VIRTUAL COMPLETE INTERSECTIONS IN $\mathbb{P}^1 \times \mathbb{P}^1$

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1. Introduction

In algebraic geometry, the study of geometric objects usually occurs through studying the vanishing set of a certain collection of polynomials, called a variety. Hilbert's Nullstellensatz establishes a correspondence between ideals and varieties. One of the most important problems in relation to a variety X is the dimension of vector spaces of homogeneous polynomials vanishing on X, which is described by the Hilbert function. In the same vein, the syzygies of a module and, consequently, its free resolutions encode similar geometric information on the Hilbert function and Hilbert polynomial. An important class of free resolutions, called Koszul complexes, corresponds to the property of being a complete intersection.

For the product of projective spaces, minimal free resolutions are usually long and contain information that is geometrically irrelevant. We hence turn to virtual resolutions that are much shorter and better expresses the geometry. Like in the case of free resolutions, we define the notion of virtual complete intersections using virtual resolutions and offer a combinatorial classification of whether a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ form a transverse virtual complete intersection.

2. Preliminaries

2.1. **Background**

A projective space $\mathbb{P}^n$ over $\mathbb{C}$ is the set of one-dimensional subspaces of the vector space $\mathbb{C}^{n+1}$ that passes through the origin. A point in $\mathbb{P}^n$ is written as a non-zero homogeneous vector $[x_0, x_1, \ldots, x_n]$. A projective variety $X \subset \mathbb{P}^n$ is the zero locus of a collection of homomgenous polynomials $f(\vec{x}) \in \mathbb{C}[x_0, x_1, \ldots, x_n]$.

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In our work, we study the generalization of a projective space $\mathbb{P}^n$ and its varieties to a multiprojective setting. In particular, we study the biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$[GV15].

Definition 2.1. The biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ is the set of equivalence classes:

$$\mathbb{P}^1 \times \mathbb{P}^1 := \{((a_0, a_1), (b_0, b_1)) \in \mathbb{C}^2 \times \mathbb{C}^2 \mid (a_0, a_1) \neq (0,0) \text{ and } (b_0, b_1) \neq (0,0)\}/\sim,$$

where the equivalence relation is defined by $x \sim y \iff x = \lambda y$ for $x, y \in \mathbb{P}^1, \lambda \in \mathbb{C}^\times$.

Let S be the polynomial ring $\mathbb{C}[x_0, x_1, y_0, y_1]$. The geometrical irrelevant ideal of this ring is $B = (x_0, x_1) \cap (y_0, y_1)$. Then the projective strong Nullstellensatz implies the correspondance between ideals in S and varieties in $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 2.2. *(Bivariate Projective Nullstellensatz) There exists an order-reversing one-to-one correspondence between proper homogeneous B-saturated radical ideals $\sqrt{I}$ and subsets of $\mathbb{P}^1 \times \mathbb{P}^1$ of the form $V(I)$.*

Here, the B-saturated ideal of I is

$$(I : B^\infty) = \{s \in S | sB^l \subseteq I \text{ for some } l > 0\}.$$

Geometrically, ideal I is equivalent to its B-saturation (or $V(I) = V((I : B^\infty)))$.

For any ideal I, we can compute the free resolution of the module $M = S/I$.

Definition 2.3. Let M be an S-module. A graded free resolution of M is an exact sequence of the form:

$$0 \longleftarrow M \xleftarrow{\varphi_0} F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \longleftarrow \cdots$$

where for all i, $F_i \cong S^{r_i}$ is a free module and each φ_i has a degree 0.

The free resolution encodes information about the connection between a module, its first syzygies, its second syzygies, and so forth. If R is a polynomial ring, we have the following result from Hilbert:

Theorem 2.4. *(Hilbert Syzygy Theorem) Let $S = \mathbb{C}[x_0, \ldots, x_n]$. Then every finitely generated S-module has a finite free resolution of length at most n.*

Hilbert's Syzygy Theorem asserts the existence of a free resolution that is short enough when working in lower-dimensional projective spaces. As we shall see in Section 2.2, an analogous result holds in the context of virtual resolutions. In the case of free resolutions, one can focus on the minimal free resolution:

Definition 2.5. A free resolution is minimal if for every $\ell \geq 1$, the nonzero entries of the graded matrix of φ_ℓ have positive degree.

