Generating Functions for f-Vectors and the cd-index of weight polytopes

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1 Introduction

A large literature has been devoted to studying the f and h polynomials of convex polytopes. Given an n-dimensional polytope P, we define its f-vector $f(P) := (f_0, f_1, \ldots, f_n)$ where f_i is the number of i-dimensional faces of P. Encoding this vector in a polynomial $f_0 + f_1t + f_2t^2 + \cdots + f_nt^n$, we can define the h-vector $h(P) = (h_0, \ldots, h_n)$ via $h_0 + h_1t + \cdots + h_nt^n = f_0 + f_1(t-1) + f_2(t-1)^2 + \cdots + f_n(t-1)^n$. In the case of a simple polytope P, $h(P)$ is nonnegative symmetric, unimodal, and can be interpreted as the rational cohomology group of an associated toric variety $X(P)$.

However, when P is not simple, the h-vector is more mysterious and encodes less information about the polytope. In search of a similar quantity, we more closely examine the poset structure of the polytope. One such invariant is the cd-index, which is a noncommutative polynomial in variables c and d that encodes the flag f-vector of an Eulerian poset. We are are interested in studying these invariants in the case of weight polytopes. Given (W, S) a finite reflection group with generating set S acting on $\mathbb{R}^n$ and a vector $\lambda \in \mathbb{R}^n$, the weight polytope $\mathcal{P}_\lambda$ is the convex hull of the W-orbit of λ. Let $J(\lambda) = \{s \in S : s(\lambda) = \lambda\}$ be the generators that fix λ. In [3], Renner uses monoid methods to describe the facial structure of $\mathcal{P}_\lambda$ in terms of $J(\lambda)$ and (W, S) when W is a Weyl group. He also gave a complete classification of simple weight polytopes for each irreducible Dynkin diagram. Using Renner's work, Golubitsky compiled generating functions of the f-polynomials for two families of type A simple weight polytopes.

In this paper, we extend Golubitsky's work to find generating functions for all possible infinite families of simple weight polytopes from irreducible Dynkin diagrams. Along the

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way, we also derive a generating function for the f-polynomials of hypersimplices. We then focus on the nonsimple polytopes. We first detail the results of Maxwell[2], and show that Renner's result holds for arbitrary finite reflection groups. We then use Maxwell to derive a generating function of the $\mathbf{cd}$-index of hypersimplices $\Delta(n,k)$. As a corollary, we recover the $\mathbf{cd}$-index for simplices from Stanley [5].

2 Preliminaries

In this section, we will give some preliminary knowledge and define some terminologies. We assume the readers are familiar with the definition of polytopes and (graded) posets/lattices.

2.1 Polytopes

A r-dimensional polytope is the convex hull of a finite number of points in $\mathbb{R}^r$ that spans the space. On the boundary of the polytope, there lies objects called **faces** of a polytope. A face is the intersection of the polytope with an arbitrary hyperplane that does not intersect the interior of the polytope. A face is a polytope itself. A j-dimensional face is called a j-face (i.e. a vertex is a 0-face, and the polytope itself is a r-face).
For a polytope P, define the **f-vector** $f(P) := (f_0, \dots, f_r)$ where f_i is the number of i-dimensional faces of P. Define its **f-polynomial** as:

$$f_P(t) = \sum_{\text{face } \Sigma \subseteq P} t^{\dim(\Sigma)} = \sum_{i=0}^{r} f_i t^i.$$

The **h-polynomial** is:

$$h_P(t) = f_P(t-1) = \sum_{i=0}^{r} f_i (t-1)^i.$$

If we expand $h_P(t) = \sum_{i=0}^{r} h_i t^i$, then $h(P) := (h_0, \dots, h_r)$ is the **h-vector** of polytope P.

Example 2.1.1. A cube has f-vector $(8, 12, 6, 1)$ and h-vector $(1, 3, 3, 1)$.

A r-dimensional polytope is called a **simple polytope** if and only if each vertex has exactly r incident edges. For example, a cube is a simple polytope. A simple polytope has extremely nice structure, which is encoded in its symmetric h-vector.

Theorem 2.1.2. *(Dehn-Sommerville equation) For a r-dimensional simple polytope P, denote its h-vector $h(P) := (h_0, \dots, h_r)$. Then for any i, $h_i = h_{r-i}$.*

2.2 Face Lattice and $\mathbf{cd}$-index

Given a polytope P with dimension r, define the **face lattice** of P as $L = \{\text{faces of } P\} \cup \{\hat{0}, \hat{1}\}$ ordered by inclusion of faces. It's known that L is a finitely graded lattice of rank $r+1$ with

$\hat{0}$ and $\hat{1}$.

We use the terminology of posets from [5]. Let ρ denote the rank function of L, where $\rho(\hat{0}) = 0$ and $\rho(\hat{1}) = r + 1$. Let $S \subseteq [n] = \{1, 2, \ldots, n\}$. The rank-selected poset L_S of L is

$$L_S = \{x \in L | \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}.$$

Define the **flag f-vector** $\alpha(S) = \alpha_L(S)$ as the number of maximal chains in L_S. Based on that, define the **flag h-vector** $\beta(S) = \beta_L(S)$ as:

$$\beta(S) = \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha(T),$$

or,

$$\alpha(S) = \sum_{T \subseteq S} \beta(T).$$

Now, we want to encode all flag h-vector into one polynomial. Consider the free algebra ring $\mathbb{C}[a, b]$ where a and b are non-commutative. Given set $S \subseteq [n]$, define its *characteristic monomial* as $u_S = u_n u_{n-1} \cdots u_1$, where

$$u_i = \begin{cases} a, & \text{if } i \notin S \\ b, & \text{if } i \in S. \end{cases}$$

