

ARBORESCENCES OF COVERING GRAPHS

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ABSTRACT. An *arborescence* of a directed graph Γ is a spanning tree directed toward a particular vertex v . The arborescences of a graph rooted at a particular vertex may be encoded as a polynomial $A_v(\Gamma)$ representing the sum of the weights of all such arborescences. The arborescences of a graph and the arborescences of a covering graph $\tilde{\Gamma}$ are closely related. Using *voltage graphs* as a means to construct arbitrary regular covers, we derive a novel explicit formula for the ratio of $A_v(\Gamma)$ to the sum of arborescences in the lift $A_{\tilde{v}}(\tilde{\Gamma})$ in terms of the determinant of Chaiken’s voltage Laplacian matrix, a generalization of the Laplacian matrix. We also provide an alternative determinantal formula to account for the non-regular covers. Chaiken’s results on the relationship between the voltage Laplacian and vector fields on Γ are reviewed, and we provide a new proof of Chaiken’s results via a deletion-contraction argument, resulting in an original proof of the Matrix Tree Theorem. Special attention is given to the case of 2-fold covers: we exhibit a non-explicit bijection between classes of arborescences of the cover $\tilde{\Gamma}$ and pairs (T, g) of arborescences T of Γ and *negative vector fields* g , as well as provide a second proof of the ratio formula in this special case.

1. INTRODUCTION

In this report, we examine the relationship between arborescences of a graph and the arborescences of a covering graph. An *arborescence* rooted at a vertex v is a spanning tree of a directed graph that is directed towards v . We say that such an arborescence is “rooted” at v . Using the Matrix Tree Theorem [FS99, Theorem 5.6.8], we can compute the sum of the weights of all arborescences of Γ rooted at a given vertex as a minor of the Laplacian matrix of Γ . We denote this weighted sum $A_v(\Gamma)$, where v is the vertex at which the arborescences are rooted.

One natural question to ask is how completely do the arborescences of a graph Γ characterize the arborescences of a covering graph $\tilde{\Gamma}$. For example, every arborescence of Γ lifts to a partial arborescence of $\tilde{\Gamma}$, and this lift is unique if the root of the arborescence in $\tilde{\Gamma}$ is fixed; conversely, every arborescence of $\tilde{\Gamma}$ descends to a subgraph of Γ containing an arborescence. Therefore, we ask whether there is a meaningful relationship between $A_v(\Gamma)$ and $A_{\tilde{v}}(\tilde{\Gamma})$, where \tilde{v} is a lift of v . We show that the answer to this question is affirmative when $\tilde{\Gamma}$ is a regular cover. In this case, $A_v(\Gamma)$ always divides $A_{\tilde{v}}(\tilde{\Gamma})$, meaning that each arborescence of Γ corresponds to a set of arborescences of $\tilde{\Gamma}$. The primary goals of this paper are to derive an explicit formula for the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ and to examine cases where this ratio is especially computationally nice.

The ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ first arose in Galashin and Pylyavskyy’s study of *R-systems* [GP19]. The *R-system* is a discrete dynamical system on an edge-weighted strongly connected simple directed graph $\Gamma = (V, E, \text{wt})$ whose state vector $X = (X_v)_{v \in V}$ evolves to its next state $X' = (X'_v)_{v \in V}$ according to the following relation:

$$(1) \quad \sum_{(u,v) \in E} \text{wt}(u,v) \frac{X_v}{X'_u} = \sum_{(v,w) \in E} \text{wt}(v,w) \frac{X_w}{X'_v}$$

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This system is homogeneous in both X and X' , so we consider solutions in projective space. Galashin and Pylyavskyy determined all solutions X' of this equation as a function of X :

Theorem 1.1. [GP19] *The system given by equation (1) has solution*

$$X'_v = \frac{X_v}{A_v(\Gamma)}.$$

This solution is unique up to scalar multiplication, yielding a unique solution to the R -system in $\mathbb{P}^{|V|}$.

However, we can see the value of X'_v in equation (1) depends only on the neighborhood of the vertex v . Thus, in the case of a covering graph $\tilde{\Gamma}$, we may find two solutions to the R -system: one by applying the previous theorem directly, and one by treating each vertex of $\tilde{\Gamma}$ locally like a vertex of Γ , and then applying the theorem. The two respective solutions are

$$X'_{\tilde{v}} = \frac{X_{\tilde{v}}}{A_{\tilde{v}}(\tilde{\Gamma})} \quad \text{and} \quad X'_{\tilde{v}} = \frac{X_{\tilde{v}}}{A_v(\Gamma)}.$$

Therefore, uniqueness of the solution implies that the vectors

$$\left(\frac{X_{\tilde{v}}}{A_{\tilde{v}}(\tilde{\Gamma})} \right)_{\tilde{v} \in \tilde{V}} \quad \text{and} \quad \left(\frac{X_{\tilde{v}}}{A_v(\Gamma)} \right)_{\tilde{v} \in \tilde{V}}$$

are scalar multiples of each other, where \tilde{v} is any lift of v . Equivalently:

Corollary 1.2. *When Γ is strongly connected and simple, the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ is independent of the choice of vertex v and of the choice of lift \tilde{v} .*

The existence of this invariance motivates finding an explicit formula for this ratio. Galashin and Pylyavskyy correctly conjectured the formula for the ratio in the case of 2-fold covers:

Theorem 1.3. *Let $\Gamma = (V, E, \text{wt}, \nu)$ be an edge-weighted $\mathbb{Z}/2\mathbb{Z}$ -voltage directed multigraph, and let $\mathcal{L}(\Gamma)$ be its voltage Laplacian matrix. Then for any vertex v of Γ and any lift \tilde{v} of v in the derived graph $\tilde{\Gamma}$ of Γ , we have*

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{2} \det \mathcal{L}(\Gamma)$$

The original goal of our project was to prove this formula, but we ultimately deduced the formula in the generality of arbitrary k -fold covering graphs, with extra interpretations in the case of regular covering graphs. In full generality, this formula is

Theorem 1.4. *Let $\Gamma = (V, E, \text{wt})$ be an edge-weighted multigraph, and let $\tilde{\Gamma}$ be a k -fold cover of Γ . Let $\mathcal{L}(\Gamma)$ be the voltage Laplacian of Γ . Then for any vertex v of Γ and any lift \tilde{v} of v in $\tilde{\Gamma}$ of Γ , we have*

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{k} \det[\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]}$$

In the above formula $\det[\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]}$ is the determinant of $\mathcal{L}(\Gamma)$ as a $\mathbb{Z}[E]$ -linear transformation when the cover is regular. We can take this determinant by restriction of scalars (see Section 5 for details). In case of arbitrary covers (including non-regular ones), the matrix $[\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]}$ can be determined concretely from the covering graph (definition 5.6).

The rest of the report will proceed as follows. Section 2 covers the background and conventions necessary to read this paper. In this section, we also discuss the Laplacian matrix and the Matrix Tree Theorem in greater detail, and give additional topological background on covering graphs. In particular, we introduce the *voltage graph*, a construction that allows us to compactly describe arbitrary regular covering graphs $\tilde{\Gamma}$ by assigning a group-valued voltage to each edge of Γ . Section 3 reviews some known results relating *vector fields* on voltage graphs to the voltage Laplacian. Vector fields are closely related to arborescences, and this

discussion especially helps to frame the results of the case of 2-fold covers. We provide an original proof of these results on vector fields, which results in a novel proof of the Matrix Tree Theorem. Section 4 details our initial results on the ratio formula in the 2-fold case, first by extending Galashin-Pylyavskyy’s invariance result to arbitrary multigraphs, and then using this invariance to give an inductive proof of the formula in the 2-fold case. The methods of this section do not generalize to higher covers, so Section 5 derives the in general case via a linear algebraic approach through the Matrix Tree Theorem. To this end, we prove several new results about how changing vector space bases affects the values of minors. We conclude with several open questions.

2. BACKGROUND

2.1. Arborescences. Let $\Gamma = (V, E, \text{wt})$ be an edge-weighted directed multigraph with a weight function on the edges $\text{wt} : E \rightarrow R$, for some ring R . In the rest of this report, we will usually abbreviate “edge-weighted” to “weighted” and “directed multigraph” to “graph.” Any instance in which we wish to consider only directed or only simple graphs will be explicitly noted. We will consider the weights of the edges of G to be indeterminates, treating the weight $\text{wt}(e)$ of an edge e as a variable. Let the set of such variables be denoted $\text{wt}(E)$. We denote the source vertex of an edge e by e_s and target vertex of e by e_t . If an edge has source v and target w , we may write $e = (v, w)$. However, note that when Γ is not necessarily simple, there may be more than one edge satisfying these properties, so (v, w) may specify multiple edges. We denote the set of outgoing edges of a vertex v by $E_s(v)$, and the set of incoming edges of v by $E_t(v)$.

Definition 2.1. An *arborescence* T of Γ rooted at $v \in V$ is a spanning tree directed towards v . That is, for all vertices w , there exists a directed path from w to v through T .¹ We denote the set of arborescences of Γ rooted at vertex v by $\mathcal{T}_v(\Gamma)$. The *weight of an arborescence* $\text{wt}(T)$ is the product of the weights of its edges:

$$\text{wt}(T) = \prod_{e \in T} \text{wt}(e)$$

We denote by $A_v(\Gamma)$ the sum of the weights of all arborescences of Γ rooted at v :

$$A_v(\Gamma) = \sum_{T \in \mathcal{T}_v(\Gamma)} \text{wt}(T)$$

$A_v(\Gamma)$ is either zero or a homogeneous polynomial of degree $|V| - 1$ in the edge weights of G .

2.2. The Laplacian Matrix and the Matrix Tree Theorem. The Matrix Tree Theorem, also known as Kirchoff’s Theorem, yields a way of computing $A_v(\Gamma)$ through the *Laplacian matrix* of Γ .

Definition 2.2. Label the vertices of Γ as v_1, v_2, \dots . The *Laplacian matrix* $L(\Gamma)$ of a graph Γ is the difference of the weighted degree matrix D and the weighted adjacency matrix A of Γ :

$$L(\Gamma) = D(\Gamma) - A(\Gamma).$$

Here, the weighted degree matrix is the diagonal matrix whose i -th entry is

$$d_{ii} = \sum_{e \in E_s(v_i)} \text{wt}(e)$$

and the weighted adjacency matrix has entries defined by

$$a_{ij} = \sum_{e=(v_i, v_j)} \text{wt}(e).$$

¹In the literature, an arborescence rooted at v is usually defined to be a spanning tree directed *away* from v , so that v is the unique source rather than the unique sink; see, for example, [KV06], [Cha82], and [GM89]. Our convention is consistent with the study of R -systems.

Since we will always be working with weighted graphs in this paper, we will usually drop the word “weighted” when talking about the Laplacian matrix. Note also the ordering of the rows and columns of the Laplacian. We will always assume that v_1 corresponds to the first row and column of $L(\Gamma)$, that v_2 corresponds to the second row and column of $L(\Gamma)$, and so on.

Theorem 2.3. (Matrix Tree Theorem) [Cha82] *The sum of the weights of arborescences rooted at v_i is equal to the the minor of $L(\Gamma)$ obtained by removing the i -th row and column:*

$$A_{v_i}(\Gamma) = \det L_i^i(\Gamma).$$

2.3. Covering graphs.

Definition 2.4. A k -fold cover of $\Gamma = (V, E)$ is a graph $\tilde{\Gamma} = (\tilde{V}, \tilde{E})$ that is a k -fold covering space of G in the topological sense that preserves edge weight. In order to use this definition, we need to find a way to formally topologize directed graphs in a way that encodes edge orientation. To avoid this, we instead give a more concrete alternative definition of a covering graph. The graph $\tilde{\Gamma} = (\tilde{V}, \tilde{E})$ is a k -fold covering graph of $\Gamma = (V, E)$ if there exists a projection map $\pi : \tilde{\Gamma} \rightarrow \Gamma$ such that

- (1) π maps vertices to vertices and edges to edges;
- (2) $|\pi^{-1}(v)| = |\pi^{-1}(e)| = k$ for all $v \in V, e \in E$;
- (3) For all $\tilde{e} \in \tilde{E}$, we have $\text{wt}(\tilde{e}) = \text{wt}(\pi(\tilde{e}))$;
- (4) π is a local homeomorphism. Equivalently, $|E_s(\tilde{v})| = |E_s(\pi(\tilde{v}))|$ and $|E_t(\tilde{v})| = |E_t(\pi(\tilde{v}))|$ for all $\tilde{v} \in \tilde{V}$.

