SMITH INVARIANTS AND DUAL GRADED GRAPHS

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ABSTRACT. The aim of this paper is to present evidence for a simple conjectural relation between eigenvalues and invariant factors of incidence matrices associated with adjacent ranks in differential posets. The conjectural relation yields the Smith invariants immediately, as the eigenvalues are completely understood [3, 15]. Furthermore, we consider more general structures: dual graded graphs [3]. In this setting, the aforementioned relations sometimes hold and other times fail.

One particularly interesting example is the Young-Fibonacci lattice $\mathbf{Y}F$ studied by Okada [14], which we show possesses the aforementioned conjectural relation.

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1. BACKGROUND

1.1. **Differential posets.** Let $\mathbf{P}$ be a locally finite poset with $\hat{0}$, having only finitely many elements of each rank. In $\mathbf{P}$ we define *up* and *down* maps U, D :

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$\mathbb{Z}\mathbf{P} \to \mathbb{Z}\mathbf{P}$ by

$$Ux = \sum_{y \succ x} y \qquad Dy = \sum_{y \succ x} x$$

for vertices x, y in $\mathbf{P}$ and extending linearly. We call $\mathbf{P}$ r-*differential*, for positive integer r, if U and D satisfy the commutation relation

$$DU - UD = rI.$$

We will often drop the r when it is arbitrary and the statement at hand does not depend on it, going against the tradition [3, 15] of its omission only when $r = 1$.

1.1.1. *Young's lattice.* The prototypical example of a 1-differential poset is *Young's lattice*, denoted by $\mathbf{Y}$. This is the set of all partitions $\mathcal{P}$, ordered by inclusion of Young diagrams; see Figure 1.

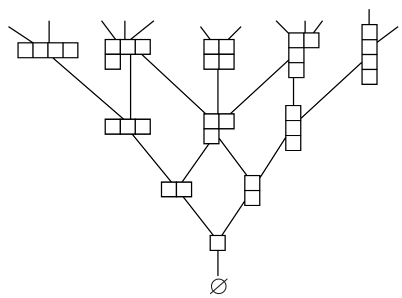


FIGURE 1. Young's lattice.

Recall that $\mathbf{Y}$ describes the branching from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$ in the representation theory of $\mathfrak{S}_n$, a fact we will later use.

1.1.2. *Fibonacci posets.* As a set, the *Fibonacci r-differential poset* $Z(r)$ consists of all finite words using alphabet $\{1_1, 1_2, \ldots, 1_r, 2\}$. For two words $w, w' \in Z(r)$, we define w to cover w' if either

(1) w' is obtained from w by changing a 2 to some 1_i, as long as only 2's occur to its left, or
(2) w' is obtained from w by deleting its first letter of the form 1_i.

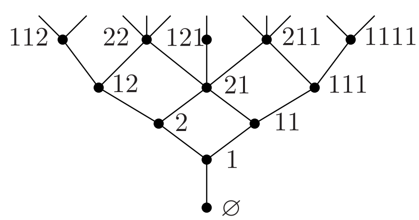


FIGURE 2. The Young-Fibonacci lattice $\mathbf{Y}F$.

It is an easy exercise to show $Z(r)$ is an r-differential poset [15]. It is also easy to see why it has such a name: the jth rank of $Z(1) = \mathbf{Y}F$ (the *Young-Fibonacci* lattice) has size f_j, the jth Fibonacci number; see Figure 2.

1.2. **Eigenvalues.** A particularly nice feature of differential posets is that the eigenvalues of DU and UD are simple to write down. To be more precise, we have the following proposition [15]:

Proposition 1.1. *Let* $\mathbf{P}$ *be an* r*-differential poset. Then*

$$\mathrm{Ch}(DU_n, \lambda) = \prod_{i=0}^{n} (\lambda - r(i+1))^{\Delta p_{n-i}}$$

and

$$\mathrm{Ch}(UD_n, \lambda) = \prod_{i=0}^{n} (\lambda - ri)^{\Delta p_{n-i}}.$$

1.3. **Smith invariants.** Henceforth, we take R to be a UFD. Furthermore, except when explicitly noted otherwise, matrices in this section are over R.

Definition 1.2. The units of $R^{n\times n}$ are called *unimodular* matrices.

It is a standard exercise to show that a matrix M is unimodular if and only if $\det M$ is a unit of R.

Definition 1.3. A (possibly rectangular) diagonal matrix D is a diagonal form for a matrix A if there exist unimodular matrices P and Q such that $D = PAQ$. It is called the (up to units) *Smith normal form* of A if the diagonal entries $d_{11}, d_{22}, \ldots$ of D are such that $d_{ii} \mid d_{jj}$ for all $i \leq j$; in this case, we say the *Smith entries* of A are $s_i = d_{ii}$.