For example, if $n = 5$ and $S = \{2, 5\}$, then $u_S = baaba$. Now we define

$$\Phi_P(a, b) = \Phi_L(a, b) = \sum_{S \subseteq [n]} \beta(S) u_S, \quad \Upsilon_P(a, b) = \Upsilon_L(a, b) = \sum_{S \subseteq [n]} \alpha(S) u_S.$$

The polynomial ϕ_P is called the **ab-index** of polytope P. We have a simple relation between Φ_P and Υ_P:

$$\Upsilon_L(a, b) = \Phi_P(a + b, b), \quad \Phi_P(a, b) = \Upsilon_P(a - b, b).$$

One amazing result is from [4, Theorem 6.1]

Theorem 2.2.1. *For any polytope P, there exists a polynomial $\Psi_P(c, d)$ in the non-commuting variables c and d such that*

$$\Phi_P(a, b) = \Psi_P(a + b, ab + ba).$$

$\Psi_P(c, d)$ is also called the **cd-index** of polytope P.

2.3 Finite Coxeter Group

In order to discuss the Maxwell's results in depth, we need to know about Coxeter groups and their properties. We now define these.

Definition 2.3.1. A **Coxeter presentation** for a group W is of the form

$$W \cong \langle s_1, \dots, s_n \mid s_i^2 = e, (s_i s_j)^{m_{ij}} = e \rangle$$

for some $m_{ij} \in \{2, 3, 4, \dots\} \cup \{\infty\}$. In such case, W is called a **Coxeter Group**. If W is finite, then W is called a **Finite Coxeter Group**. $S = \{s_1, s_2, \dots, s_n\}$ is called the **Generating Set** of W.

One way to view a Coxeter group that will be useful later on is the following

Definition 2.3.2. Given a Coxeter presentation (W, S), we can encapsulate it in the **Coxeter Diagram**, denoted $\Gamma(W)$, a graph with $V = S$ and if $m_{ij} = 3$, s_i and s_j are connected with no label and if $m_{ij} > 3$, s_i and s_j are connected with label m_{ij}.

From now on, we work with a Coxeter Group W with generating set S.

Definition 2.3.3. For any set $X \subseteq S$, define $W_X = \langle X \rangle$ as the subgroup generated by elements in X. Specifically, $W = W_S$. First, define X_T to be the largest subset $X_T \subseteq X$ such that no connected component of X_T on the Coxeter diagram lies entirely in T. Define the set $X^\perp \subseteq S$ as the set of those $s \in S$ which commute with every element of X. We also define $X^* := X_T \cup (X_T)^\perp$.

Definition 2.3.4. For any $w \in W$, define the length of w, or $\text{len}(w)$, as the minimal integer such that w can be decomposed into the product of $\text{len}(w)$ elements in S.

Proposition 2.3.5. *For any $w \in W$, if $w = s_1 s_2 \cdots s_{\text{len}(w)}$, then the set $\{s_1, s_2, \dots, s_{\text{len}(w)}\}$ is independent of the choice of such decomposition. We denote that set as S_w.*

Proposition 2.3.6. *For any $w \in W$ and $X \subseteq W$,*

$$w \in W_X \iff S_w \subseteq X.$$

Proposition 2.3.7. *For any $w \in W$ and $s \in S$, if w commutes with s, then either $s \in S_w$ or $s \in S_w^\perp$.*

Proposition 2.3.8. *For any $X, Y \subseteq S$ and $a \in W$, double coset $W_X a W_Y$ contains a unique element w of minimal length. When $w = a$, we say a is (X, Y)-reduced.*

Theorem 2.3.9. *Finite Coxeter Groups W are reflection groups.*

We can thus think of any Coxeter group as acting on a Euclidean space, so we can consider the orbits of vectors under these groups.

Definition 2.3.10. For $T \subseteq S$, define the shadow poset $L(W, T) = \{gW_X W_T | g \in W, X \subseteq S\}$ ordered by inclusion.

Theorem 2.3.11 ([2], Theorem 5.11)**.** *$L(W, T)$ is isomorphic to the face lattice of the polytope P_λ, where $T \subset S$ such that W_T is the stabilizer of λ.*

In order to actually understand the poset, we emphasize the following results of Maxwell

Proposition 2.3.12 ([2], Proposition 2.2)**.** *We have $gW_X W_T \subset hW_Y W_T$ iff $X_T \subset Y_T$ and $S_w \subset X^T$, where w is the (X, Y)-reduction of $g^{-1}h$*

Proposition 2.3.13 ([2], Corollary 2.3)**.** *We have $gW_X W_T = hW_Y W_T$ iff $X_T = Y_T$ (so that $X^* = Y^*$) and $gW_{X^*} = hW_{Y^*}$.*

2.4 Coxeter Invariant Polytopes

Given a finite Coxeter system (W, S), we know from the previous that it can be thought of as a finite reflection group in some Euclidean $\mathbb{R}^n$.

Definition 2.4.1. Given $\lambda \in \mathbb{R}^n$, we define the **weight polytope** P_λ to be the convex hull of $\{w \cdot \lambda \mid w \in W\}$.

Definition 2.4.2. If λ is chosen so that P_λ is a simple polytope, we call $J(\lambda)$ **combinatorially smooth**.