We do not require a covering graph to be connected—results about arborescences are trivial in the disconnected case anyways.

2.4. Voltage graphs and derived graphs.

Definition 2.5. Let G be a finite group. A *weighted G -voltage graph* $\Gamma = (V, E, \text{wt}, \nu)$ is a weighted directed multigraph with each edge e also labeled by an element $\nu(e)$ of G . This labeling is called the *voltage* of the edge e . Note that the voltage of an edge e is entirely distinct from the weight of e .

Definition 2.6. Given a G -voltage graph Γ , we may construct an $|G|$ -fold covering graph of Γ known as the *derived graph* $\tilde{\Gamma} = (\tilde{V}, \tilde{E}, \text{wt})$. The derived graph of a voltage graph is a graph with vertex set $\tilde{V} = V \times G$ and edge set

$$\tilde{E} := \{[v \times x, w \times (gx)] : x \in G, e = (v, w) \in \Gamma, \nu(e) = g \in G\}.$$

Example 2.7. Let $G = \mathbb{Z}/3\mathbb{Z} = \{1, g, g^2\}$, and let Γ be the following G -volted graph, where edges labeled (x, y) have edge weight x and voltage y :

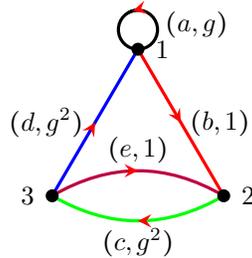


FIGURE 1. A $\mathbb{Z}/3\mathbb{Z}$ -voltage graph Γ

Then the derived graph $\tilde{\Gamma}$, with vertices $(v, x) = v_x$ and with edges labeled by weight, is:

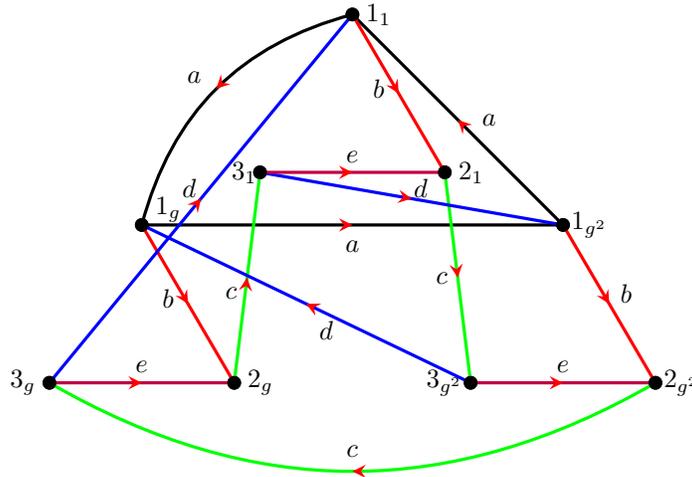


FIGURE 2. The derived covering graph $\tilde{\Gamma}$ of Γ . Edge colors denote correspondence to the edges of Γ via the quotient map.

While derived graphs might seem to be a very special subset of covering graphs, they in fact give rise to a broad class of covering graphs called *regular covering graphs*.

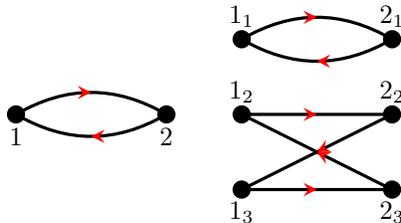
Definition 2.8. Given a graph Γ and a covering graph $\tilde{\Gamma}$, the *deck group* $\text{Aut}(\pi)$ of $\tilde{\Gamma}$ is the group of automorphisms on $\tilde{\Gamma}$ that preserve the fibers of the projection map π .

Definition 2.9. A *regular cover* $\tilde{\Gamma}$, sometimes known as a *Galois cover*, of a graph Γ is a covering graph whose deck group is transitive on each fiber $\pi^{-1}(v)$ for each $v \in V$.

Example 2.10. The derived graph in example 2.7 is a regular cover because the cyclic permutation σ that sends each $v_{i,x}$ to $v_{i,gx}$ is in $\text{Aut}(\pi)$, which shows that $\text{Aut}(\pi)$ is transitive on each fiber $\pi^{-1}(v)$.

Example 2.11. The following is a simple example of a graph (left) and a non-regular covering graph (right):

No automorphism maps vertex 1_1 to vertex 1_2 , since, for example, 1_1 is part of a 2-cycle and 1_2 is not, so $\text{Aut}(\pi)$ is not transitive on $\pi^{-1}(1)$. Nevertheless, all criteria necessary to be a covering graph are met.



Theorem 2.12. (Theorems 3 and 4 in [GT75]) *Every regular cover $\tilde{\Gamma}$ of a graph Γ may be realized as a derived cover of Γ with voltage assignments in $\text{Aut}(\pi)$. Conversely, every derived graph is a regular cover.*

The main focus of this paper is to explore the relationship between the arborescences of a voltage graph Γ and the arborescences of its derived graph $\tilde{\Gamma}$. Theorem 2.12 allows us to deal with all regular covering graphs in the framework of a voltage. It turns out that regularity is not necessary for Theorem 1.4, which holds for all k -fold covers; however, the results of this main theorem have nice interpretations in terms of the voltage Laplacian in the regular case.

2.5. The reduced group algebra. We wish to define a matrix similar to the Laplacian matrix that tracks all the relevant information in an G -voltage graph. In order to do so in general, we need to extend our field of coefficients in order to codify the data given by the voltage function ν . Following the language of Reiner and Tseng in [RT14]:

Definition 2.13. The *reduced group algebra* of a finite group G over a ring R is the quotient

$$\overline{R[G]} = \frac{R[G]}{\langle \sum_{g \in G} h \rangle},$$

where $R[G]$ is the group algebra of G over R . That is, we quotient the group algebra by the all-ones vector with respect to the basis given by G .

For simplicity, in the remainder of this paper we take $R = \mathbb{Z}$. In practice, we will always be dealing with integers. Note that if $G \cong \mathbb{Z}/2\mathbb{Z}$, then $\overline{\mathbb{Z}[G]} \cong \mathbb{Z}$, with the non-identity element of G identified with -1 . Similarly, if $G \cong \mathbb{Z}/p\mathbb{Z}$, with p prime, then $\overline{\mathbb{Z}[G]} \cong \mathbb{Z}(\zeta_p)$, where ζ_p is a primitive p -th root of unity and the generator g of G is identified with ζ_p . (To see this, note that both rings arise by adjoining to \mathbb{Z} an element with minimal polynomial $\sum_{i=0}^{p-1} x^i$.) The fact that the reduced group algebra of prime cyclic G lies in a field extension over $\mathbb{Q} \supseteq \mathbb{Z}$ will be important later in giving us nice formulas for the ratio of arborescences described in the introduction.

2.6. The voltage Laplacian matrix. We now define a generalization of the Laplacian matrix that takes into account voltages:

Definition 2.14. The *voltage adjacency matrix* $\mathcal{A}(G)$ has entries given by

$$a_{ij} = \sum_{e=(v_i, v_j) \in E} \nu(e) \text{wt}(e),$$

where we consider $\nu(e)$ as an element of the reduced group algebra $\overline{\mathbb{Z}[G]}$. That is, the i, j -th entry consists of sum of the *volted* weights of all edges going from the i -th vertex to the j -th vertex. The *voltage Laplacian matrix* $\mathcal{L}(\Gamma)$ is defined as

$$\mathcal{L}(\Gamma) = D(\Gamma) - \mathcal{A}(\Gamma)$$

where $D(\Gamma)$ is the (unvolted) weighted degree matrix as described in Definition 2.2.

Note that when every edge has trivial voltage, then $\mathcal{L}(\Gamma) = L(\Gamma)$, so that the voltage Laplacian is indeed a generalization of the Laplacian. Since we consider the edge weights of Γ as indeterminates, we treat the entries of $\mathcal{L}(G)$ as elements of $\overline{\mathbb{Z}[G]}[\text{wt}(E)] \subset \mathbb{Z}[\overline{G}][\text{wt}(E)]$ —that is, the polynomial ring of edge weights with coefficients in the reduced group algebra.

Example 2.15. Let Γ the $\mathbb{Z}/3\mathbb{Z}$ -voltage graph in Example 2.7. Under the identification $\overline{\mathbb{Z}[\mathbb{Z}/3\mathbb{Z}]} \cong \mathbb{Z}(\zeta_3)$, the voltage Laplacian of Γ is

$$\begin{aligned} \mathcal{L}(\Gamma) &= \begin{bmatrix} a+b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d+e \end{bmatrix} - \begin{bmatrix} \zeta_3 a & b & 0 \\ 0 & 0 & \zeta_3^2 c \\ \zeta_3^2 d & e & 0 \end{bmatrix} \\ &= \begin{bmatrix} (1-\zeta_3)a+b & -b & 0 \\ 0 & c & -\zeta_3^2 c \\ -\zeta_3^2 d & -e & d+e \end{bmatrix} \end{aligned}$$

3. THE VOLTAGE LAPLACIAN AND VECTOR FIELDS

Definition 3.1. A *vector field* γ of a directed graph Γ is a subgraph of Γ such that every vertex of Γ has outdegree 1 in γ . Similarly to arborescences, we define the *weight* of a vector field $\text{wt}(\gamma) := \prod_{e \in \gamma} \text{wt}(e)$ of a vector field be the product of its edge weights, so that $\text{wt}(\gamma)$ is a degree $|V|$ monomial with respect to the edge weights of Γ . Write $C(\gamma)$ for the set of cycles in a vector field γ . If G is abelian, and if c is a cycle of γ then defined the voltage of c as $\nu(c) := \prod_{e \in c} \nu(e)$ —this product is only well-defined when G is abelian.

The determinant of $\mathcal{L}(\Gamma)$ counts vector fields of Γ in the following way:

Theorem 3.2. *Let G be an abelian group, and let Γ be an edge-weighted G -voltage graph. Then*

$$\sum_{\gamma \in \Gamma} \left[\text{wt}(\gamma) \prod_{c \in C(\gamma)} (1 - \nu(c)) \right] = \det \mathcal{L}(\Gamma)$$

where the sum ranges over all vector fields γ of Γ .

Example 3.3. Let Γ be the $\mathbb{Z}/3\mathbb{Z}$ -voltage graph of example 2.7. There are four distinct vector fields of Γ (see Figure 3).

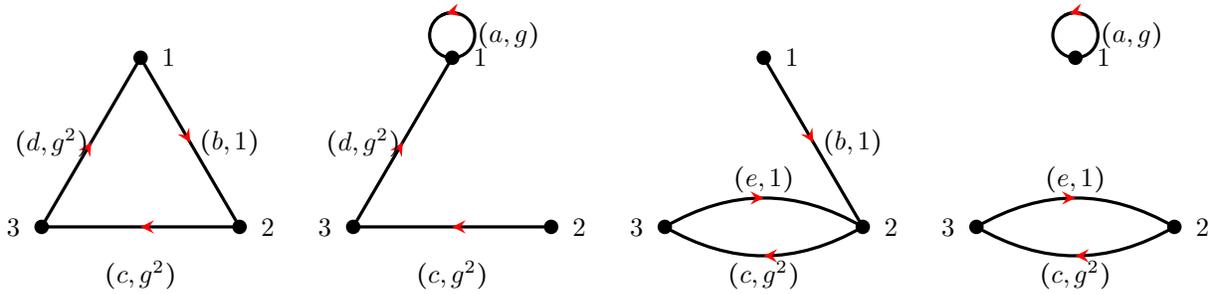


FIGURE 3. The four vector fields of Γ

The first three of these vector fields contain one cycle; from left to right, these unique cycles have weights ζ_3, ζ_3 , and ζ_3^2 . The rightmost vector field has two cycles, one with weight ζ and the other of weight ζ^2 . From Example 2.15, we have

$$\det \mathcal{L}(\Gamma) = (1 - \zeta_3)bcd + (1 - \zeta_3)acd + (1 - \zeta_3^2)bce + (1 - \zeta_3)(1 - \zeta_3^2)ace$$

The four terms in this expression correspond to the four vector fields of Γ as described by the theorem.