The minimal free resolution is unique up to isomorphism. In particular, it is closely related to the Hilbert function:

Definition 2.6. If $M = S/I$ is a finitely generated graded module over S, then the Hilbert function $H_M(d) : \mathbb{Z} \to \mathbb{N}$ is defined by:

$$H_M(d) = \dim_{\mathbb{C}} M_d,$$

where $\dim_{\mathbb{C}}$ is the dimension of the degree d piece of M as a vector space over $\mathbb{C}$.

The Hilbert function eventually becomes a polynomial. For an ideal of points, the Hilbert polynomial is equal to the number of points in $V(I)$. Given a graded minimal free resolution, we can take the degree d part of the resolution and use the alternating sum formulation to get the Hilbert function:

Theorem 2.7.

$$H_M(d) = \sum_i (-1)^i H_{F_i}(d)$$

Next we move on to the notion of complete intersections, which is formally defined by the existence of a regular sequence:

Definition 2.8. If $I \subseteq S$ is a bihomogeneous ideal and $M = S/I$. Then, a sequence of elements $f_1, f_2, \ldots, f_d$ of I is a M-regular sequence modulo S if an only if the following conditions hold:

(i) $(f_1, \ldots, f_d)M \neq M$

(ii) f_1 is not a zero-divisor in S/I.

(iii) f_i is not a zero-divisor in $S/(I, f_1, \ldots, f_{i-1})$ for $1 < i \leq d$.

For a projective variety V, the property of being a complete intersection is determined by its associated ideal $I(V)$.

Definition 2.9. An ideal $I \subseteq S$ is a complete intersection if it is generated by a regular sequence. A set of points $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a complete intersection if $I(X)$ is a complete intersection.

In the next section, we present an alternative characterization of complete intersections and use the result as a basis for our definition of virtual complete intersections.

2.2. Virtual Complete Intersection

Analogous to free resolutions in a projective space, we introduce virtual resolutions in a biprojective space, up to the irrelevant ideal $B = (x_0, x_1) \cap (y_0, y_1)$.

Definition 2.10. A virtual resolution for an ideal I in the biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ is a free complex

$$0 \longleftarrow S \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} F_2 \xleftarrow{\varphi_3} \cdots$$

such that F_i are free modules for $i \geq 0$, $\operatorname{ann}\left(\frac{\ker(\varphi_i)}{\operatorname{im}(\varphi_{i+1})}\right) \supseteq B^l$ for some $l > 0$, and $\operatorname{im}(\varphi_1) : B^N = I : B^{N'}$ for $N, N' \gg 0$, where $I : J = \{s \in S | sJ \subseteq I\}$.

Theorem 2.11. *(Virtual Hilbert Syzygy Theorem [BES17]) Let $Y \subset \mathbb{P}^{\mathbf{n}}$ be a zero-dimensional scheme and I be its associated B-saturated S-ideal. Then, S/I has a virtual resolution of length at most $|\mathbf{n}| = n_1 + n_2 + \cdots + n_r$.*

As a consequence of Theorem 2.11, for any graded module M of $S_{\mathbb{P}^1 \times \mathbb{P}^1}$, the shortest virtual resolution of M has length less than or equal to 2.

To use this result as a basis for the geometric notion of virtual complete intersection, first observe the following:

Lemma 2.12. *In $\mathbb{P}^1 \times \mathbb{P}^1$, the minimal free resolution of an ideal I is a Koszul complex if and only if I is a complete intersection, i.e. I is generated by a regular sequence.*

A proof of this result can be found in [Pee11]. In $\mathbb{P}^1 \times \mathbb{P}^1$ which has dimension 2, a Koszul complex has the form

$$0 \leftarrow S^1 \xleftarrow{\begin{bmatrix} f & g \end{bmatrix}} S^2 \xleftarrow{\begin{bmatrix} -f \\ g \end{bmatrix}} S^1 \leftarrow 0$$

for some bihomogeneous polynomials f and g. Thus, for the product of projective spaces, we can define the notion of a virtual complete intersection:

Definition 2.13. Let I be an ideal of points in $\mathbb{P}^1 \times \mathbb{P}^1$. We say I is a virtual complete intersection (VCI) if I has a virtual resolution that is a Koszul complex. In particular, $V(I) = V(f) \cap V(g)$.