For a simple polytope, the **h-vector** encodes a tremendous amount of geometric information about P_λ. In order to derive generating functions for their f-polynomials, we need a theorem of Renner, which gives the facial structure of a weight polytope P_λ in terms of the Weyl group (W, S) and $J(\lambda) = \{s \in S \mid s(\lambda) = \lambda\}$.

Theorem 2.4.3 (Renner, [3] Corollary 1.3)**.** *Let W be a Weyl group and let $r : W \to GL(V)$ be the usual reflection representation of W. Let $\mathcal{C} \subset V$ be the rational Weyl chamber and let $\lambda \in \mathcal{C}$. Then*

- *$\mathcal{P}_\lambda$ has exactly one W orbit of faces for each $I \subset S$ such that no connected component of I lies entirely in $J(\lambda)$. The collection of all such I is $S(\lambda)$.*
- *This W-orbit is represented by a face F_I whose relative interior intersects $\overline{C}$ and has parabolic subgroup $W_I = \langle s \rangle_{s \in I}$ stabilizing F_I, but acting nontrivially on F_I.*
- *The W-stabilizer of F_I is W_{I^*} where $I^* = I \cup \{s \in J(\lambda) : st = ts, \forall\, t \in I\}$.*

Remark 2.4.4. *Renner's result is stated for Weyl groups, not all finite reflection groups. Our calculations of generating functions for f-vectors in Section 3 only uses Weyl groups, but more generally the structural result holds for all finite reflection groups. See Section 4 for details about this.*

This theorem is the primary method of deriving generating functions. More specifically, the above statement about stabilizers implies the W-orbit of F_I for $I \in S(\lambda)$ in $\mathcal{P}_\lambda$ has the coset structure W/W_{I^*}. We then have

Corollary 2.4.5.

$$f_i(P_\lambda) = \sum_{I \in S(\lambda), |I| = i} \frac{|W|}{|W_{I^*}|}$$

and

$$f_{P_\lambda}(t) = \sum_{I \in S(\lambda)} \frac{|W|}{|W_{I^*}|} t^{|I|}$$

3 Results on Simple Polytopes

In this section, we derive generating function expressions for new cases of simple polytopes. Renner's classification result gives the exact cases for simple weight polytopes arising from irreducible Dynkin diagrams:

Theorem 3.0.1 ([3], Corollary 3.5)**.** *For the infinite families A_n, B_n, D_n, the subsets $\{J \subset S \mid J$ is combinatorially smooth$\}$ are the following*

1. A_n
   (a) $J = \emptyset$
   (b) $J = \{s_1, \ldots, s_i\}$, $1 \le i < n$
   (c) $J = \{s_j, \ldots, s_n\}$, $1 < j \le n$.
   (d) $J = \{s_1, \ldots, s_i, s_j, \ldots, s_n\}$, $1 \le i$, $i \le j-3$, *and* $j \le n$.
2. B_n
   (a) $J = \emptyset$
   (b) $J = \{s_1, \ldots, s_i\}$, $1 \le i \le n$
   (c) $J = \{s_n\}$
   (d) $J = \{s_1, \ldots, s_i, s_n\}$, $1 \le i$, $i \le n-3$
3. D_n
   (a) $J = \emptyset$
   (b) $J = \{s_1, \ldots, s_i\}$, $1 \le i < n-1$
   (c) $J = \{s_{n-1}\}$
   (d) $J = \{s_n\}$
   (e) $J = \{s_1 \ldots, s_i, s_{n-1}\}$, $i \le n-4$
   (f) $J = \{s_1 \ldots, s_i, s_n\}$, $i \le n-4$

Here we use the convention that for type B/D, the fork and doubled edge are on the right. In [1], Golubitsky derives the following generating functions for the weight polytopes of type A_n in the case that $J(\lambda) = \emptyset$ and $J(\lambda) = \{1, \ldots, k\}$:

Theorem 3.0.2 (Golubitsky)**.** *For A_n, denote the f-polynomial as $F_n(t)$ for $J = \emptyset$ and $F_{n,k}(t)$ for $J = \{1, \ldots, k\}$. Then we have the generating functions*

$$\sum_{n \ge 0} F_n(t) \cdot \frac{x^{n+1}}{(n+1)!} = \frac{e^{tx} - 1}{t + 1 - e^{tx}}$$

and

$$\sum_{n \ge k \ge 0} F_{n,k}(t) \cdot \frac{x^{n+1} y^k}{(n+1)!} = \frac{e^{xy}}{y - 1} \cdot \left(y + \frac{e^{txy} - t - 1}{t + 1 - e^{tx}} \right) - 1$$

Renner's classification implies that there is one more type A class of simple polytopes. We find a generating function for f-polynomial of these simple polytopes.

Theorem 3.0.3. *Let* $F_{n,a,b}(t)$ *be the* f*-polynomial associate to the type-*A_n *polytope with* $J(\lambda) = \{1, \ldots, a, \ldots, n-b+1, \ldots n\}$. *Then we have the generating function*

$$\sum_{a,b \geq 0} \sum_{n > a+b} F_{n,a,b}(t) \cdot \frac{x^{n+1} y^a z^b}{(n+1)!} = \frac{1}{y^2 - y} \left(x + \frac{(xy - e^{xy} + 1)(xz - e^{xz)})}{y} \right.$$
$$+ \frac{\left(tz + (t+1)e^{xz} - t - e^{(t+1)xz}\right)\left(\frac{ty+(t+1)e^{(xy)} - t - e^{((t+1)xy)}}{\left(t - e^{(tx)} + 1\right)y} - e^{(xy)}\right)}{t(y-1)z}$$
$$\left. + \frac{e^{(xy+xz)}}{ty} + \frac{\left(ze^{(txy)} - ye^{(txz)}\right)e^{(xy+xz)}}{t(y-z)y} \right)$$

To prove this fact, we will rely on this incredibly useful recurrence.