The following special case of the above, with $G = \mathbb{Z}/2\mathbb{Z}$, will be useful later in our first proof of the ratio of arborescences formula for signed graphs:

Corollary 3.4. *Suppose that Γ is a $\mathbb{Z}/2\mathbb{Z}$ -voltage graph, i.e. a signed graph. Define a negative vector field γ of Γ to be a vector field such that every cycle c of γ has an odd number of negative edges, so that $\nu(c) = -1$. Denote the set of negative vector fields of G by $\mathcal{N}(\Gamma)$. Then*

$$\sum_{\gamma \in \mathcal{N}(\Gamma)} 2^{\#C(\gamma)} \text{wt}(\gamma) = \det \mathcal{L}(\Gamma)$$

We present two proofs of Theorem 3.2: one original, as far as we are aware, and the other dating back to Chaiken.

The first proof proceeds by deletion-contraction, and requires the following lemma.

Lemma 3.5. *Let Γ be as in Theorem 3.2 with voltage function $\nu : E \rightarrow \overline{\mathbb{Z}[G]}$, let v be any vertex of Γ , and let $g \in G$. We define a new voltage function $\nu_{v,g}$ given by*

$$\nu_{v,g}(e) = \begin{cases} g\nu(e) & \text{if } e \in E_i(v), e \notin E_t(v) \\ g^{-1}\nu(e) & \text{if } e \in E_t(v), e \notin E_s(v) \\ \nu(e) & \text{else} \end{cases}$$

- (a) *for any cycle c of Γ , we have $\nu(c) = \nu_{v,g}(c)$; and*
(b) *the determinant of the voltage Laplacian of Γ with respect to the voltage ν is equal to the determinant of the voltage Laplacian of Γ with respect to $\nu_{v,g}$. That is,*

$$\det \mathcal{L}(V, E, \text{wt}, \nu) = \det \mathcal{L}(V, E, \text{wt}, \nu_{v,g})$$

Proof.

- (a) If c does not contain the vertex v , or if c is a loop at v , then the voltages of all edges in c remain unchanged. Otherwise, c contains exactly one ingoing edge e of v and one outgoing edge f of v , so that

$$\begin{aligned} \nu_{v,g}(c) &= \frac{\nu(c)}{\nu(e)\nu(f)} [g\nu(e)][g^{-1}\nu(f)] \\ &= \nu(c) \end{aligned}$$

as desired.

- (b) The matrix $\mathcal{L}(V, E, \text{wt}, \nu)$ may be transformed into the matrix $\mathcal{L}(V, E, \text{wt}, \nu_{v,g})$ by multiplying the row corresponding to v by g and multiplying the column corresponding to v by g^{-1} , so the determinant remains unchanged. ■

This lemma will allow us some freedom to change the voltage of Γ as needed.

Proof. (First proof of Theorem 3.2). Denote the left-hand side of the theorem as

$$\Omega(\Gamma) := \sum_{\gamma \in \Gamma} \left[\text{wt}(\gamma) \prod_{c \in \mathcal{C}(\gamma)} (1 - \nu(c)) \right]$$

We proceed by deletion-contraction. Our base case will be when the only edges of Γ are loops. When this happens, $\mathcal{L}(\Gamma)$ is diagonal, with

$$l_{ii} = \sum_{e=(v_i, v_i) \in E} (1 - \nu(e)) \text{wt}(e).$$

Thus we have

$$\det \mathcal{L}(\Gamma) = \prod_{i=1}^{|V|} \left(\sum_{e=(v_i, v_i) \in E} [1 - \nu(e)] \text{wt}(e) \right)$$

If we expand the product above, each term will correspond to a unique combination of one loop per vertex of Γ . But such combinations are precisely the vector fields of Γ , so we obtain

$$\det \mathcal{L}(\Gamma) = \Omega(\Gamma)$$

For the inductive step, assume that there exists at least one edge e between distinct vertices, and assume that the proposition holds for graphs with fewer non-loop edges than Γ . Using the lemma, we may change the voltage of Γ so that e has voltage 1 without changing either $\Omega(\Gamma)$ or $\det \mathcal{L}(\Gamma)$. Without loss of generality, let $v_1 = E_s$ and $v_2 = e_t$.

If γ is a vector field of Γ , then γ either contains e or it does not. In the latter case, γ is also a vector field of $\Gamma \setminus e$. Clearly all such γ arise uniquely from a vector field of $\Gamma \setminus e$. Therefore, there is a weight-preserving bijection between the vector fields of Γ not containing e and the vector fields of $\Gamma \setminus e$.

Otherwise, if $e \in \gamma$, then no other edge of the form (v_1, v_j) is in γ . We define a special type of contraction: let $\Gamma/_{1e} := (\Gamma/e) \setminus E_s(v_1)$. That is, we contract along e , and delete all other edges originally in $E_s(v_1)$. Note that the contraction process merges vertices v_1 and v_2 into a “supervertex,” which we denote v_{12} .

Then vector field γ descends uniquely to a vector field $\bar{\gamma}$ on $\Gamma/_{1e}$. Every vector field $\bar{\gamma}$ in $\Gamma/_{1e}$ corresponds uniquely to a vector field of Γ containing e , obtained by letting the unique edge coming out of the supervertex v_{12} in $\bar{\gamma}$ be the unique edge coming out of the vertex v_2 in γ , and letting e be the unique edge with source at v_1 in γ . This inverse map shows that the vector fields of Γ containing e are in bijection with the vector fields of $\Gamma/_{1e}$. This bijection is weight-preserving up to a factor of $\text{wt}(e)$. Finally, note that γ and its contraction $\bar{\gamma}$ have the same number of cycles, with the same voltages. If a cycle contains e in γ , then that cycle is made one edge shorter in $\bar{\gamma}$, but still has positive length since e is assumed to not be a loop. If c is a cycle containing e in γ , then since e has voltage 1, the cycle voltage $\nu(c/e)$ of the contracted version of c is equal to the cycle voltage before contraction. Thus, we may write

$$\Omega(\Gamma) = \Omega(\Gamma \setminus e) + \text{wt}(e)\Omega(\Gamma/_{1e})$$

By inductive hypothesis, since $\Gamma \setminus e$ and $\Gamma/_{1e}$ have strictly fewer non-loop edges than Γ , we have

$$\Omega(\Gamma \setminus e) + \text{wt}(e)\Omega(\Gamma/_{1e}) = \det \mathcal{L}(\Gamma \setminus e) + \text{wt}(e) \det \mathcal{L}(\Gamma/_{1e})$$

Note that $\mathcal{L}(\Gamma \setminus e)$ is equal to $\mathcal{L}(\Gamma)$ with $\text{wt}(e)$ deleted from both the 1,1- and 1,2-entries. Therefore, via expansion by minors, we obtain

$$(2) \quad \det L(\Gamma \setminus e) + \text{wt}(e) \det L_1^1(\Gamma) + \text{wt}(e) \det L_1^2(\Gamma) = \det \mathcal{L}(\Gamma)$$

where $\mathcal{L}_i^j(\Gamma)$ is the submatrix of $\mathcal{L}(\Gamma)$ obtained by removing the i -th row and the j -th column.

To construct $\mathcal{L}(\Gamma/_{1e})$ from $\mathcal{L}(\Gamma)$, we disregard the first row of $\mathcal{L}(\Gamma)$, since the special contraction $\Gamma/_{1e}$ simply removes the outgoing edges $E_s(v_1)$. Then, we combine the first two columns of $\mathcal{L}(\Gamma)$ by making their sum the first column of $\mathcal{L}(\Gamma/_{1e})$, since when we perform a contraction that merges v_1 and v_2 into v_{12} , we also have $E_t(v_1) \cup E_t(v_2) = E_t(v_{12})$. Thus $\mathcal{L}(\Gamma/_{1e})$ is a $(|V| - 1) \times (|V| - 1)$ matrix that agrees with both $\mathcal{L}_1^1(\Gamma)$ and $\mathcal{L}_1^2(\Gamma)$ on its last $|V| - 2$ columns, and whose first column is the sum of the first columns of $\mathcal{L}_1^1(\Gamma)$ and $\mathcal{L}_1^2(\Gamma)$. Therefore,

$$\det \mathcal{L}(\Gamma/_{1e}) = \det \mathcal{L}_1^1(\Gamma) + \det \mathcal{L}_1^2(\Gamma)$$

Substituting into (2), we obtain

$$\begin{aligned} \det \mathcal{L}(\Gamma) &= \det \mathcal{L}(\Gamma \setminus e) + \text{wt}(e) \det \mathcal{L}(\Gamma/_{1e}) \\ &= \Omega(\Gamma \setminus e) + \text{wt}(e)\Omega(\Gamma/_{1e}) \\ &= \Omega(\Gamma) \end{aligned}$$

as desired. ■

The second proof of the theorem follows a style similar to Chaiken’s proof of the Matrix Tree Theorem in [Cha82]. Chaiken actually proves a more general identity, which he calls the “All-Minors Matrix Tree Theorem,” that gives a combinatorial formula for any minor of the voltage Laplacian. We do not reproduce such generality here, but instead follow a simplified version of his proof, more along the lines of Stanton and White’s version of Chaiken’s proof of the Matrix Tree Theorem but with more generality [SW86].

Proof. (Second proof of Theorem 3.2) (Chaiken). Let Γ have n vertices. For simplicity, assume that Γ has no multiple edges, since we can always decomposed $\det \mathcal{L}(\Gamma)$ into a sum of determinants of voltage Laplacians of simple subgraphs of Γ , which also partitions the sum given in the theorem. We also assume that Γ is a complete bidirected graph, since we can ignore edges not in Γ by just considering them to have edge weight

0. Write $\mathcal{L}(\Gamma) = (\ell_{ij})$, write $D(\Gamma) = (d_{ij})$, and write $\mathcal{A}(\Gamma) = (a_{ij})$, so that $\ell_{ij} = \delta_{ij}d_{ii} - a_{ij}$. Then the determinant of $\mathcal{L}(\Gamma)$ may be decomposed as

$$\det \mathcal{L}(\Gamma) = \det(\delta_{ij}d_{ii} - a_{ij}) = \sum_{S \subseteq [n]} \left[\sum_{\pi \in P(S)} (-1)^{\#C(\pi)} \text{wt}_\nu(\pi) \prod_{i \in [n]-S} d_{ii} \right]$$

where $P(S)$ denotes the set of permutations of S , the set $C(\pi)$ is set of cycles of π , and $\text{wt}_\nu(\pi) := \prod_{i \in S} a_{i, \pi(i)}$. The product of the d_{ii} may be rewritten as a sum over functions $[n] - S \rightarrow [n]$, yielding

$$\begin{aligned} \det \mathcal{L}(\Gamma) &= \sum_{S \subseteq [n]} \sum_{\pi \in P(S)} (-1)^{c(\pi)} \text{wt}_\nu(\pi) \sum_{f: [n]-S \rightarrow [n]} \text{wt}(f) \\ (3) \qquad &= \sum_{S \subseteq [n]} \sum_{\pi \in P(S)} \sum_{f: [n]-S \rightarrow [n]} (-1)^{c(\pi)} \text{wt}_\nu(\pi) \text{wt}(f) \end{aligned}$$

where $\text{wt}(f)$ denotes the *unvolted* weight of the edge set corresponding to the function f , since this part of the product ultimately comes from the degree matrix. Thus, the determinant may be expressed as a sum of triples (S, π, f) of the above form—that is, we let S be an arbitrary subset of $[n]$, we let π be a permutation on S , and we let f be a function $[n] - S \mapsto [n]$.

The permutation π may always be decomposed into cycles, and f will sometimes have cycles as well—that is, sometimes we have $f^{(m)}(k) = k$ for some $k \in \mathbb{Z}$ and $k \in [n] - S$. We can “swap” cycles between π and f . Suppose c is a cycle of f that we want to swap into π . Let the subset of $[n]$ on which c is defined be denoted W . Then we may obtain from our old triple a new triple $(S \sqcup W, \pi \sqcup c, f|_{[n]-S-W})$, where $\pi \sqcup c$ denotes the permutation on $S \sqcup W$ given by $(\pi \sqcup c)(v) = \pi(v)$ if $v \in S$ and $(\pi \sqcup c)(v) = c(v)$ if $v \in W$. That is, we “move” C from f to π . Similarly, if c is a cycle of π , then we can obtain a new triple $(S - W, \pi|_{S-W}, f \sqcup c)$. Note that these two operations are inverses.