With this definition in hand, we have an analogous connection between virtual reolution and its geometry.

Theorem 2.14 (Don't know how to prove this)**.** *In $\mathbb{P}^1 \times \mathbb{P}^1$, assume I is a B-saturated radical ideal of a set of points. Then, I will have a virtual Koszul resolution $0 \longleftarrow I \longleftarrow S \longleftarrow S^2 \longleftarrow S$ if and only if I is a virtual complete intersection $V(I) = V(f) \cap V(g)$ with multiplicity* 1.

3. Non-Combinatorial Determinants of the Minimal Free Resolution

Before giving a combinatorial classification of virtual complete intersections, we answer the question of to what extent configurations determine the minimal free resolution and virtual resolutions of a variety. We first investigated the minimal free resolution of an ideal of a set of points. We consider two sets of points X, Y equivalent up to configuration if they are the same under permutation and relabeling of the rulings. In other words, there exists two bijections $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ and $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$Y = \{(\phi(a), \psi(b)) | (a, b) \in X\} \text{ or } Y = \{(\phi(b), \psi(a)) | (a, b) \in X\}.$$

It turns out that the size of a minimal free resolution was not solely dependent on the configuration of points, but in some cases also depended on the actual values of the coordinates. Our data is presented below:

Example 3.1. For the following four-point configuration, the minimal free resolution as well as virtual resolution vary depending on the specific set of values of the coordinates:

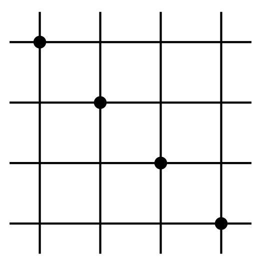

FIGURE 1. A four-point configuration whose minimal free resolution depends on the cross ratio. Each ruling correspond to one copy of $\mathbb{P}^1$.

When the points are assigned coordinates such that the cross ratio is the same on both copies of $\mathbb{P}^1$ (e.g. $([1:1],[1:1]),([1:2],[1:2]),([1:3],[1:3]),([1:4],[1:4])$), *Macaulay2* [M2] shows that the minimal free resolution of S/I has the form:

$$S^1 \leftarrow S^6 \leftarrow S^8 \leftarrow S^3 \leftarrow 0. \tag{1}$$

Accordingly, a virtual resolution has the simpler form:

$$S(1,5)^1 \leftarrow \begin{matrix} S(-1,-1)^1 \\ \oplus \\ S(0,-4)^1 \end{matrix} \leftarrow S(0,0)^1 \leftarrow 0 \tag{2}$$

When given the set of points has different cross ratios on the two copies of $\mathbb{P}^1$ (e.g. $([1:1],[1:1]),([1:-1],[1:2]),([1:3],[1:3]),([1:4],[1:4])$), the minimal free resolution will have the form:

$$S^1 \leftarrow S^6 \leftarrow S^7 \leftarrow S^2 \leftarrow 0, \tag{3}$$

and the corresponding virtual resolution has the form:

$$S(2,4)^1 \leftarrow S(-1,-2)^2 \leftarrow S(0,0)^1 \leftarrow 0 \tag{4}$$

Further, the minimal free resolution of bigger collections of points will vary depending on the value of the coordinates if the configuration contains four points that don't share any coordinates.

Definition 3.2. If the four points have homogeneous coordinates $[a : a'], [b : b'], [c : c'], [d : d']$, their cross ratio is:

$$\frac{(ca' - ac')(db' - bd')}{(da' - ad')(cb' - bc')}.$$

If a point is in the form of $\lambda[1 : 0]$, then the terms involving this point are dropped from both the numerator and the denominator.

The cross ratio of a set of four points in $\mathbb{P}^1$ is an invariant under projective change of coordinates [Har95].