Proposition 3.0.4. *We have the recurrence of* f*-polynomials as*

$$F_{n,a,b}(t) = F_{n,a,b-1}(t) - F_{n-b-1,a,0}(t) \cdot \binom{n+1}{b+1} \cdot \frac{(t+1)^{b+1} - t - 1}{t}$$

Proof. Label the vertices of the Dynkin diagram $[1], \ldots, [n]$. Let $S_{a,b,n}$ be the permissible subsets such that no connected components lies in $J = \{1, \ldots, a, n-b+1, \ldots, n\}$. Then $S_{a,b,n} \subset S_{a,b-1,n}$, so it suffices to find the discrepancy between these two sets. The elements of $S_{a,b-1,n} \backslash S_{a,b,n}$ are subsets T such that $[n-b] \notin T$ but $[n-b+1], [n-b+2], \ldots, [n-b+k] \in T$ such $1 \leq k \leq b$, and these are the only elements larger than $n-b$ in T. The first $n-b-1$ vertices are free to choose. Using Corollary 2.3.5, for fixed k we can write the f-polynomial as $F_{n-b-1,a,0}(t) \cdot t^k \cdot \frac{(n+1)!}{(n-b)!} \cdot \frac{1}{(k+1)!(b-k)!}$, so summing over all n yields

$$F_{n-b-1,a,0}(t) \cdot \left(\sum_{k=1}^{b} \binom{n+1}{b+1}\binom{b+1}{k+1} t^k \right) = f_{n-b-1,a,0} \cdot \binom{n+1}{b+1} \cdot \frac{(t+1)^{b+1} - (b+1)t - 1}{t}$$

Meanwhile, we also need to consider cases in $S_{a,b-1,n}$ where the coefficient from Corollary 2.3.5 changes. Note that $|I|$ and $|W|$ are unchanged, but the set I^* changes depending on whether $J = \{1, \ldots, a, n-b+2, \ldots, n\}$ or $J = \{1, \ldots, a, n-b+1, \ldots, n\}$. The elements such that the coefficient changes are those where $[n-b+1]$ commutes with T, which are subsets where the last $[n-b], [n-b+1], \ldots, [n]$ are not in the set. In one case, there is an extra element s such that $st = ts$ for all $t \in T$, so $|W_{I^*}|$ should be multiplied by $b+1$. We can write the difference between the two f-polynomials of these cases as

$$F_{n-b-1,a,0}(t) \cdot \left(\frac{(n+1)!}{(n-b)!(b+1)!} - \frac{(n+1)!}{(n-b)!b!} \right) = F_{n-b-1,a,0}(t) \cdot \binom{n+1}{b+1} \cdot (1 - (b+1))$$

Combining all these terms yields

$$\begin{aligned} F_{n,a,b}(t) &= F_{n,a,b-1}(t) - F_{n-b-1,a,0}(t) \cdot \binom{n+1}{b+1} \cdot \frac{(t+1)^{b+1} - (b+1)t - 1}{t} - F_{n-b-1,a,0}(t) \cdot \binom{n+1}{b+1} \cdot b \\ &= F_{n,a,b-1}(t) - F_{n-b-1,a,0}(t) \cdot \binom{n+1}{b+1} \cdot \frac{(t+1)^{b+1} - t - 1}{t} \end{aligned}$$

□

Note that this sum includes cases where $n = a + b + 1$, which are actually nonsimple. The reason for this is that it significantly simplifies our recursion.

In order to prove this, we need to understand a preliminary result about hypersimplices:

Proposition 3.0.5.

$$\sum_{a,b\geq 0} F_{a+b+1,a,b}(t) \frac{y^a z^b}{(a+b+2)!} = e^{y+z} \left(\frac{ze^{ty} - ye^{tz} + y - z}{tyz(y-z)} \right)$$

Proof. Note that subsets that have no connected component in $J(\lambda)$ subsets T that are connected components of the graph and have $[a+1] \in T$. Let the endpoints of T be $[u+1], [n-v]$. Then $|W_{I^*}| = u!v!(n-u-v+1)!$ and $|W| = (n+1)!$, so that

$$F_{a+b+1,a,b}(t) = \sum_{\substack{u,v\geq 0, w\geq 1 \\ u+v+w=n+1}} \frac{(a+b+2)!}{u!v!w!} t^{w-1}$$