This process is always weight-preserving—it does not matter whether c is considered as part of π or as part of f , since it will always contribute $\text{wt}(c)$ to the product. However, one iteration of this map will swap the sign of $(-1)^{\#C(\pi)}$, and will also remove or add a factor from $\text{wt}_\nu(\pi)$ corresponding to the voltage of c . If π and f have k cycles amongst both of them, then there are 2^k possibilities for swaps, yielding a free action of $(\mathbb{Z}/2\mathbb{Z})^k$. If we start from the case π is the empty partition, then the sign $(-1)^{\#C(\pi)}$ starts at 1. Every time we choose to swap a cycle c into π from f , we flip this sign and multiply by $\nu(c)$, effectively multiplying by $-\nu(c)$. Thus, the sum of terms in (3) coming from the orbit of the action of $(\mathbb{Z}/2\mathbb{Z})^k$ on (S, f, π) is

$$\text{wt}(\pi) \text{wt}(f) \prod_{c \in C(\pi) \cup C(f)} (1 - \nu(c))$$

where $\text{wt}(\pi)$ is now unvolted. This orbit class corresponds to the contribution of one vector field γ of Γ to the overall sum, where γ is the unique vector field such that $\text{wt}(\gamma) = \text{wt}(\pi) \text{wt}(f)$. Thus, summing over all orbit classes, we obtain the desired formula:

$$\det \mathcal{L}(\Gamma) = \sum_{\gamma \in \Gamma} \left[\text{wt}(\gamma) \prod_{c \in C(\gamma)} (1 - \nu(c)) \right]$$

■

Corollary 3.4 gives a fast proof of the Matrix Tree Theorem (Theorem 2.3). Indeed, Chaiken considers Theorem 3.2 to be a generalization of the Matrix Tree Theorem.

Proof. (Matrix Tree Theorem) The set $\mathcal{T}_{v_j}(\Gamma)$ of arborescences of the graph Γ rooted at v remains the same if we remove all edges in $E_s(v_j)$ and replace them with a single loop e , so let this be the case. We assign a $\mathbb{Z}/2\mathbb{Z}$ -voltage to Γ : let all edges of Γ be positive except e , which is negative. Then the negative vector fields of Γ are precisely the arborescences of Γ plus the edge e —no other configurations are possible, since any

cycle other than the loop e would be positive. Since every such negative vector field has exactly one cycle (the loop e), by the corollary the sum of the weights of the arborescences of Γ is given by

$$A_{v_j}(\Gamma) = \frac{\det \mathcal{L}(\Gamma)}{2 \text{wt}(e)}.$$

However, the row in $\mathcal{L}(\Gamma)$ corresponding to v_j consists of all zeroes except in the column corresponding to v , which contains $2 \text{wt}(e)$. Thus, $\det \mathcal{L}(\Gamma)$ is given by $2 \text{wt}(e) \det \mathcal{L}_j^j(\Gamma)$, where $\det \mathcal{L}_j^j(\Gamma)$ is the minor of $\mathcal{L}(\Gamma)$ corresponding to removing the j -th row and column. Thus,

$$A_v(\Gamma) = \det \mathcal{L}_v^v(\Gamma)$$

as desired. ■

4. THE RATIO FORMULA FOR 2-FOLD COVERS

We now state and prove a result about the arborescences of any 2-fold covering graph. Recall in Corollary 1.2, we showed that the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ is well-defined and independent of the choice of vertex v when Γ is simple and strongly connected. Galashin and Pylyavskyy's study of R -systems occurs almost exclusively in the context of strongly connected simple digraphs, but with the corollary in hand we no longer need to consider the relevant R -system. Thus, we may extend the proposition to any directed multigraph:

Corollary 4.1. (Invariance under rerooting). *Corollary 1.2 extends to arbitrary multigraphs whenever possible. That is, if Γ is any multigraph (not necessarily simple or strongly connected), and $\tilde{\Gamma}$ is any covering graph of Γ , we still have that the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ is independent of the choice of vertex v and its lift \tilde{v} as long as this ratio is defined (i.e. $A_v(\Gamma) \neq 0$).*

Proof. Suppose Γ is simple but not necessarily strongly connected. We may consider Γ as a subgraph of the complete graph $K_{|V|}$. Denote the complement of the edge set of G in $K_{|V|}$ as $E_{K \setminus \Gamma}$. By Proposition 1.2, we know that $\frac{A_{\tilde{v}}(\widetilde{K_{|V|}})}{A_v(K_{|V|})}$ is well-defined and independent of the choice of v .

Now note that whenever an edge of Γ has weight 0, any arborescence containing that edge vanishes in the polynomial $A_v(\Gamma)$. Thus, let φ be the evaluation homomorphism that maps the weight of every edge in $E_{K \setminus \Gamma}$ to 0, so that

$$(4) \quad \varphi(A_v(K_{|V|})) = A_v(\Gamma)$$

$$(5) \quad \varphi(A_{\tilde{v}}(\widetilde{K_{|V|}})) = A_{\tilde{v}}(\tilde{\Gamma})$$

since every arborescence of Γ rooted at v is also an arborescence of $K_{|V|}$ rooted at v , and this set of arborescences is precisely the set of arborescences not containing any edge in $E_{K \setminus \Gamma}$ (and similarly for $\widetilde{K_{|V|}}$ and $\tilde{\Gamma}$). Since ratio of the left-hand sides of equations (4) and (5) is invariant under changing root, so is the ratio of the right-hand sides.

In the additional case that Γ is not simple, we can augment Γ to a graph Γ^+ by placing a vertex on the midpoint of each edge of Γ —given $e = (u, v)$, we add a vertex v_e with unique ingoing edge (u, v_e) and unique outgoing edge (v_e, v) . Set $\nu(u, v_e) = \nu(u, v)$ and $\nu(v_e, v) = 1$. Then Γ^+ is a simple graph, since every edge is either of the form (v_e, v) or (u, v_e) , and we know $|E_s(v_e)| = |E_t(v_e)| = 1$. We therefore know that Proposition 1.2 holds for Γ^+ . However, note that whenever we root at some vertex $v \in V(\Gamma) \subseteq V(\Gamma^+)$, every arborescence must contain every edge of the form (v_e, v) , since this is the only outgoing edge of v_e ; similarly, every arborescence of the cover of Γ^+ contains both lifts of every (v_e, v) . Since these edges must always be used in both of these constructions, we may freely contract along them without cutting out or merging any arborescences in both Γ^+ and $\tilde{\Gamma}^+$. Contracting along every such edge transforms Γ^+ back into Γ . Therefore, if we let $\varphi : \text{wt}(v_e, v) \mapsto 1$ and $\varphi : \text{wt}(u, v_e) = \text{wt}(u, v)$, we have $A_v(\Gamma) = \varphi(A_v(\Gamma^+))$ and $A_{\tilde{v}}(\tilde{\Gamma}) = \varphi(A_{\tilde{v}}(\tilde{\Gamma}^+))$. Therefore, Proposition 1.2 result holds for non-simple Γ as well.

■

We now turn to the first proof of Theorem 1.3, which provides a formula for $\frac{A_v(\tilde{\Gamma})}{A_v(\Gamma)}$ when $\tilde{\Gamma}$ is a 2-fold cover, which we now restate:

Theorem 4.2. *Let Γ be an edge-weighted $\mathbb{Z}/2\mathbb{Z}$ -volted directed multigraph—that is, a signed graph. For any vertex v of Γ and any lift \tilde{v} of v to the derived graph $\tilde{\Gamma}$ of Γ , we have*

$$A_v(\Gamma) \det \mathcal{L}(\Gamma) = 2A_{\tilde{v}}(\tilde{\Gamma})$$

Equivalently, either $A_v(\Gamma) = 0$ or we have

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{2} \det \mathcal{L}(\Gamma).$$

Proof. We proceed by strong induction, and by relying heavily on the fact that we can usually prove the formula rooting at a specific vertex to proliferate the formula to any vertex. We will also apply the results about negative vector fields from Corollary 3.4.

4.1. Base cases. First, note that if some vertex of Γ has outdegree 0, then both sides of the above identity are 0, no matter which root is chosen, since a row of $\det \mathcal{L}(\Gamma)$ will be the zero vector and at least two vertices in $\tilde{\Gamma}$ will have outdegree 0.

Next, suppose that every vertex of Γ has outdegree exactly 1. Choose any $v \in G$. Then there is only one candidate for a negative vector field of Γ , and only one candidate for an arborescence of Γ rooted at v . If Γ has more than one cycle, then Γ is disconnected, so that $A_v(\Gamma) = 0$ and $A_{\tilde{v}}(\tilde{\Gamma}) = 0$. Assume Γ has exactly one cycle. If this cycle does not contain v , then no path from the vertices in the cycle to v exists, so no arborescence rooted at v exists, and $A_v(\Gamma) = A_{\tilde{v}}(\tilde{\Gamma}) = 0$. Now assume that the unique cycle of Γ contains v . If this cycle is positive, then no negative vector fields exist, and thus $\det \mathcal{L}(\Gamma) = 0$ by Corollary 3.4. Furthermore, there exist two disjoint lifts of this cycle to $\tilde{\Gamma}$, which again means that $\tilde{\Gamma}$ is disconnected, so that $A_{\tilde{v}}(\tilde{\Gamma}) = 0$. Thus, the statement holds in these cases.

The only remaining case is if Γ has a unique negative cycle that contains v . Then this cycle lifts to a cycle twice as long in $\tilde{\Gamma}$ containing both lifts v' and v'' of v , which is the unique cycle in \tilde{G} . Therefore, the edges of \tilde{G} , except the unique edge in $E_i(\tilde{v})$, form a unique arborescence \tilde{T} rooted at \tilde{v} . Similarly, the edges of G , except the unique edge in $E_s(v)$, form a unique arborescence T rooted at v ; and Γ forms the unique negative vector field γ of itself. Thus $\text{wt}(T) \text{wt}(\gamma) = \text{wt}(\tilde{T})$. Since $\det \mathcal{L}(\Gamma) = 2 \text{wt}(\gamma)$ by Corollary 3.4, previous proposition, we conclude that $A_v(\Gamma) \det \mathcal{L}(\Gamma) = 2A_{\tilde{v}}(\tilde{\Gamma})$. This proves the identity when $|E| \leq |V|$.

4.2. Main inductive step. Now suppose that the identity holds whenever $|E| \leq k$ for some $k \geq |V|$, and let G have $k+1$ edges. By the pigeonhole principle, at least one vertex v of G satisfies $|E_s(v)| \geq 2$. Assume further that we can choose such v with $A_v(\Gamma) \neq 0$.

Let e be any edge in $E_s(v)$, and define $E_s^e(v) := E_s(v) \setminus e$. Then both $G \setminus e$ and $G \setminus E_s(v)$ have at most k edges, since $|E_s(v)| \geq 2$. By inductive hypothesis,

$$\begin{aligned} A_v(\Gamma \setminus e) \det \mathcal{L}(\Gamma \setminus e) &= 2A_{\tilde{v}}(\widetilde{\Gamma \setminus e}) \\ A_v(\Gamma \setminus E_s^e(v)) \det \mathcal{L}(\Gamma \setminus E_s^e(v)) &= 2A_{\tilde{v}}(\widetilde{\Gamma \setminus E_s^e(v)}) \end{aligned}$$

Without loss of generality, let $\tilde{v} = v'$, and let e', e'' be the lifts of e with sources at v', v'' , respectively. Every arborescence of $\tilde{\Gamma}$ rooted at v' contains exactly one edge in $E_s(v'')$. This edge is either e'' or it is not, so we may partition such arborescences into two disjoint classes based on whether they include e'' —that is,

$$A_{v'}(\tilde{\Gamma}) = A_{v'}(\tilde{\Gamma} \setminus e'') + A_{v'}(\tilde{\Gamma} \setminus E_s^{e''}(v''))$$

However, note that

$$A_{v'}(\tilde{\Gamma} \setminus e'') = A_{v'}(\tilde{\Gamma} \setminus \{e', e''\}) = A_{v'}(\widetilde{\Gamma \setminus e})$$

No arborescence rooted at v' utilizes any edge with source at v' , so we may simply delete the edge e' from Γ as it suits us. Similarly,

$$\begin{aligned} A_{v'}(\tilde{\Gamma} \setminus E_s^{e''}(v'')) &= A_{v'}(\tilde{\Gamma} \setminus (E_s^{e''}(v'') \cup E_i^{e'}(v'))) \\ &= A_{v'}(\Gamma \setminus \widetilde{E_i^e(v)}) \end{aligned}$$