Lemma 3.3. *If coordinates are changed so that three of the points have homogeneous coordinates* $[0 : 1], [1 : 1], [1 : 0]$*, the cross ratio is the ratio of the homogeneous coordinates of the fourth point.*

Note that the cross ratio is dependent on the order of the points, so the same set of points can have six different cross ratios.

Lemma 3.4. *Given a configuration of four points with distinct coordinates in* $\mathbb{P}^1 \times \mathbb{P}^1$*, consider the two cross ratios of the projection of the points into each copy of* $\mathbb{P}^1$ *(preserving the order of the points). The minimal resolution of these points depends on whether or not these two cross ratios are the same.*

Proof. Consider the form $x_0y_1 - x_1y_0$. This will vanish if and only if the two cross ratios are the same. Therefore, the degree of the $(1, 1)$ graded piece of the ideals differs in the two cases. Since the Hilbert Function is recoverable from the minimal resolution, the resolution must also differ. [this proof needs work] □

The fact that free resolutions depend not just on configurations but also on coordinates suggests that they are limited in their geometric applicability. Although virtual resolutions are not a complete solution, they are an improvement in this regard.

However, it is worth noting that virtual resolutions is not a combinatorial invariant, despite that they convey more condensed geometric structure and appears to be a better indicator of the combinatorics. For in-depth discussion, see Remark 5.11.

4. Reduced Points and the Set-Theoretic Approach

First, we consider sets of points which are the intersection of two polynomial forms without consideration of the multiplicities of the intersection. This is the "reduced" case, and corresponds to taking the radical of the ideal of the points before taking the virtual resolution.

Remark 4.1. In the reduced case, all configurations of points are virtual complete intersections.

Proof. Choose f to be the product of the smallest set of $(1,0)$-forms needed to cover the set of points (i.e. vertical lines in every ruling containing points). Let k be the degree of f, and let n be the maximum number of points in a single vertical ruling. Assign multiplicities to each point, such that the sum of the multiplicities of points in each vertical ruling is n (for example, in the ruling with n points, each point will have multiplicity 1). Now, construct n k-tuples each containing one point from each vertical ruling, such that each point appears as many times as its multiplicity. For each k-tuple, we can use Lagrangian interpolation to find a polynomial that will pass through those k points, and will therefore intersect f exactly at points of the configuration. The product of all such polynomials will give a valid g. □

Next, we will consider the "non-reduced" case, and account for the multiplicity of the intersection points. Given a configuration X, we assume that all points are simple intersections. Considering multiplicity corresponds to a scheme-theoretic view of intersections, which requires that the homogeneous ideal generated by the two intersecting forms equal the defining ideal of X, instead of just having the same radical. As we will see, this transversality condition will lead to a richer classification of configurations into VCIs, non-VCIs, and coordinate dependent cases.

5. Determination of VCIs

Theorem 5.1. *(Generalized Bézout's Theorem [Šaf13]) Let $f, g \in S = \mathbb{C}[x_0, x_1, y_0, y_1]$ be two bihomogeneous forms in $\mathbb{P}^1 \times \mathbb{P}^1$. If f and g are in generic position of multidegree (a, b) and (c, d) respectively, then $|V(f) \cap V(g)| = ad + bc$ counting multiplicities.*

The bigraded Bézout theorem will be used extensively as a tool to combinatorially determine virtual complete intersections. For brevity, we will refer to Theorem 5.1 as "Bézout's Theorem" from this point onward.

Lemma 5.2. *Let f and g be two projective curves in generic position of multidegree (a, b) and (c, d), respectively. Given a configuration of finitely many points in $\mathbb{P}^1 \times \mathbb{P}^1$, let m be the maximum number of points on the* *same horizontal ruling, and n be the maximum number of points on the same vertical ruling. If $X = V(f) \cap V(g)$, then $\max(a, c) \geq m$ and $\max(b, d) \geq n$.*