Summing over all possible a, b yields

$$\begin{aligned} \sum_{a,b\geq 0} F_{a+b+1,a,b}(t) \frac{y^a z^b}{(a+b+1)!} &= \sum_{a,b\geq 0} \sum_{\substack{u,v\geq 0, w\geq 1 \\ u\leq a, v\leq b \\ u+v+w=a+b+2}} \frac{(a+b+2)!}{u!v!w!} t^{w-1} \frac{y^a z^b}{(a+b+2)!} \\ &= \sum_{a,b\geq 0} \sum_{\substack{u,v\geq 0, w\geq 1 \\ u\leq a, v\leq b \\ u+v+w=a+b+2}} \frac{t^{w-1}}{u!v!w!} y^a z^b \\ &= \sum_{r,s\geq 0} \sum_{u,v\geq 0} \frac{t^{r+s+1} y^u z^v}{u!v!(r+s+2)!} y^r z^s \\ &= \sum_{u,v\geq 0} \frac{y^u z^v}{u!v!} \cdot \sum_{r,s\geq 0} \frac{y^r z^s t^{r+s+1}}{(r+s+2)!} \\ &= \sum_{u,v\geq 0} \frac{y^u z^v}{u!v!} \cdot \sum_{r,s\geq 0, r+s=n} \frac{t^{n+1}}{(n+2)!} \sum_{k=0}^{n} y^k z^{n-k} \\ &= \frac{1}{t(y-z)} \cdot \left(\sum_{u\geq 0} \frac{y^u}{u!} \cdot \sum_{v\geq 0} \frac{z^v}{v!} \sum_{n\geq 0} \frac{t^{n+2}}{(n+2)!} \cdot (y^{n+1} - z^{n+1}) \right) \end{aligned}$$

$$= \frac{e^{y+z}}{tyz(y-z)} \left(z \sum_{n \geq 0} \frac{t^{n+2} y^{n+2}}{(n+2)!} - y \sum_{n \geq 0} \frac{t^{n+2} z^{n+2}}{(n+2)!} \right)$$
$$= \frac{e^{y+z}}{tyz(y-z)} \left(z(e^{ty} - ty - 1) - y(e^{tz} - tz - 1) \right)$$
$$= \frac{e^{y+z}}{tyz(y-z)} \left(ze^{ty} - z - ye^{tz} + y \right)$$

as desired. □

Proof of Theorem 3.0.3. Using the the recurrence from Proposition 3.0.3, when summing over all x, y, z we get a nice recurrence which only depends on the ability to calculate the sum in Proposition 3.0.5 and Golubitsky's formula for $J(\lambda) = \{1, \dots, k\}$. Solving for our generating function yields the desired formula. □

We obtain similar generating functions for the simple polytope cases of B_n, according to Renner's classification:

Theorem 3.0.6. *In B_n, denote the f-polynomial as $F_n(t)$ when $J = \emptyset$, and as $F_{n,k}(t)$ when $J = \{1, \dots, k\}$ Then we have the generating functions*

$$\sum_{n \geq 1} F_n(t) \cdot \frac{x^n}{n!} = \frac{te^{tx}}{t + 1 - e^{2tx}} - 1$$

$$\sum_{n > k \geq 0} F_{n,k}(t) \cdot \frac{x^n y^k}{n!} = \frac{1}{y-1} \left(e^{(t+2)xy} + \frac{e^{tx} \cdot \left(e^{2(t+1)xy} - (t+1)\, e^{2xy} + t - ty \right)}{(t + 1 - e^{2tx})y} \right)$$

We use the following lemma, similar to Lemma 3.0.4

Lemma 3.0.7. *Define $H_k = \sum_{n>k} F_{n,k}(t) \frac{x^n}{n!}$, then*

$$F_{n,k-1} - F_{n,k} = F_{n-k-1,0} \cdot 2^{k+1} \binom{n}{k+1} \cdot \frac{(t+1)^{k+1} - (t+1)}{t}.$$

$$H_{k-1} - H_k = (t+2)^k \frac{x^k}{k!} + (H_0 + 1) \cdot \frac{2^{k+1} x^{k+1}}{(k+1)!} \cdot \frac{(t+1)^{k+1} - (t+1)}{t}.$$

Theorem 3.0.8. *In B_n, denote the f-polynomial as $F_n(t)$ when $J = \{n\}$, and as $F_{n,k}(t)$ when $J = \{1, \dots, k, n\}$. Then we have the generating functions*

$$\sum_{n \geq 1} F_n(t) \cdot \frac{x^n}{n!} = \frac{te^{tx} - (t+1)tx}{t + 1 - e^{2tx}} - 1$$

$$\sum_{n-2>k\geq 0} F_{n,k}(t)\frac{x^{n+1}y^k}{(n+1)!} = \frac{1}{y^2-y}\Bigg(xy+\Big(y+\frac{(t+1)e^{(2xy)}}{t}-\frac{e^{(2(t+1)xy)}}{t}-1\Big)\Big(\frac{(t+1)tx-te^{(tx)}}{t-e^{(2tx)}+1}+1\Big)$$
$$-x-\frac{\left((t+1)xy+\frac{1}{t}+1\right)e^{(2xy)}-\frac{e^{(2(t+1)xy)}}{t}-e^{((t+2)xy)}}{y}\Bigg)$$

We actually end up having the same recurrence as in the case with no $\{n\} \subset J$, the only difference being the initial conditions.

Lemma 3.0.9. *Define* $H_k = \sum_{n\geq k+2} F_{n,k}(t)\frac{x^n}{n!}$*, then*

$$F_{n,k-1} - F_{n,k} = F_{n-k-1,0}\cdot 2^{k+1}\binom{n}{k+1}\cdot\frac{(t+1)^{k+1}-(t+1)}{t}.$$

$$H_{k-1} - H_k = F_{k+1,k-1}\frac{x^{k+1}}{(k+1)!} + (H_0 + x)\cdot\frac{2^{k+1}x^{k+1}}{(k+1)!}\cdot\frac{(t+1)^{k+1}-(t+1)}{t},$$

where

$$F_{k+1,k-1}(t) = -2^k(k+1)(t+1) + 2^{k+1}\frac{(t+1)^{k+1}-1}{t} + (t+2)^{k+1} - 2^{k+1}.$$