Thus,

$$\begin{aligned} A_{\tilde{v}}(\tilde{\Gamma}) &= A_{\tilde{v}}(\widetilde{\Gamma \setminus e}) + A_{\tilde{v}}(\Gamma \setminus \widetilde{E_s^e(v)}) \\ &= \frac{1}{2} A_v(\Gamma \setminus e) \det \mathcal{L}(\Gamma \setminus e) + \frac{1}{2} A_v(\Gamma \setminus E_s^e(v)) \det \mathcal{L}(\Gamma \setminus E_s^e(v)) \end{aligned}$$

Now, note that $A_v(\Gamma \setminus e) = A_v(\Gamma \setminus E_s^e(v)) = A_v(\Gamma)$ —again, no arborescence rooted at v utilizes any edge in $E_s(v)$. Thus,

$$2A_{\tilde{v}}(\tilde{\Gamma}) = A_v(\Gamma) (\det \mathcal{L}(\Gamma \setminus e) + \det \mathcal{L}(\Gamma \setminus E_s^e(v)))$$

Finally, note that the matrix $\mathcal{L}(\Gamma)$, the matrix $\mathcal{L}(G \setminus e)$, and the matrix $L(G \setminus E_s^e(v))$ are all equal except in the row corresponding to v , and that the sum of the v -th rows of $\mathcal{L}(\Gamma \setminus e)$ and $L(\Gamma \setminus E_s^e(v))$ is equal to the v -th row of $\mathcal{L}(\Gamma)$. Thus, $\det \mathcal{L}(\Gamma) = \det \mathcal{L}(\Gamma \setminus e) + \det L(\Gamma \setminus E_s^e(v))$, so that

$$2A_{\tilde{v}}(\tilde{\Gamma}) = A_v(\Gamma) \det \mathcal{L}(\Gamma)$$

as desired. By Corollary 4.1, we conclude that

$$2A_{\tilde{u}}(\tilde{\Gamma}) = A_u(\Gamma) \det \mathcal{L}(\Gamma)$$

for any choice of $u \in V$.

4.3. Exceptional cases. We must choose v to be some vertex with $|E_s(v)| \geq 2$, but what if all such vertices satisfy $A_v(\Gamma) = 0$? Then either $A_u(\Gamma) = 0$ for all $u \in \Gamma$, in which case the theorem is trivially satisfied, or there exists some vertex u with outdegree exactly 1 such that $A_u(\Gamma) \neq 0$.

Suppose that in the latter case we can choose u such that there exist two distinct arborescences T_1 and T_2 rooted at u . Then there must exist some vertex w such that the outgoing edge e of w in T_1 is distinct from the outgoing edge f of w in T_2 . Define Γ^+ to be the graph obtained by Γ by adding an auxiliary edge a from u to w , so that Γ^+ has $k+2$ edges, and therefore $\Gamma^+ \setminus e$ and $\Gamma^+ \setminus E_s^e(w)$ both have at most $k+1$ edges. Since u has outdegree 2 in both $\Gamma^+ \setminus e$ and $\Gamma^+ \setminus E_s^e(w)$, we may apply the inductive step to conclude

$$\begin{aligned} A_u(\Gamma^+ \setminus e) \det \mathcal{L}(\Gamma^+ \setminus e) &= 2A_{\tilde{u}}(\widetilde{\Gamma^+ \setminus e}) \\ A_u(\Gamma^+ \setminus E_s^e(w)) \det \mathcal{L}(\Gamma^+ \setminus E_s^e(w)) &= 2A_{\tilde{u}}(\Gamma^+ \setminus \widetilde{E_s^e(w)}) \end{aligned}$$

Note that $A_u(\Gamma^+ \setminus E_s^e(w)) \neq 0$, since by assumption there exists at least one arborescence T_1 rooted at u using the edge e , so that T_1 remains an arborescence even after removing the edges $E_s^e(v)$. Similarly, $A_u(\Gamma^+ \setminus e) \neq 0$. Therefore, we may apply Proposition 4.1 to conclude

$$\begin{aligned} A_w(\Gamma^+ \setminus e) \det \mathcal{L}(\Gamma^+ \setminus e) &= A_{\tilde{w}}(\widetilde{\Gamma^+ \setminus e}) \\ A_w(\Gamma^+ \setminus E_s^e(w)) \det \mathcal{L}(\Gamma^+ \setminus E_s^e(w)) &= A_{\tilde{w}}(\Gamma^+ \setminus \widetilde{E_s^e(w)}) \end{aligned}$$

Since e and the edges of $E_s^e(w)$ are elements of $E_s(w)$, we can apply the same arguments as we did in the original inductive step. Then it follows that $A_w(\Gamma^+ \setminus e) = A_w(\Gamma^+ \setminus E_s^e(w)) = A_w(\Gamma^+)$, that $A_{\tilde{w}}(\widetilde{\Gamma^+ \setminus e}) + A_{\tilde{w}}(\Gamma^+ \setminus \widetilde{E_s^e(w)}) = A_{\tilde{w}}(\tilde{\Gamma}^+)$, and ultimately that

$$A_w(\Gamma^+) \det \mathcal{L}(\Gamma^+) = 2A_{\tilde{w}}(\tilde{\Gamma}^+)$$

Since $A_w(\Gamma^+) \neq 0$ —the auxiliary edge a ensures that any arborescence rooted at u may be modified into an arborescence rooted at w —we may reroot to conclude

$$A_u(\Gamma^+) \det \mathcal{L}(\Gamma^+) = 2A_{\tilde{u}}(\tilde{\Gamma}^+)$$

Note that every arborescence $T \in \mathcal{T}_u(\Gamma)$ lifts uniquely to an arborescence $T^+ \in \mathcal{T}_u(\Gamma^+)$ not containing a , and conversely that every arborescence $T^+ \in \mathcal{T}_u(\Gamma^+)$ not containing a descends uniquely to an arborescence $T \in \mathcal{T}_u(\Gamma)$. We therefore perform the same trick that we did in the proof of Corollary 4.1. Let φ be the evaluation homomorphism mapping $\text{wt}(a) \mapsto 0$. Then have $\varphi(A_u(\Gamma^+)) = A_u(\Gamma)$, $\varphi(\mathcal{L}(\Gamma^+)) = \mathcal{L}(\Gamma)$, and $\varphi(A_{\tilde{u}}(\tilde{\Gamma}^+)) = A_{\tilde{u}}(\tilde{\Gamma})$. Since φ is a homomorphism, we conclude

$$A_u(\Gamma) \det \mathcal{L}(\Gamma) = 2A_{\tilde{u}}(\tilde{\Gamma})$$

Thus, the formula is proven.

4.4. Rooted tree case. Finally, we consider the case where

- (1) No vertices with outdegree ≥ 2 root an arborescence;
- (2) There exists at least one arborescence rooted at some vertex; *and*
- (3) All vertices with outdegree 1 root no more than 1 arborescence?

In this case, Γ must have a structure similar to a rooted tree. Let u be a vertex with outdegree 1 that roots exactly one arborescence T . Without loss of generality, u is the only vertex of outdegree 1—we may contract along the unique outgoing edge e of any other such vertex u' to yield a graph with fewer edges otherwise, since

- Every arborescence of Γ rooted at u passes through e , so that $A_u(\Gamma) = \text{wt}(e)A_u(\Gamma/e)$;
- Every arborescence of $\tilde{\Gamma}$ passes through both lifts of e , so that $A_{\tilde{u}}(\tilde{\Gamma}) = \text{wt}(e)^2 A_{\tilde{u}}(\tilde{\Gamma}/e)$; and
- $\det \mathcal{L}(\Gamma) = \text{wt}(e) \det \mathcal{L}(\Gamma/e)$ via expansion by minors along the row corresponding to u' .

Therefore, the unique outgoing edge of u must be a loop, since otherwise the terminal vertex of this edge roots an arborescence, violating condition 1 above since all vertices other than u have outdegree ≥ 2 . We may treat Γ as a Hasse diagram for the poset defined by T , with u the unique minimal element. Every other vertex v of Γ has exactly one edge belonging to the arborescence T , and all other edges of v must point to some $v \geq u$. Otherwise, a non-cyclic path from v to u distinct from the one given by T would exist, violating the uniqueness of the arborescence T .

Take any vertex $w \neq u$. Let e be the edge of w belonging to T . Define Γ^+ to be the graph obtained from Γ by adding an auxiliary edge a from u to w . Then we apply the same trick with the arborescences of the cover to conclude that $\Gamma^+ \setminus e$ and $\Gamma^+ \setminus E_s^e(w)$ satisfy the formula when rooted at u . Note that $A_u(\Gamma^+ \setminus E_s^e(w))$ is never zero, since T remains an arborescence in $\Gamma^+ \setminus E_s^e(w)$, so we may apply Corollary 4.1 to conclude that $A_v(\Gamma^+ \setminus E_s^e(w)) \det \mathcal{L}(\Gamma^+ \setminus E_s^e(w)) = A_{\tilde{w}}(\Gamma^+ \setminus \widetilde{E_s^e(w)})$.

If any edge in $E_s^e(w)$ does not point towards w , it points to some vertex $w' > w$, so that w' roots an arborescence by modifying T to pass through a and the edge $(w, w') \in E_s^e(w)$. Since w' has outdegree ≥ 2 and roots at least one arborescence, we conclude that the desired identity also holds on $\Gamma^+ \setminus e$ when rooting at w instead. In this case, we know the formula holds for $\Gamma^+ \setminus e$ and $\Gamma^+ \setminus E_s^e(w)$ when rooting at w , so now we may apply the same logic as the inductive step to conclude that the formula holds for Γ^+ when rooting at w , and therefore when rooting at u . Setting $\text{wt}(a) = 0$ then shows that the formula holds for Γ rooting at u .

If this process goes through for at least one vertex $w \neq u$, then we are done. Otherwise, we conclude that edge set of Γ consists only of the tree T plus loops, in which case we may prove the formula directly. Without loss of generality, all loops are negative, since positive loops do not contribute to either the negative vector fields of Γ nor the arborescences of $\tilde{\Gamma}$. Then every arborescence of $\tilde{\Gamma}$ contains at least one lift of every edge in T , but this is the only condition on the arborescences—as long as the lift of the same negative loop is not used twice, there can be no cycles. For every loop besides the one at u , there are two choices of lifts. Thus, for each negative vector field $\gamma \in \Gamma$, we obtain $2^{\#C(\gamma)-1}$ arborescences of $\tilde{\Gamma}$ —one factor of two for each

loop of γ other than the one at u . Since this process uniquely describes all arborescences of $\tilde{\Gamma}$, we have by Corollary 3.4 that

$$\begin{aligned} 2A_{\tilde{u}}(\tilde{\Gamma}) &= \text{wt}(T) \sum_{\gamma \in \tilde{\Gamma}} 2^{\#C(\gamma)} \text{wt}(\gamma) \\ &= A_u(\Gamma) \det \mathcal{L}(\Gamma) \end{aligned}$$

This exhausts all possible exceptions to the inductive step, completing the proof. ■

5. GENERALIZATION TO HIGHER COVERS

The preceding proof unfortunately does not generalize to k -fold covers for $k > 2$. In these cases, the main inductive step fails because there are too many lifts of an outgoing edge of v , even if we disregard the outgoing edges of \tilde{v} . In this section, we present a more algebraic approach that generalizes to higher covers.

5.1. Restriction of scalars.

Definition 5.1. Let R be a commutative ring, and let S be a free R -algebra of finite rank. Let T be an S -linear transformation on a free S -module M of finite rank. Then we may also consider M as a free R -module of finite rank, and T as an R -linear transformation; this is known as *restriction of scalars*. We write $\det_R T$ to denote the determinant of T as an R -linear transformation.

Recall that the voltage Laplacian $\mathcal{L}(\Gamma)$ has entries in the reduced group algebra augmented by edge weights: $S = \mathbb{Z}[G][E]$. Letting $R = \mathbb{Z}[E]$, we may also consider $\mathcal{L}(\Gamma)$ as an R -linear transformation on a R -module of rank $(|G| - 1)|V|$.