Proof. Assume, for the sake of contradiction, that $a, c < m$. Without loss of generality, we can change coordinates to assume that the m points are on the ruling with coordinates $[1 : 0]$. We can restrict f to the ruling $[1 : 0]$ by substituting $x_0 = 1, x_1 = 0$, yielding a single variable polynomial of degree a with m roots. By our assumption that $a < m$, this restriction of m must be identically 0, and so $V(f)$ contains the entire ruling $[1 : 0]$. By an identical argument on g using $c < m$, we have $V(g)$ also containing the entire ruling $[1 : 0]$. Therefore, $V(f) \cap V(g)$ contains that entire ruling, and so cannot be the original set of points. Thus, our assumption that $a, c < m$ was false, and so $\max(a, c) \geq m$. The proof that $\max(b, d) \geq n$ is exactly analogous. □

Lemma 5.3. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let m be the maximum number of points on a single horizontal ruling, and let n be the maximum number of points on a single vertical ruling. If X is a VCI of polynomials f and g with multidegrees (a, b) and (c, d) respectively, and $|X| < mn$, then:*

- *Either (i) $a \geq m$ and $b \geq n$, or (ii) $c \geq m$ and $d \geq n$.*
- *In the first case, g has horizontal components on the lines containing the m points, and vertical components on the lines containing the n points. In the second case, the same is true of f.*

Proof. By Lemma 5.2, we have

$$\max(a, c) \geq m, \max(b, d) \geq n.$$

Without loss generality, suppose $a \geq c, d \geq b$. Then, $a \geq m, d \geq n$. However, in this case $ad \geq mn$, so $ad + bc \geq mn$, contradicting $|X| < mn$. Therefore, we must have $a \geq c, b \geq d$. Then, we must have $a \geq m, b \geq n$. This proves 5.3. If g does not contain the entire line of the m collinear points, then g restricted to that line is a nonzero polynomial with m roots, and so has degree at least m. This means that $c \geq m$, which gives the contradiction $|X| = ad + bc \geq bc \geq mn$. Similarly, if g does not contain the ruling with n points, then its restriction to that line must have degree at least n giving the contradiction $|X| = ad + bc \geq ad \geq mn$. This completes the proof. □

Based on Lemma 5.2 and 5.3, our main result is the following:

Theorem 5.4. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let m be the maximum number of points of X on a single horizontal ruling, and let n be the maximum number of points on a single vertical ruling. If $|X| < mn$, and there is at least one point in X that is on a horizontal ruling with m points and a vertical ruling with n points, then X is not a VCI.*

Proof. Assume $X = V(f) \cap V(g)$, where f is a curve of multidegree (a, b) and g is a curve of multidegree (c, d). By Lemma 5.3, we must have $a \geq m, b \geq n$.

Suppose $V(g)$ includes s horizontal lines and t vertical lines. By the previous lemma, $s, t \geq 1$, and by the condition of the theorem, the intersection of these $s + t$ lines contains at least one point of X. Factoring g, we get: $g = (x_1 - \alpha_1 x_0)(x_1 - \alpha_2 x_0) \cdots (x_1 - \alpha_s x_0)(y_1 - \beta_1 y_0) \cdots (y_1 - \beta_t y_0) \cdot g_0$. Denote the degree of g_0 by (p, q). Let $Y \subseteq X$ be points covered by the $s + t$ components of g. We have $|Y| \leq ms + nt - 1$, since we are certainly double counting the point on the vertical and horizontal rulings containing most points. The remaining set of points $X \setminus Y$ must be precisely the intersection of f and g_0, whose cardinality is $aq + bp$ according to Bézout's Theorem.

Applying Bézout's Theorem again to f and g, it follows that

$$a(s + q) + b(t + p) = |X| \leq ms + nt + aq + bp - 1.$$

Simplifying the inequality above yields:

$$as + bt \leq ms + nt - 1$$

Since $a \geq m, b \geq n$, and $s, t \geq 1$, we have a contradiction. Thus, X cannot be a virtual complete intersection. □

Corollary 5.5. *If X is a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ forming a Ferrers diagram, then X is a virtual complete intersection if and only if it is a rectangle.*

Proof. Defining m and n as before, if a Ferrers diagram is not a rectangle, the number of points is strictly lower than mn, and the corner of the diagram is one of m points on its horizontal ruling and n points on its vertical ruling, so the Theorem 5.4 applies. If the configuration is a rectangle, then $\deg(f) = (m, 0), \deg(g) = (0, n)$ and (f, g) forms a regular sequence, indicating a complete intersection. □