4 Maxwell implies Renner

This section provides a complete proof of Renner's result about the facial structure of weight polytopes. First, we prove the following fact

Proposition 4.0.1. *Let* $(g, X, T) \leq (h, Y, T)$ *under the Maxwell poset. Say a face* (g', X', T) *is* X*-type if* $X'_T = X_T$*. Then, a* X*-type face is inside a* Y*-type face iff* $X_T \subseteq Y_T$*, and the number of* X*-type face inside a fixed* Y*-type face is*

$$\frac{|W_Y|}{|W_{X_T}|\cdot|W_{Y\cap X^{\perp}}|}$$

Proof. By 2.3.12, we have that $gW_XW_T \subset hW_YW_T$ iff $X_T \subset Y_T$ and $S_w \subset X^{\perp}$. In other words, $w \in W_XW_{X^{\perp}}W_Y$. WLOG, by writing $h^{-1}gW_XW_T \subset W_YW_T$ we can assume that $h = 1$. Therefore, the number of possible orbits is the size $|W_XW_{X^{\perp}}W_Y|$. By definition of $X^{\perp}$, $W_XW_{X^{\perp}} = W_{X^{\perp}}W_X$ commute, so this size is actually $|W_X^{\perp}W_XW_Y| = |W_{X^{\perp}}W_Y|$ since $X \subset Y$. By 2.3.13, the stability consists of all elements such that $h^{-1}gW_{X^*} = W_{X^*}$, which implies $g^{-1}h \in W_{X^*}$. So the size of the stabilizer is $|W_{X^*}|$. We thus have that the number of faces is the number equality modulo the stabilizer, which is

$$\frac{|W_X^{\perp}W_Y|}{|W_{X^{\perp}}|} = \frac{|W_{X^{\perp}}||W_Y|}{|W_{X^{\perp}\cap Y}|} = \frac{|W_Y|}{|W_{X^{\perp}\cap Y}|}$$

□

Note that if $Y = S$, then we just get the formula $\frac{|W|}{|W_{X_T}||X^{\perp}|} = \frac{|W|}{|W_{X^*}|}$, which the formula derived from Renner's theorem. We are now in a position to prove Renner's theorem.

proof of 2.4.3. We have a natural action of W on $L(W, T)$ via $w(hW_X W_T) \to whW_X W_T$. Fixing $T = J(\lambda)$, note that the subsets that have no connected component lying entirely in J are precisely the sets X such that $X_J = X$. Given subsets $K, L \subset S$, if $K_J \neq L_J$ then 2.3.13 implies they cannot be in the same orbit. Alternatively, if $K_J = L_J$ then we claim any two sets $gW_K W_J$ and $hW_L W_J$ are in the same W-orbit. This statement boils down to the fact that $W_K W_J = W_L W_J$, which is true by 2.3.13.
For the second statement we consider the face $W_I W_J$ where $I_T = I$. Through Maxwell's bijection, one fundamental chamber corresponds to elements $(\mathrm{id}, X) = W_X W_J$. Therefore, taking $X = J$ gives the desired face. Note then that W_I acts on $W_I W_J$, but non trivially as the action of a nontrivial group on itself permutes the elements.
The third statement about the stabilizer follows immediately from setting $Y = S$ in 2.3.13.
□

5 The nonsimple case: flag-f vectors and ab-index

We now turn to the case of nonsimple weight polytopes. We wish to find generating functions for the ab and cd-indices, as these carry more information about the polytope than the f and h-polynomials.

5.1 Hyper-simplices

Definition 5.1.1. For integers n, k where $n \geq k \geq 1$, the (n, k)-hypersimplex, denoted $\Delta(n, k)$, is the convex hull of the S_n orbit of $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$ where there are k 1's and $n - k$ 0's.

Note that $k = 1$ yields the normal n-dimensional simplices. When looking at the A_{n-1} diagram, these are the weight polytopes where $J(\lambda) = S \setminus \{s_k\}$, where we have included all the generators except for the kth element. By Renner's classification of combinatorially smooth $J(\lambda)$ for the type A Dynkin diagram, $\Delta(n, k)$ is nonsimple unless $k = 1, n$. Recall in Section 3 that in finding a generating function for the last simple case, we actually found a generating function for the f-polynomial for $\Delta(n, k)$. But instead, we would like to have a generating function for cd-index. For the sake of notational convenience, we define

$$D(x) := \frac{e^{(a-b)x} - 1}{a - b} = \sum_{k=1}^{\infty} (a - b)^{k-1} \frac{x^k}{k!}$$

We now give our generating function for the ab-index of the hypersimplex

Theorem 5.1.2. *Let* $\Phi_{n,k}(a,b)$ *denote the* ab*-index of the hypersimplex* $\Delta(n,k)$*. Then we have the generating function*

$$\sum_{n\geq k\geq 1}\Phi_{n,k}(a,b)\frac{y^k}{(n+1)!}=(1-D(y+1)\cdot b)^{-1}(D(y)D(1)b+\frac{1}{y-1}(D(y)-yD(1)))$$

Proof. we have the following simple recurrence for Υ_P:

$$\Upsilon_P=\Upsilon_{\hat{0}\hat{1}}=\sum_{\hat{0}<x<\hat{1}}a^{\rho(x,\hat{1})-1}b\Upsilon_{\hat{0}x}+a^{\rho(\hat{0},\hat{1})-1}.$$

This recurrence comes from the fact that Υ_P is a sum over all possible chains that doesn't contain $\hat{0}$ or $\hat{1}$ in P. Now, we group all chains according to the highest-ranked element. We say it is x. Then, the sum of all chains in this group is the same as $\Upsilon_{\hat{0}x}$, which gives us the recurrence formula. The coefficient $a^{\rho(x,\hat{1})-1}b$ means x is *being there* while all higher-ranked elements are *absent*. The constant term comes from the unique empty chain.