Example 5.2. Returning to Example 2.15, the voltage Laplacian $\mathcal{L}(\Gamma)$ is a matrix that represents a linear transformation on a $\mathbb{Z}(\zeta_3)[E]$ -module with basis vectors indexed by the three vertices of Γ :

$$\mathcal{L}(\Gamma) = \begin{bmatrix} (1 - \zeta_3)a + b & -b & 0 \\ 0 & c & -\zeta_3^2 c \\ -\zeta_3^2 d & -e & d + e \end{bmatrix}$$

We may consider this same module as a $\mathbb{Z}[E]$ -module instead, simply by disallowing scalar multiplication outside of the subring $\mathbb{Z}[E] \subseteq \mathbb{Z}(\zeta_3)[E]$. Now we look at the basis vectors of the $\mathbb{Z}[E]$ -module. Since the $\mathbb{Z}[E]$ -span of a set of vectors is smaller than its $\mathbb{Z}(\zeta_3)[E]$ -span, however, we will need more basis vectors than before in order to span the entire module. One basis for this module has basis vectors doubly indexed by vertices and the two non-identity group elements of $\mathbb{Z}/3\mathbb{Z}$, which shows that the module has $\mathbb{Z}[E]$ -rank 6. Ordering basis vectors as $v_1^g, v_2^g, v_3^g, v_1^{g^2}, v_2^{g^2}, v_3^{g^2}$, the voltage Laplacian may be considered as a $\mathbb{Z}[E]$ -linear transformation, with matrix

$$[\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]} = \begin{bmatrix} a + b & -b & 0 & a & 0 & 0 \\ 0 & c & c & 0 & 0 & -c \\ d & -e & d + e & -d & 0 & 0 \\ -a & 0 & 0 & 2a + b & -b & 0 \\ 0 & 0 & c & 0 & c & 0 \\ d & 0 & 0 & 0 & -e & d + e \end{bmatrix}$$

and the $\mathbb{Z}[E]$ -determinant of this transformation is

$$\begin{aligned} \det_{\mathbb{Z}[E]} \mathcal{L}(\Gamma) &:= \det [\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]} \\ &= 3a^2 c^2 d^2 + 3b^2 c^2 d^2 + 6abc^2 d^2 + 9a^2 c^2 e^2 + 3b^2 c^2 e^2 + 9abc^2 e^2 + 9a^2 c^2 de + 3b^2 c^2 de + 12abc^2 de \end{aligned}$$

With restriction of scalars in hand, we restate Theorem 1.4 in the case of regular covering graphs before examining the particularly nice case of G prime cyclic:

Theorem 1.4. (Regular covering graph case) *Let $\Gamma = (V, E, \text{wt}, \nu)$ be an edge-weighted G -voltage multigraph, and let $\tilde{\Gamma}$ be its derived covering graph. Then for any vertex v of Γ and any lift \tilde{v} of v , we have*

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{|G|} \det_{\mathbb{Z}[E]} \mathcal{L}(\Gamma)$$

5.2. The prime cyclic case. In the case where G is prime cyclic, the theorem has an especially nice interpretation:

Corollary 5.3. *Let G be the prime cyclic group of order p , and let Γ be as in the theorem. Then for any vertex v of Γ and any lift \tilde{v} of v in the derived graph $\tilde{\Gamma}$ of Γ , we have*

$$\begin{aligned} \frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} &= \frac{1}{|G|} N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}[\det \mathcal{L}(\Gamma)] \\ &= \frac{1}{|G|} \prod_{i=1}^{p-1} \det[\sigma_i(\mathcal{L}(\Gamma))] \end{aligned}$$

where $N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}[\det \mathcal{L}(\Gamma)]$ denotes the field norm of $\mathbb{Q}(\zeta_p)$ over \mathbb{Q} , naturally extended to a norm on $\mathbb{Q}(\zeta_p)[E]$, and σ_i is the field automorphism mapping $\zeta_p \mapsto \zeta_p^i$.

Proof. The corollary follows from the theorem if we can show that $\det_{\mathbb{Z}[E]} \mathcal{L}(\Gamma) = N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}[\det \mathcal{L}(\Gamma)]$. Theorem 1 of [Sil00] states that if A is a commutative ring, if B is a commutative subring of $\text{Mat}_n(A)$, and if $M \in \text{Mat}_m(B)$, then

$$\det_A M = \det_A (\det_B M)$$

In this case, let $A := \mathbb{Q}[E]$. The reduced group algebra $B := \mathbb{Q}(\zeta_p)[E]$ may be realized as a subring of $\text{Mat}_{p-1}(A)$, with an element α of B being identified with the A -matrix corresponding to multiplication by α in B , where we view B as an A -module. Note that A and B are both commutative. Finally, we let $M = \mathcal{L}(\Gamma)$. But the field norm $N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}}(\alpha)$ is defined as the determinant of the map $x \mapsto \alpha x$ as a \mathbb{Q} -linear transformation, or, equivalently in our case, a \mathbb{Z} -linear transformation. When extended to $\mathbb{Q}(\zeta_p)[E]$, this definition shows that

$$\det_{\mathbb{Z}[E]} \left(\det_{\mathbb{Z}(\zeta_p)[E]} \mathcal{L}(\Gamma) \right) = N_{\mathbb{Q}(\zeta_p)/\mathbb{Q}} \left[\det_{\mathbb{Q}(\zeta_p)[E]} \mathcal{L}(\Gamma) \right]$$

as desired. ■

Example 5.4. Let Γ be the graph from example 2.7. We computed $\det \mathcal{L}$ in Example 3.3:

$$\det \mathcal{L}(\Gamma) = (1 - \zeta_3)bcd + (1 - \zeta_3)acd + (1 - \zeta_3^2)bce + (1 - \zeta_3)(1 - \zeta_3^2)ace$$

Since voltage is given by $\mathbb{Z}/3\mathbb{Z}$, the reduced group algebra is $\mathbb{Z}(\zeta_3)[E] \subset \mathbb{Q}(\zeta_3)[E]$, which we treat as an extension over \mathbb{Q} . The Galois norm in this case is the same as the complex norm, since the Galois conjugate of an element of $\mathbb{Q}(\zeta_3)[E]$ is the same as its complex conjugate. This norm is

$$\begin{aligned} \det \mathcal{L}(\Gamma) \det \overline{\mathcal{L}(\Gamma)} &= ((1 - \zeta_3)bcd + (1 - \zeta_3)acd + (1 - \zeta_3^2)bce + (1 - \zeta_3)(1 - \zeta_3^2)ace) \\ &\quad \cdot ((1 - \zeta_3^2)bcd + (1 - \zeta_3^2)acd + (1 - \zeta_3)bce + (1 - \zeta_3^2)(1 - \zeta_3)ace) \\ &= 3a^2c^2d^2 + 3b^2c^2d^2 + 6abc^2d^2 + 9a^2c^2e^2 + 3b^2c^2e^2 + 9abc^2e^2 + 9a^2c^2de + 3b^2c^2de + 12abc^2de \end{aligned}$$

which matches $\det_{\mathbb{Z}[E]} \mathcal{L}(\Gamma)$ from Example 5.2.

5.3. Triangularization. To prove Theorem 1.4, we carefully apply a change of basis to the Laplacian matrix of $\tilde{\Gamma}$. In order to do this, we need to be a little more careful in how we write this matrix and in which basis vectors we are using. The (ordinary) Laplacian matrix $L(\tilde{\Gamma})$ acts on the module $\mathbb{Z}[E]^{\tilde{V}}$. We have been implicitly writing this matrix with respect to the standard basis $\{v_i^g\}_{i \in [|V|], g \in G}$. We now want to enforce an ordering on the vertices of $\tilde{\Gamma}$, and thus on these basis vectors. Fix an ordering on the vertices V of Γ , and fix an ordering of the elements of G such that the identity of G is first. Then order the basis vectors in colexicographic order based on their labeling (v, g) , so that the first $|V|$ basis vectors are $b_1^{1_G}, \dots, b_n^{1_G}$.

It is also beneficial to write an explicit basis for the $\mathbb{Z}[E]$ -module of rank $(|G| - 1)|V|$ that $\mathcal{L}(\Gamma)$ acts on via restriction of scalars from $\overline{\mathbb{Z}[G]}[E]$ to $\mathbb{Z}[E]$. The $\overline{\mathbb{Z}[G]}[E]$ -span of a vector m is equal to the $\mathbb{Z}[E]$ -span of the set $\{gm\}_{g \in G, g \neq 1_G}$: note that $m = -\sum_{g \in G, g \neq 1_G} gm$ by the definition of the reduced group algebra. Therefore, given the standard $\overline{\mathbb{Z}[G]}[E]$ -basis $\{v_i\}_{i=1}^{|V|}$ corresponding to the vertices of Γ , we derive a standard $\mathbb{Z}[E]$ -basis $\{v_i^g\}_{i \in [|V|], g \neq 1_G}$. Again, we order this basis in colexicographic order. Note that this basis corresponds to the last $(|G| - 1)|V|$ vectors in the standard basis we have defined for $L(\tilde{\Gamma})$.

Having written $L(\tilde{\Gamma})$ with respect to our ordering of basis vectors, we wish to perform the change of basis described by the following lemma:

Lemma 5.5. *Let G and Γ be as in Theorem 1.4. Write $L(\tilde{\Gamma})$ with basis vectors ordered as above. Let*

$$S = \begin{bmatrix} \mathbf{id}_{|V|} & 0_{|V|} & \dots & 0_{|V|} \\ \mathbf{id}_{|V|} & & & \\ \vdots & & \mathbf{id}_{(|G|-1)|V|} & \\ \mathbf{id}_{|V|} & & & \end{bmatrix}$$

Then the change of basis given by S yields the following block triangularization of $L(\tilde{\Gamma})$:

$$(6) \quad S^{-1}L(\tilde{\Gamma})S = \begin{bmatrix} L(\Gamma) & * \\ 0 & [\mathcal{L}(\Gamma)]_{\mathbb{Z}} \end{bmatrix}$$

Proof. Let $\beta_i = \sum_{g \in G} v_i^g$. Conjugation by S^{-1} corresponds to a change of basis that maps $v_i^{1_G} \mapsto \beta_i$ and $v_i^g \mapsto v_i^g$ when $g \neq 1_G$. Therefore, all we need to do is examine the action of the linear transformation corresponding to the matrix $L(\tilde{\Gamma})$ on this new basis. Denote this linear transformation as T .

First, we show that

$$T(\beta_j) = \sum_{i=1}^{|V|} \ell_{ij} \beta_i$$

where ℓ_{ij} is the i, j -entry of $L(\Gamma)$. To see this, consider the $|G|$ columns of $L(\tilde{\Gamma})$ corresponding to the fiber $\{v_{j,g}\}_{g \in G}$ of v_j . The sum of these columns is equal to $T(\beta_j)$, expressed as a column vector with respect to the standard basis. If we have an edge $e = (v_i, v_j)$, and if $\nu(e) = g$, then we have a term of $\text{wt}(e)$ in each $(i, h) \times (i, h)$ -entry of $L(\tilde{\Gamma})$ for each $h \in G$, as well as a term of $-\text{wt}(e)$ in each $(i, h) \times (j, gh)$ -entry of $L(\tilde{\Gamma})$ for each $h \in G$. Thus, the sum of the $|G|$ columns of $L(\tilde{\Gamma})$ corresponding to the fiber $\{v_{j,g}\}_{g \in G}$ is precisely

$$T(\beta_j) = \sum_{i=1}^{|V|} \sum_{g \in G} \ell_{ij} v_i^g = \sum_{i=1}^{|V|} \ell_{i,j} \beta_i$$

as desired. Therefore, the left block column of (6) is correct.

The effect of our change of basis on the lower-right $(|G| - 1)|V| \times (|G| - 1)|V|$ block of $L(\tilde{\Gamma})$ is to add the i -th row of $L(\tilde{\Gamma})$, for $i \in [|V|]$, to rows $i + |V|, i + 2|V|, \dots, i + (|G| - 1)|V|$ —that is, we add the 1_G -components of the linear transformation of $\mathcal{L}'(\Gamma)$ to the g -components for each $g \neq 1_G$ in G . But this mimics the structure of the *reduced* group algebra; that is, what we have actually done in the lower-right hand block is to write $\mathcal{L}(\Gamma)$ as a \mathbb{Z} -matrix, as desired. \blacksquare

Here is an alternative description of the lower-right hand matrix above.