Lemma 5.6. *Let X be a VCI, m, and n be defined as above, and $|X| < mn$. Assume, without loss of generality, that $a \geq m, b \geq n$. Then:*

- *$a = m$, $b = n$*
- *$V(g)$ has vertical components exactly on rulings with n points of X, and has horizontal components exactly on rulings with m points of X.*

Proof. As above, let s be the number of horizontal lines and t be the number of vertical lines of $V(g)$; let Y be the number of points of g covered by those lines; let (p, q) be the multidegree of the remaining components of g. From our earlier work, we have $s, t \geq 1$ and:

$$as + bt = |Y| \leq ms + nt$$

We have $a \geq m, b \geq n$, and either of these being strict would contradict the above, so $a = m, b = n$. This tells us

$$ms + nt = |Y|$$

Thus, each vertical component of $V(g)$ must contain n points of X and each horizontal component must contain m points of X (by the previous theorem, these points cannot overlap). □

Note that when $|X| < mn$, the values of s and t are intrinsically determined by the configuration of X: they are equal to the maximum number of points on any horizontal and vertical ruling respectively.

Theorem 5.7. *If $|X| < mn$ and $\gcd(m, n)$ does not divide $|X|$, then X is not a VCI.*

Proof. Suppose $X = V(f) \cap V(g)$. From the previous lemma, we can assume f has multidegree (m, n). Letting g have multidegree (c, d). By Bézout's theorem, $|X| = dm + cn$. This is divisible by $\gcd(m, n)$, which is a contradiction. □

Lemma 5.8. *If X is a VCI with $X = V(f) \cap V(g)$, $|X| < mn$, $\gcd(m, n) = 1$, and the multidegree of f is (m, n), then the multidegree of g is (c, d) with:*

$$c = n^{-1}|X| \mod m$$

$$d = m^{-1}|X| \mod n$$

(where $0 \leq c < m$ and $0 \leq d < n$).

Proof. By Bézout's theorem,

$$dm + cn = |X|$$

Considering modulo m and n, we have:

$$c \equiv n^{-1}|X| \mod m$$

$$d \equiv m^{-1}|X| \mod n$$

Since $cn, dm < mn$ we must have $c < m, d < n$, and so c and d must have the desired values. □

Lemma 5.9. *If X is a configuration with $|X| < mn$, $\gcd(m, n) = 1$, let*

$$c = n^{-1}|X| \mod m$$

$$d = m^{-1}|X| \mod n$$

Let s and t be defined as before, and let $p = d - s$ and $q = c - t$. If any of the following are true, X will not be a VCI:

- $dm + cn > |X|$
- $d < s$ *or* $c < t$

- *There is a horizontal ruling with strictly between q and m points of X, or a vertical ruling with strictly between p and n points of X.*

Proof. By our previous lemma, we know that c and d are uniquely determined if a VCI exists, and by the Chinese Remainder Theorem, we have:

$$|X| \equiv dm + cn \mod mn$$

If $dm + cn > |X|$, it is impossible to obtain $|X|$ by adding positive integer multiples of m and n (see the Chicken McNugget Theorem and related analysis).

If X is a VCI of f and g, then g has s horizontal line components, so we would have $d \geq s$. Similarly since it has t vertical line components, we would have $c \geq t$.

By a previous lemma, any horizontal ruling with fewer than m points of X cannot be contained in $V(g)$. However, a polynomial of multidegree (p, q) cannot vanish on more than p points of a horizontal ruling without containing the entire ruling. Analogously, g cannot vanish on between q and n points of a vertical ruling. □

Note that when X is a VCI of f and g with $|X| < mn$, we have determined not only the multidegree of g but also the multidegree of components that are not degree 1 lines - p and q are intrinsically determined.

Theorem 5.10. *If X has the same number (n) of points in each vertical (or each horizontal) ruling, it is a VCI.*

Proof. We will prove the vertical case, and horizontal will follow analogously. Let f be the polynomial such that $V(f)$ is comprised of all the vertical rulings that contain points of X. Using Lagrangian interpolation, there exists a polynomial that intersects each of these rulings once at any given point. By labeling the points in each vertical ruling from 1 to n, let g_i be the polynomial vanishing on all the points labeled i. Multiplying together the g_i yields a form whose variety intersects $V(f)$ exactly at X. □

Remark 5.11. When $|X| \geq mn$, VCIs are not always determined by configuration. That is, the same configuration may be a VCI with some coordinates, but not with others.