To derive a similar recurrence for the ab-index, replace a with $a-b$ and we have

$$\Phi_P=\Phi_{\hat{0}\hat{1}}=\sum_{\hat{0}<x<\hat{1}}(a-b)^{\rho(x,\hat{1})-1}b\Phi_{\hat{0}x}+(a-b)^{\rho(\hat{0},\hat{1})-1}.$$

According to Maxwell's Theorem, when $P=L(n,k)$, all sub-lattices $[\hat{0},x]$ are isomorphic to the face lattice of some hyper-simplices as well, and the number of $x\in P$ such that $[\hat{0},x]\cong\Phi_{L(n',k')}$ is equal to

$$\frac{(n+1)!}{(n'+1)!(k-k')!(n-n'+k'-k)!}.$$

If we plug in this fact into the recurrence for $\Phi_{n,k}\frac{y^k}{(n+1)!}$, then in the recurrence expansion, the coefficient of $\Phi_{n',k'}\frac{y^{k'}}{(n'+1)!}$ will be

$$\frac{(a-b)^{k-k'}y^{k-k'}}{(k-k')!}\cdot\frac{(a-b)^{n-n'+k'-k}}{(n-n'+k'-k)!}\cdot(a-b)^{-1}b.$$

Therefore, if we define $F=\sum_{n\geq k\geq 1}\Phi_{n,k}(c,d)\frac{y^k}{(n+1)!}$, then we can show the following equation is true be comparing coefficients (the first term comes from the the analysis above, the second term comes from the terms that looks like a^tb and rhe last term comes from terms that look like a^t):

$$\begin{aligned}F&=\left[\left(\sum_{p\geq 0}\frac{(a-b)^py^p}{p!}\right)\left(\sum_{q\geq 0}\frac{(a-b)^q}{q!}\right)-1\right](a-b)^{-1}b\cdot F+\sum_{n\geq k\geq 1}\frac{(a-b)^{n-1}by^k}{k!(n-k+1)!}\\&\quad+\sum_{n\geq k\geq 1}\frac{(a-b)^ny^k}{(n+1)!}\\&=D(y+1)\cdot F+D(y)D(1)b+\frac{1}{y-1}(D(y)-yD(1)).\end{aligned}$$

which conclude our proof. ∎

Note that since a, b do not commute, it is crucial to write the fraction with an inverse on the left. As a corollary, we derive a nice generating function for the normal simplices, is:

Corollary 5.1.3. *Let $\Phi(n)(a,b)$ denote the element ab-index of the n-dimensional simplex. Then we have the generating function*

$$\sum_{n\geq 1} \frac{\Phi_n(a,b)}{(n+1)!} = (1 - D(1)b)^{-1} \cdot D(1)$$

This result is essentially the same formula as [5] Corollary 1.4.

6 cd-index

Given our formula for the generating function for the ab-index, one might hope we can rewrite the generating function as a cd-index. The closest we can get to this task is to write the generating function in terms of

$$c(x) := \cosh((a-b)x) = \frac{1}{2}(e^{(a-b)x} + e^{(b-a)x}) = \sum_{j=0}^{\infty} (a-b)^{2j} \frac{x^{2j}}{(2j)!} = \sum_{j=0}^{\infty} (c^2 - 2d)^j \frac{x^{2j}}{(2j)!}$$

and

$$s(x) := \frac{\sinh((a-b)x)}{a-b} = \frac{e^{(a-b)x} - e^{(b-a)x}}{2(a-b)} = \sum_{j=0}^{\infty} (a-b)^{2j} \frac{x^{2j+1}}{(2j+1)!} = \sum_{j=0}^{\infty} (c^2 - 2d)^j \frac{x^{2j+1}}{(2j+1)!},$$

which are clearly functions of c, d. Denote these functions as $c(x), s(x)$, respectively. We then have

Theorem 6.0.1.

$$\sum_{n\geq k\geq 1} \Psi_{n,k}(c,d) \frac{y^k}{(n+1)!} = (1 - s(y+1) \cdot c + c(y+1))^{-1} \cdot \left(\frac{c(y+1) - c(y) - c(1) + 1}{c^2 - 2d} \cdot c \right.$$
$$\left. - s(y+1) + \frac{y+1}{y-1} \cdot (s(y) - s(1)) \right)$$

First Proof. First we "multiply" numerator and denominator by $z := 1 + e^{(b-a)(y+1)}$. That is, we have the equality

$$(z(1 - D(y+1) \cdot b))^{-1} \cdot z(D(y)D(1)b + \frac{1}{y-1}(D(y) - yD(1)))$$

Now it suffices to write this numerator and denominator in terms of $c, d, c(x), s(x)$. First,

$$\begin{aligned}
(1+e^{(b-a)(y+1)})(1-D(y+1)\cdot b) &= 1-(e^{(a-b)(y+1)}-1)\cdot\frac{b}{a-b}+e^{(b-a)(y+1)}-(1-e^{(b-a)(y+1)})\cdot\frac{b}{a-b}\\
&= 1-(e^{(a-b)(y+1)}\cdot\frac{b}{a-b}-e^{(b-a)(y+1)})-e^{(b-a)(y+1)}\\
&= 1-s(y+1)\cdot 2b+c(y+1)-s(y+1)\cdot(a-b)\\
&= 1-s(y+1)\cdot c+c(y+1)
\end{aligned}$$