Definition 5.6. Let $\{v_1, \dots, v_n\}$ be the set of vertices of our graph Γ , let $\tilde{\Gamma}$ be a k -fold cover of Γ , where vertex v_i is lifted to v_i^1, \dots, v_i^k . Define $n(k-1) \times n(k-1)$ matrices D and A with basis $v_1^2, \dots, v_n^2, v_1^3, \dots, v_n^3, \dots, v_1^k, \dots, v_n^k$ as follows.

$$\begin{aligned} A[v_i^t, v_j^r] &= \sum_{e=(v_i^t, v_j^r)} \text{wt}(e) - \sum_{e=(v_i^1, v_j^r)} \text{wt}(e) \\ D[v_i^t, v_i^t] &= \sum_{e \in E_s(v_i^t)} \text{wt}(e) \end{aligned}$$

for $1 < t, r \leq k$. Finally, we define

$$[\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]} := D - A.$$

Note that this interpretation works even for non-regular cover, though in this case the matrix cannot be interpreted as the \mathbb{Z} -linearization of a voltage Laplacian. Nevertheless, we can use this alternate description to extend the definition of $[\mathcal{L}(\Gamma)]_{\mathbb{Z}}$ to account for even the case of non-regular covers, which will make Lemma 5.5 true for arbitrary covers, and thus will make Theorem 1.4 true for arbitrary finite covers—the remaining lemmata do not make use of regularity. Thus, we have the most general form of the main theorem:

Theorem 1.4. (General version). *Let $\Gamma = (V, E, \text{wt})$ be an edge-weighted multigraph, and let $\tilde{\Gamma}$ be a k -fold covering graph of Γ . Then for any vertex v of Γ and any lift \tilde{v} of v , we have*

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{k} \det[\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]}$$

with $[\mathcal{L}(\Gamma)]_{\mathbb{Z}[E]}$ defined in definition 5.6.

This proof of Lemma 5.5 finally explains why it is necessary to go through the trouble of working in the reduced algebra all this time, rather than the ordinary group algebra—we need to lower the \mathbb{Z} -rank of our algebra by 1 in order to get the triangularization described by the lemma. This triangularization is very close to giving us what we want for Theorem 1.4, but unfortunately taking minors and change of basis do not commute. Besides, we need to find a factor of $|G|$ somewhere along the way. Define $U = S^{-1}L(\tilde{\Gamma})S$. Without loss of generality, assume that we want to root our arborescences of Γ at vertex v_1 and our arborescences of $\tilde{\Gamma}$ at vertex $v_{1,1_G}$. Then want to show that

$$\det U_1^1 = |G| \det L_1^1(\tilde{\Gamma}),$$

for then the theorem follows from the lemma, since $L_1^1(\tilde{\Gamma}) = A_{(1,1_G)}(\tilde{\Gamma})$ and $\det U_1^1 = A_1(\Gamma) \det_{\mathbb{Z}[E]} \mathcal{L}(\Gamma)$. We do so by performing the change of basis S into two steps: the first will multiply the minor by $|G|$, and the second will leave it unchanged.

5.4. The two-step change of basis. Here is the first step:

Lemma 5.7. *Let L be the Laplacian matrix of some graph $\Gamma = (V, E, \text{wt})$. Let P be the change of basis matrix that maps $v_i \mapsto v_i + \sum_{j \in J} \alpha_j v_j$, for some $J \subseteq |V| \setminus \{i\}$ and $\alpha_j \in \mathbb{R}$ for $j \in J$, but leaves all other basis vectors unchanged. That is, P is the identity matrix but with α_j in entry (i, j) for each $j \in J$. Then*

$$\det(P^{-1}LP)_i^i = A_{v_i}(\Gamma) + \sum_{j \in J} \alpha_j A_{v_j}(\Gamma)$$

Proof. Without loss of generality, let $i = 1$. We first consider the case that $|J| = 1$, and we will assume $J = \{2\}$. $P^{-1}L(\tilde{\Gamma})$ differs from $L(\tilde{\Gamma})$ in that the second row of $P^{-1}L(\tilde{\Gamma})$ is the second row of $L(\tilde{\Gamma})$ with α_2 times the first row of $L(\tilde{\Gamma})$ subtracted from it. $P^{-1}L(\tilde{\Gamma})P$ differs from $P^{-1}L(\tilde{\Gamma})$ in that the first column of $P^{-1}L(\tilde{\Gamma})P$ is the first column of $P^{-1}L(\tilde{\Gamma})$ with α_2 times the second column of $P^{-1}L(\tilde{\Gamma})$ added to it. However, since we are finding the determinant of $P^{-1}L(\tilde{\Gamma})P$ with the first row and column removed, we are only interested in the lower-right hand $(|V| - 1) \times (|V| - 1)$ submatrix and can ignore this column operation.

We may interpret $(P^{-1}LP)_1^1$ as a submatrix of the Laplacian of a different graph, which we will denote as Γ' . We construct Γ' as follows. The vertices of Γ' are $v'_1, \dots, v'_{|V|}$. If there is an edge $v_r \rightarrow v_s$ in Γ , then there is an edge $v'_r \rightarrow v'_s$ in Γ' , so Γ' contains Γ as a subgraph. For each edge $e \in E_i(v_1)$ that is not a loop, if $e_t \neq v_2$, we add the edge (v'_2, e'_t) to Γ' with weight $-\alpha_2 \text{wt}(e)$ and the edge (v'_2, v'_1) with $\alpha_2 \text{wt}(e)$. The first of these edges will be called an edge of type 1 and the second an edge of type 2. For each edge $e \in E_i(v_1)$ where $e_t = v_2$, we add the edge (v'_2, v'_1) with weight $\alpha_2 \text{wt}(e)$. Call this an edge of type 3. We can see that $L_1^1(\Gamma') = (P^{-1}LP)_1^1$, so $\det(P^{-1}LP)_1^1$ counts the arborescences of Γ' rooted at v'_1 .

We will divide the arborescences of Γ' into four categories (See Figure 4).

- (1) Arborescences that do not contain any type 1, type 2, or type 3 edges. The weighted sum of these arborescences is counted by $A_{v_1}(\Gamma)$ because they are exactly the arborescences that use only edges in the subgraph Γ of Γ' .
- (2) Arborescences that contain a type 1 edge and arborescences that differ from these by replacing the type 1 edge with a type 2 edge of the same weight with opposite sign. For every type 1 edge, there is a type 2 edge of the same weight with opposite sign. This means that for every arborescence that contains a type 1 edge, there is an arborescence that is the same, except instead of the type 1 edge it has a type 2 edge of the same weight with opposite sign. The weights of these arborescences cancel out, so the weighted sum of all of these arborescences is 0.
- (3) Arborescences that contain a type 3 edge. By removing the edge of type 3 and replacing it with the corresponding edge pointing in the opposite direction, we obtain an arborescence with weight divided by α_2 rooted at v'_2 . This arborescence does not contain any edges of types 1, 2, or 3, so it corresponds to an arborescence in Γ rooted at v_2 . Similarly, given an arborescence in Γ rooted at v_2 with an edge from v_1 to v_2 , we can reverse this process. So, the weighted sum of these arborescences is α_2 times the weighted sum of arborescences rooted at v_2 in Γ that contain an edge from v_1 to v_2 .
- (4) Arborescences that contain an edge of type 2 that are not counted in (2). These are arborescences where removing the edge of type 2 and replacing it with the corresponding edge $e' = (v'_2, w')$ of type 1 does not give an arborescence. This means the only path in the arborescence from w' to v'_1 goes through v'_2 . Removing the type 2 edge gives two disconnected components, one directed towards v'_1 and one directed towards v'_2 . The edge e' originally came from the edge $e = (v_1, w) \in \Gamma$. Consider our arborescence without the type 2 edge but with the edge (v_1, w) that has the weight of our type 2 edge divided by α_2 . We now have an arborescence rooted at v'_2 that has no type 1, 2, or 3 edges. This arborescence corresponds to an arborescence in Γ rooted at v_2 . Similarly, given an arborescence in Γ rooted at v_2 with no edge from v_1 to v_2 , we can reverse this process. So, the weighted sum of these arborescences is α_2 times the weighted sum of arborescences rooted at v_2 in Γ that do not contain an edge from v_1 to v_2 .

Adding the weighted sums of the arborescences in these four categories, we find $A_{v_1}(\Gamma') = A_{v_1}(\Gamma) + \alpha_2 A_{v_2}(\Gamma)$.

We will proceed by induction. To do this, we first need to show that for Γ' as constructed above, $A_{v_\ell}(\Gamma') = A_{v_\ell}(\Gamma)$ whenever $\ell \neq 1$. This is true when $\ell = 2$, since every new edge we have added is in $E_s(v_2)$. For $\ell \neq 1, 2$, note that $L_\ell^\ell(\Gamma')$ differs from $L_\ell^\ell(\Gamma)$ only in the second row: the second row of $L(\Gamma')$ is the difference of the first two rows of $L(\Gamma)$. Thus, we may expand the determinant along the second row to write

$$\det L_\ell^\ell(\Gamma') = \det L_\ell^\ell(\Gamma) + \det M$$

where M is the matrix obtained by replacing the second row of $L_\ell^\ell(\Gamma)$ with $-\alpha_2$ times the first row of $L_\ell^\ell(\Gamma)$. Since M has two rows that are scalar multiples of each other, it has determinant zero. Therefore, $\det L_\ell^\ell(\Gamma') = \det L_\ell^\ell(\Gamma)$ and, by the Matrix Tree Theorem, $A_{v_\ell}(\Gamma') = A_{v_\ell}(\Gamma)$.

We now perform the change of basis one step at a time. Let $J = \{j_1, \dots, j_n\}$. Suppose that when P_k is the change of basis matrix mapping $b_{v_1} \mapsto b_{v_1} + \sum_{m=1}^k \alpha_{j_m} b_{v_{j_m}}$, we know that $\det(P_k^{-1}L(\Gamma)P_k)_1^1 = A_{v_1}(\Gamma) + \sum_{m=1}^k \alpha_{j_m} A_{v_{j_m}}(\Gamma)$, and that $(P_k^{-1}L(\Gamma)P_k)_1^1$ is the submatrix of the Laplacian of some graph Γ' that satisfies $A_{v_1}(\Gamma') = A_{v_1}(\Gamma) + \sum_{m=1}^k \alpha_{j_m} A_{v_{j_m}}(\Gamma)$ and $A_{v_\ell}(\Gamma') = A_{v_\ell}(\Gamma)$ for $\ell \neq 1$. Let P'_k be the change of basis matrix

mapping $b_{v_1} \mapsto b_{v_1} + \alpha_{j_{k+1}} b_{v_{j_{k+1}}}$. Then applying our construction from the first part of the proof on Γ' , we conclude $((P'_k)^{-1} P_k^{-1} L(\Gamma) P_k P'_k)$ is the submatrix of the Laplacian of some graph Γ'' satisfying

$$\begin{aligned} A_{v_1}(\Gamma'') &= A_{v_1}(\Gamma') + \alpha_{j_{k+1}} A_{v_{j_{k+1}}}(\Gamma') \\ &= A_{v_1}(\Gamma) + \sum_{m=1}^{k+1} \alpha_{j_m} A_{v_{j_m}} \end{aligned}$$

and also $A_{v_\ell}(\Gamma'') = A_{v_\ell}(\Gamma') = A_{v_\ell}(\Gamma)$ for $\ell \neq 1$.

Therefore, by induction on the size of J , we conclude that

$$\det(P^{-1} L(\Gamma) P)_1^1 = A_{v_1}(\Gamma) + \sum_{j \in J} \alpha_j A_{v_j}(\Gamma)$$

as desired. ■

Here is the second step of the change of basis:

Lemma 5.8. *Let R be a commutative ring and let $M \in \text{Mat}_n(R)$. Let $Q \in GL_n(R)$ such that Q fixes the i -th unit basis vector e_i . Then*

$$\det(Q^{-1} M Q)_i^i = \det M_i^i$$

In other words, the change of basis given by Q commutes with taking the minor of M corresponding to removing the i -th row and column.