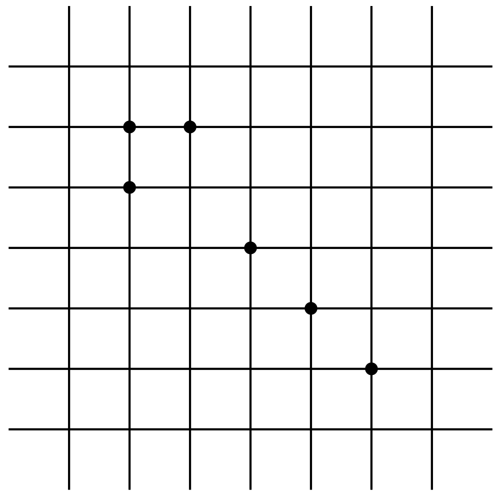

FIGURE 2

Proof. Consider the configuration of six points above, and suppose it is the VCI of f and g with multidegree (a, b) and (c, d) respectively. If any of the degrees were 0, say a, then $V(f)$ would be parallel lines, and since there are five distinct coordinates, f would have degree $(0, 5)$. There is no choice of c, d such that $ad + bc = 6$, so none of the degrees are 0. Furthermore, applying our lemma, we find that the only possible multidegrees (up to permutation) are $(2, 1)$ for f and $(2, 2)$ for g. Since there are two points sharing a ruling (both vertically and horizontally), f must have a degree $(0, 1)$ component passing through the vertical one, and therefore must have a degree $(2, 0)$ component passing through the remaining four points. But since conics are determined by three points, this is impossible in most cases, and the set of points could not be a VCI. However, in the cases where the remaining four points do lie on a conic, the points may be a VCI. For instance, if the points have coordinates:

$$([1 : 1], [1 : 1]), ([2 : 1], [1 : 2]), ([3 : 1], [1 : 3]), ([4 : 1], [1 : 4]), ([1 : 0], [1 : 1]), ([1 : 0], [1 : 0])$$

then this set of points is the intersection of the varieties of

$$x_0x_1x_2 - x_1^2x_3$$

and

$$24x_1^2x_2^2 - x_0^2x_2x_3 - 50x_1^2x_2x_3 + x_0^2x_3^2 - 9x_0x_1x_3^2 + 35x_1^2x_3^2$$

□

6. Hilbert Function and Regularity

Berkesch Zamaere et al. [BES17] proved that for a B-saturated S-module M, virtual resolutions can be constructed as a subcomplex of a minimal free resolution of M given an element d in the regularity of M. Furthermore, taking the set of minimal elements r in the regularity of M and computing the virtual resolution of the pairs (M, r), we have that the sum of twists in the virtual resolutions is equal to the sum of twists in the minimal free resolution.

To further investigate how minimal elements in the regularity interact with the geometric aspect of a point set variety, we compare the minimum degrees of and the multi-graded betti table of a virtual resolution that explicitly indicates the existence of VCI.

Remark 6.1. The regularity of an ideal does not reflect whether a variety is a VCI or not. For instance, the "hook" configuration and the four-point "box" configuration in Example 6.2 have the same regularity $(1, 1) + \mathbb{N}^2$. However, we know from Corollary 5.5 that the former is not VCI, whereas the latter is indeed a complete intersection.