Now, for the numerator we have

$$\begin{aligned}
&(1+e^{(b-a)(y+1)})(D(y)D(1)b+\frac{1}{y-1}(D(y)-yD(1)))\\
&= (1+e^{(b-a)(y+1)})(e^{(a-b)(y+1)}-e^{(a-b)y}-e^{(a-b)}+1)(a-b)^{-2}b\\
&\quad +(1+e^{(b-a)(y+1)})((y-1)(a-b))^{-1}(e^{(a-b)y}-1+y-ye^{a-b})\\
&= (e^{(a-b)(y+1)}+1-e^{(a-b)y}-e^{(b-a)}-e^{(a-b)}-e^{(b-a)y}+1+e^{(b-a)(y+1)})(a-b)^{-2}b\\
&\quad +(1+e^{(b-a)(y+1)})/(a-b)+((y-1)(a-b))^{-1}(e^{(a-b)y}+e^{b-a}-ye^{a-b}-ye^{(b-a)y})\\
&= (c(y+1)\cdot 2b-c(y)\cdot 2b-c(1)\cdot 2b+2b+a-b+(a-b)c(y+1))(a-b)^{-2}-s(y+1)\\
&\quad +((y-1)(a-b))^{-1}\Bigg(\frac{y+1}{2}(e^{(a-b)y}-e^{(b-a)y})-\frac{y-1}{2}(e^{(a-b)y}+e^{(b-a)y})\\
&\qquad -\frac{y+1}{2}(e^{(a-b)}-e^{(b-a)})-\frac{y-1}{2}(e^{(a-b)}+e^{(b-a)})\Bigg)\\
&= (c(y+1)\cdot c-(c(y)+c(1))\cdot 2b+c)(a-b)^{-2}-\frac{c(1)+c(y)}{(a-b)}+\frac{y+1}{y-1}(s(y)-s(1))-s(y+1)\\
&= \frac{c(y+1)-c(y)-c(1)+1}{c^2-2d}\cdot c-s(y+1)+\frac{y+1}{y-1}(s(y)-s(1))
\end{aligned}$$

The result thus follows.

□

Second Proof. This result can also be derived directly from Stanley's recurrence function for cd-indices. In [5] Theorem 1.1, we have

$$2\Psi_P = 2\Psi_{\hat{0}\hat{1}} = \sum_{\substack{\hat{0}<x<\hat{1}\\ \rho(x,\hat{1})=2j-1}} (c^2-2d)^{j-1}c\Psi_{\hat{0}x} + \sum_{\substack{\hat{0}<x<\hat{1}\\ \rho(x,\hat{1})=2j}} (c^2-2d)^{j}\Psi_{\hat{0}x} + \begin{cases} 2(c^2-2d)^{k-1} & \text{if } \rho(\hat{0},\hat{1})=2k-1\\ 0 & \text{if } \rho(\hat{0},\hat{1})=2k.\end{cases}$$

According to Maxwell's Theorem, when $P = L(n,k)$, all sub-lattices $[\hat{0}, x]$ are isomorphic to the face lattice of some hyper-simplices as well, and the number of $x \in P$ such that $[\hat{0}, x] \cong L(n', k')$ is equal to

$$\frac{(n+1)!}{(n'+1)!(k-k')!(n-n'+k'-k)!}.$$

Therefore, if we define $F = \sum_{n\geq k\geq 1} \Psi_{n,k}(c,d)\frac{y^k}{(n+1)!}$, then

$$\begin{aligned}2F &= (\cosh(y)\sinh(1) + \sinh(y)\cosh(1))c \cdot F\\ &\quad - (\sinh(y)\sinh(1)(c^2-2d) + \cosh(y)\cosh(1) - 1)\cdot F\\ &\quad + (\overline{\cosh}(y)\overline{\cosh}(1)(c^2-2d) + \sinh(y)\sinh(1))c\\ &\quad - (\overline{\cosh}(y)\sinh(1) + \sinh(y)\overline{\cosh}(1))(c^2-2d) + \frac{2}{y-1}(\sinh(y) - y\cdot\sinh(1)).\end{aligned}$$

where

$$\sinh(t) = \sum_{j\geq 0}(c^2-2d)^j\frac{t^{2j+1}}{(2j+1)!},$$

$$\cosh(t) = \sum_{j\geq 0}(c^2-2d)^j\frac{t^{2j}}{(2j)!},$$

$$\overline{\cosh}(t) = \sum_{j\geq 1}(c^2-2d)^{j-1}\frac{t^{2j}}{(2j)!}.$$

□

which further reduces to the form in Theorem 6.0.1.

7 Future Directions

Here are some ideas for future work:

- [4] and [5] discuss a correspondence between the ab-index of ordinary simplices and simsun permutations, a special kind of alternating permutation. We attempted to find a correspondence between pairs of simsun permutations and ab-indices of hypersimplices, but were unable to do so. Exploring more in this direction could be interesting.
- Find the generating functions for f-polynomials of type D, although we may do this later.
- We wrote a much faster program for determining the f-polynomials and cd-index without drawing a polytope. From here we can actually calculate all possible f-polynomials and cd-indices for the exceptional Coxeter groups. Expect such data in a later edition of the report.

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