Proof. Let V be the free R -module of rank n on which M acts. Taking the minor $\det M_i^i$ corresponding to removing the first row and column is equivalent to evaluating the determinant of $\overline{T_M}$, the linear transformation corresponding to M descended to the quotient space $V/\langle e_i \rangle$. That is, ignoring a row and its corresponding column is equivalent to considering the transformation on this quotient space. Thus, if a change of basis leaves the basis vector corresponding to the i -th row and column unchanged, the determinant on this quotient space will not change either, since the relevant quotient space will not change, and $\det \overline{T_M}$ does not depend on the basis chosen for $V/\langle e_i \rangle$. Thus, the minor remains unchanged under this change of basis. ■

5.5. Proof of Theorem 1.4.

Proof. In Lemma 5.7, let P be the change of basis that maps $v_1^{1G} \mapsto \beta_1 := \sum_{g \in G} v_1^g$, and let Q be the change of basis that maps $v_i^{1G} \mapsto \sum_{g \in G} v_i^g$ for $i > 1$, which satisfies the hypotheses of Lemma 5.8. Letting S be the matrix from Lemma 5.5, we have $S = QP$. Thus, by Lemmata 5.7 and 5.8,

$$\begin{aligned} \det U_1^1 &= \det(QPL(\tilde{\Gamma})P^{-1}Q^{-1})_1^1 \\ &= \det(PL(\tilde{\Gamma})P^{-1})_1^1 \\ &= \sum_{g \in G} A_{v_{1,g}}(\Gamma) \end{aligned}$$

By symmetry,

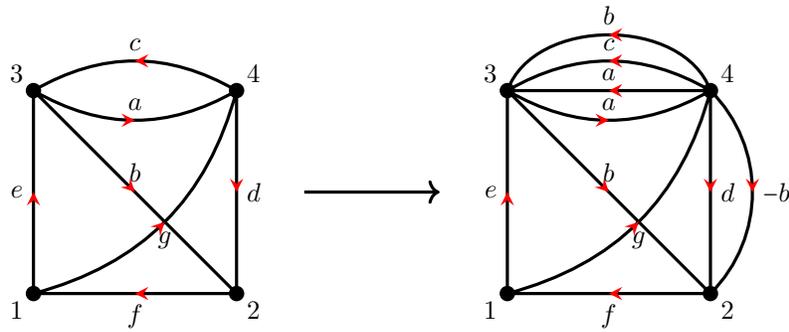
$$\sum_{g \in G} A_{v_{1,g}}(\Gamma) = |G| A_{v_{1,1G}}(\Gamma)$$

However, from the triangularization given by Lemma 5.5, and by the Matrix Tree Theorem, we know that

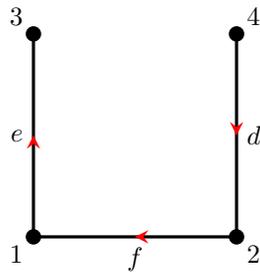
$$\begin{aligned} \det U_1^1 &= A_{v_1}(\Gamma) \det_{\mathbb{Z}[E]} \mathcal{L}(\Gamma) \\ \det L_1^1(\tilde{\Gamma}) &= A_{(v_1, 1G)}(\tilde{\Gamma}) \end{aligned}$$

Therefore,

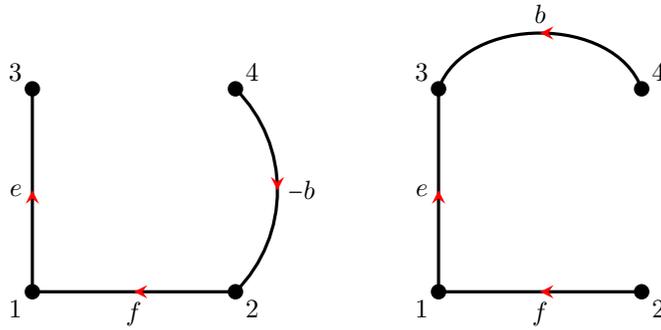
$$|G| A_{(v_1, 1G)}(\tilde{\Gamma}) = A_{v_1}(\Gamma) \det_{\mathbb{Z}[E]} \mathcal{L}(\Gamma)$$



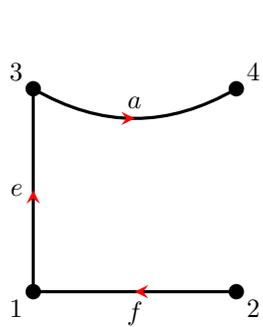
A graph Γ and the corresponding graph Γ'



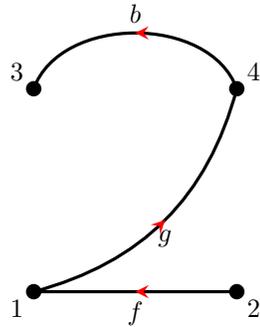
An arborescence of type 1.



A pair of arborescences of type 2.



An arborescence of type 3.



An arborescence of type 4.

FIGURE 4. Types of arborescences for Γ'

as desired. ■

6. FURTHER DISCUSSION AND QUESTIONS

6.1. Possible Combinatorial Interpretation. Theorem 1.4 makes it clear that the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ is a homogeneous polynomial with integer coefficients—writing $A_{\tilde{v}}(\tilde{\Gamma}) = \frac{1}{|\tilde{\Gamma}|} A_v(\Gamma) \det_{\mathbb{Z}[E]} \mathcal{L}(\tilde{\Gamma})$, the factor of $\frac{1}{|\tilde{\Gamma}|}$ must divide $\det_{\mathbb{Z}[E]} \mathcal{L}(\tilde{\Gamma})$ because every coefficient of $A_v(\Gamma)$ is 1. We further conjecture:

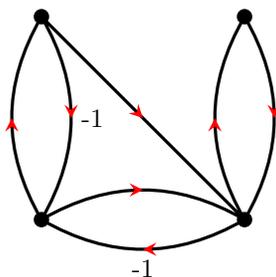
Conjecture 6.1. *The ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ has positive integer coefficients.*

Corollary 3.4 along with Theorem 1.3 proves the conjecture in the 2-fold case, and yields a nice combinatorial interpretation of our result: we obtain several arborescences of $\tilde{\Gamma}$ by combining an arborescence T of Γ with a negative vector field γ of Γ . More precisely, if γ has k cycles, then the pair (T, γ) corresponds to 2^{k-1} arborescences of $\tilde{\Gamma}$ each with weight $\text{wt}(T) \text{wt}(\gamma)$. However, it is not clear how to explicitly exhibit this correspondence. Our original method of trying to prove Theorem 1.3 was to derive such a correspondence, which ultimately failed. The following is a plausible but incorrect candidate for the desired correspondence between pairs (T, γ) and arborescences of \tilde{T} of $\tilde{\Gamma}$: Given (T, γ) , construct \tilde{T} by taking the unique lift of T into $\tilde{\Gamma}$ and then letting the outedges of the remaining vertices of \tilde{T} come from γ . We then need to apply some free action of $\mathbb{Z}/2\mathbb{Z}^{k-1}$ to obtain all 2^{k-1} corresponding arborescences. Candidates for this action included:

- Swapping the lifts of edges the components of γ not containing v .
- Swapping the lifts of edges in the cycles of the components of γ not containing v .
- In either of the above, swapping only negative edges.

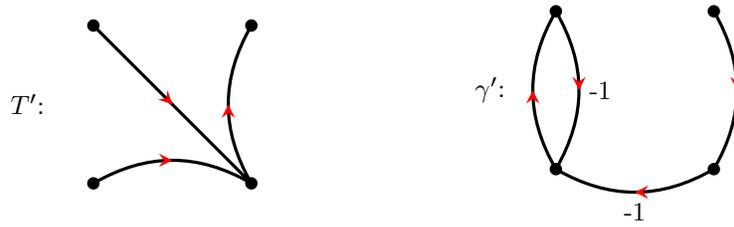
However, regardless of which action we choose, this construction fails to be injective even at the first step—that is, there exist examples of distinct pairs $(T, \gamma), (T', \gamma')$ both mapping to the same \tilde{T} .

Consider the following $\mathbb{Z}/2\mathbb{Z}$ -voltage graph, where every edge has trivial voltage unless labeled with -1 :

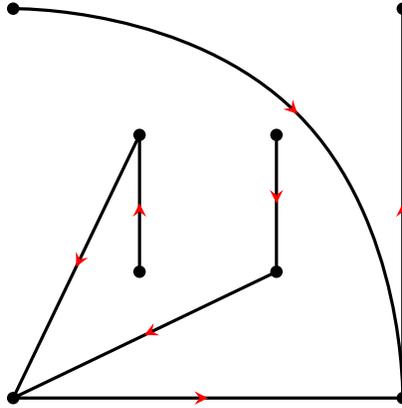


We have the following pairs (T, γ) and (T', γ') , with the arborescences rooted at the upper-right vertex:





Both of these pairs give rise to the following arborescence in the covering graph:



Another potential candidate would be to perform a similar construction by first lifting γ rather than T —the connected lift of a negative vector field is always acyclic—any cycle without repeated lifts in $\tilde{\Gamma}$ necessarily descends to a positive cycle in Γ . We have not explored this candidate in depth, since we eventually shifted our focus away from the 2-fold case. There may be a similarly easy counterexample for this construction.

Thus, the problem remains:

Problem 6.2. Let Γ be a $\mathbb{Z}/2\mathbb{Z}$ -voltage graph and let $\tilde{\Gamma}$ be its derived cover. Find a combinatorial correspondence between:

- (1) Pairs (T, γ) of arborescences and negative vector fields of Γ ; and
- (2) Arborescences \tilde{T} of $\tilde{\Gamma}$,

where each pair (T, γ) corresponds to $C(\gamma)$ arborescences \tilde{T} , with $C(\gamma)$ being the number of cycles of γ .

In the case of general regular covers, we do not even have a concrete combinatorial interpretation of $\det_{\mathbb{Z}[E]} \mathcal{L}(\Gamma)$. Such an interpretation would probably be the cleanest way to prove Conjecture 6.1.

Problem 6.3. Find a combinatorial interpretation of the polynomial $\det_{\mathbb{Z}[E]} \mathcal{L}(\Gamma) = \frac{A_{\tilde{\Gamma}}}{A_{\Gamma}}$, assuming Conjecture 6.1 is true. Then find an explicit combinatorial construction that yields a correspondence as in Problem 6.2.

6.2. Interpreting the restriction-of-scalars determinant. In the case where the voltage group G is prime cyclic, Corollary 5.3 yields a computationally nice interpretation of Theorem 1.4: the \mathbb{Z} -determinant is really a field norm, which may be computed in ways other than restriction of scalars—for example, as a product of Galois conjugates. This result could be extended if there existed an analogue to the field norm for arbitrary reduced group algebras, or indeed for general free algebras of finite rank. If this is too hard, we could instead focus on the reduced group algebras of abelian groups only.

Problem 6.4. Let R be a commutative ring, and let A be a free algebra over R of finite rank. Let $\alpha \in A$. Find an alternative expression or interpretation of $\det_R \alpha$, where the multiplicative action of α is viewed as a linear transformation on the R -module A , analogous to a field norm. Useful special cases include $R = \mathbb{Z}$

or \mathbb{Q} , when A is commutative, and/or when A is the group algebra or reduced group algebra of some finite group G .

6.3. Iterated covers for solvable voltage groups.

Theorem 6.5. [Tan] (Galois Theory of Covering Spaces) *Let X be a topological space, and let \tilde{X} be a regular (Galois) covering space with covering map $p: \tilde{X} \rightarrow X$. Suppose that the deck group $G = \text{Aut}(p)$ acts properly discontinuously on \tilde{X} . Let H be a subgroup of G . Then:*

- (1) *The space \tilde{X} is a regular covering space of the quotient space \tilde{X}/H (obtained by identifying the elements in each orbit of H), and its deck group is H .*
- (2) *If H is a normal subgroup of G , then \tilde{X}/H is a regular covering space of X with deck group isomorphic to G/H .*

Due to Theorem 2.12, the voltage group G plays the role of the deck group in our discussion of covering graphs. Thus, if G is not simple, we can factor the ratio of arborescences even further by constructing an intermediate covering graph. In particular, if G is solvable, we can use the Jordan-Hölder series of G to construct a series of regular covers of prime degree to obtain $\tilde{\Gamma}$, with p -prime, so that at each step we use a $\mathbb{Z}/p\mathbb{Z}$ -voltage graph. This may simplify the calculation of Theorem 1.4, since we may apply Corollary 5.3 to each step, with the ratio $\frac{A_v(\tilde{\Gamma})}{A_v(\Gamma)}$ telescoping along the way.

6.4. Random covers.

Conjecture 6.6. *Let $\Gamma = (E, V)$ be a graph, fix a vertex v with non-trivial arborescence. Let Γ' be a random k -fold cover of Γ , assuming uniform distribution. Then the expected value of the ratio of arborescence is*

$$\mathbb{E} \left[\frac{A_{v'}(\Gamma')}{A_v(\Gamma)} \right] = \frac{1}{k} \prod_{w \in V} \left(\sum_{\alpha \in E_s(w)} \text{wt}(\alpha) \right)^{k-1}$$

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