Example 6.2. Two configurations with the same regularity:

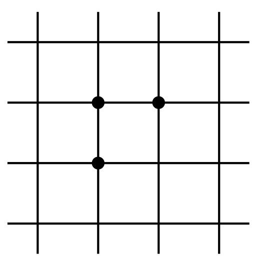

FIGURE 3. The "hook" configuration

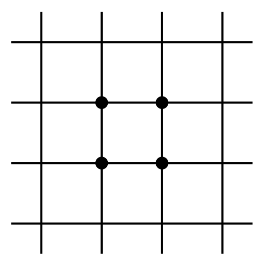

FIGURE 4. The "box" configuration

APPENDIX A. CATALOGUE OF SMALL CONFIGURATIONS

A.1. **Three-Point Configurations**

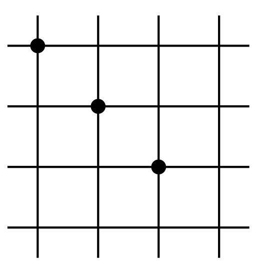

FIGURE 5

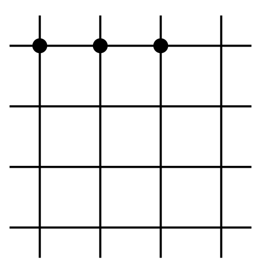

FIGURE 6

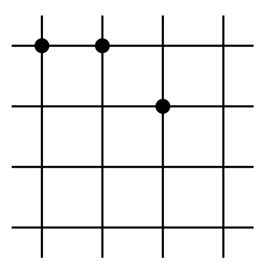

FIGURE 7

A.1.1. *VCIs.*

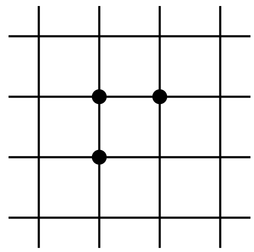

Figure 8

A.1.2. *Not VCI.*

A.2. **Four-Point Configurations**

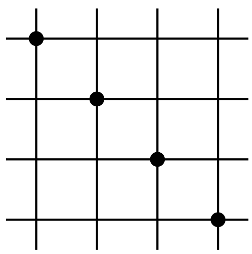

Figure 9

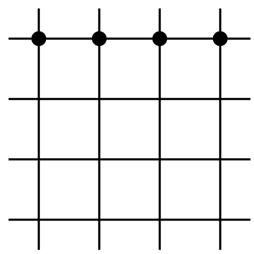

Figure 10

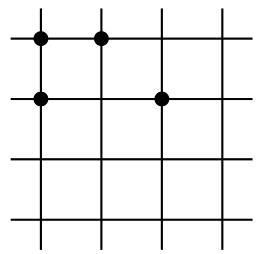

Figure 11

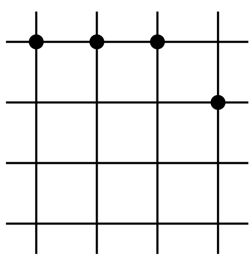

Figure 12

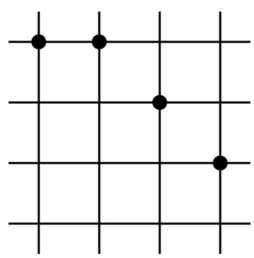

Figure 13

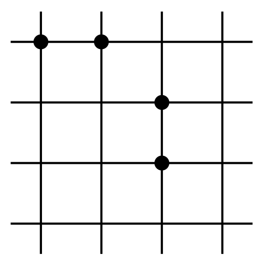

Figure 14

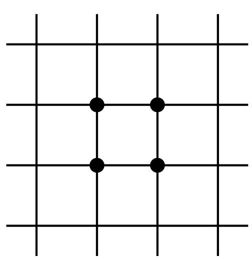

Figure 15

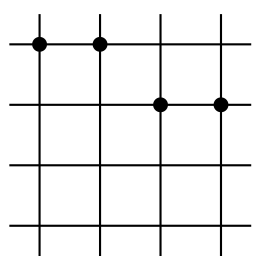

Figure 16

A.2.1. *VCIs.*

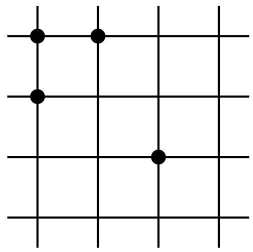

FIGURE 17

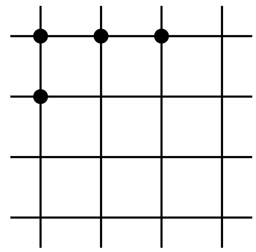

FIGURE 18

A.2.2. *Not VCI.*

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