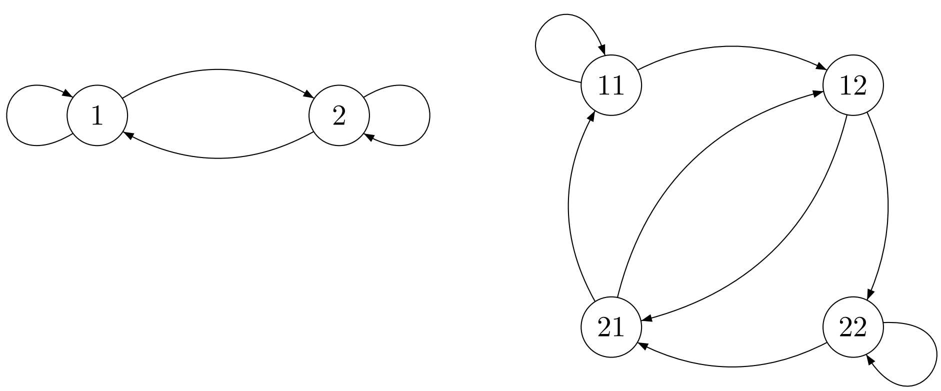

Figure 6: On the left, the complete directed graph G on two vertices. On the right, its line digraph $L(G)$, solution of maximal size of $(A-2)A^2 = 0$.

If we want the solution of maximal size of $(A-2)A^3 = 0$, we only have to compute the line digraph of $L(G)$, and so on.

The algorithm to find the solutions of maximal size of $(A-a)A^k = 0$ consists in computing the $(k-1)$-th iterated line digraph of the complete directed graph on a vertices.

Conclusion

If it was relatively easy to find all the solutions in the case of polynomials of degree 2, the problem gets much more complicated with polynomials of higher degree. Even in simple cases, we can in the best case only find the solutions of maximal size, and we were not able to find solutions in most of the cases. So this problem seems really difficult and will presumably need much more work to get solved.

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