It should be noted that a matrix A need not have a Smith normal form, as R is only a UFD, not a PID. However, if it does have a Smith form, then it is unique up to units.

For an integral matrix A, let $d_i(A)$ be the greatest common divisor of the determinants of all the $i \times i$ minors of A, where $d_i(A) = 0$ if all such $i \times i$ determinants are zero. The number $d_k(A)$ is called the kth *determinantal divisor* of A. The following is quite useful when studying Smith invariants.

Theorem 1.4. *The Smith normal form entries* $(s_1, s_2, \ldots)$ *of a matrix* $A \in \mathbb{Z}^{n\times n}$ *are given by the equation*

$$s_j(A) = \frac{d_j(A)}{d_{j-1}(A)},$$

where $d_0(A)$ *is taken to be* 1.

From this, when A is integral and invertible we get a useful description for the largest Smith invariant.

Proposition 1.5. *Let* $A \in \mathbb{Z}^{n\times n}$ *be nonsingular. Then* s_n *is the smallest positive integer for which*

$$s_n A^{-1} \in \mathbb{Z}^{n\times n}.$$

Proof. From basic linear algebra we know that

$$A^{-1}_{ij} = (-1)^{j+i} \frac{M_{ji}}{\det A},$$

where M_{ij} is defined to be the determinant of the $(n-1)\times(n-1)$ matrix resulting from the deletion of row i and column j in A. Hence

$$\begin{aligned} sA^{-1}\in\mathbb{Z}^{n\times n} &\Leftrightarrow\quad s\frac{M_{ij}}{\det A}\in\mathbb{Z}\quad\forall i,j\\ &\Leftrightarrow\quad \det A\mid sM_{ij}\quad\forall i,j\\ &\Leftrightarrow\quad \det A\mid \gcd_{i,j}(sM_{ij})=s\gcd_{i,j}(M_{ij})\\ &\Leftrightarrow\quad s\frac{\gcd_{i,j}(M_{ij})}{\det A}\in\mathbb{Z}\\ &\Leftrightarrow\quad \frac{s}{s_n}\in\mathbb{Z}, \end{aligned}$$

since Theorem 1.4 implies

$$s_n=\frac{d_n(A)}{d_{n-1}(A)}=\frac{\det A}{d_{n-1}(A)}=\frac{\det A}{\gcd M_{ij}}.$$

□

2. Conjecture for r-differential posets

The following definitions are central to our conjectures.

Definition 2.1. Let φ be an endomorphism of a rank n free R-module, with all eigenvalues in R. To φ we associate a partition $\mathcal{E}$ of its eigenvalues, defined to be the multiset of sets

$$E_i=\{\text{eigenvalues with multiplicity at least } i\},$$

where $1\le i\le n$.

Definition 2.2. Let $\varphi\in R^{n\times n}$. Then φ is said to possess the *Smith-eigenvalue relation* if it has all its eigenvalues in R and it has a Smith form over R, with

$$s_{n+1-i}=\prod_{\lambda\in E_i}\lambda,$$

taking the empty product to be 1.

Note that $\varphi+tI$ possessing the Smith-eigenvalue relation over $R[t]$ implies $\varphi+kI$ has the Smith-eigenvalue relation for all $k\in R$.

Example 2.3. Let $\varphi=\begin{pmatrix}2&1\\1&2\end{pmatrix}$. Then we have $\varphi+tI$ has eigenvalues $t+1$ and $t+3$. Thus, $E_1=\{t+1,t+3\}$ and $E_2=\varnothing$. Computing, we have

$$\begin{pmatrix}2+t&1\\1&2+t\end{pmatrix}\sim\begin{pmatrix}0&1-(2+t)^2\\1&2+t\end{pmatrix}\sim\begin{pmatrix}0&1-(2+t)^2\\1&0\end{pmatrix}\sim\begin{pmatrix}1&0\\0&(t+1)(t+3)\end{pmatrix},$$

where the last step simply involves swapping rows and factoring. Looking back at Definition 2.2, one can now see that $\varphi+tI$ possesses the Smith-eigenvalue relation over $\mathbb{Z}[t]$.

The following proposition shows that the Smith-eigenvalue relation is special for even diagonal matrices.

Proposition 2.4. *For diagonal matrices $\varphi\in\mathbb{Z}^{n\times n}$ over $\mathbb{Z}$*

(i) *φ has the Smith-eigenvalue relation if and only if every pair of distinct eigenvalues $\lambda_i \neq \lambda_j$ are relatively prime. (In particular, a singular φ has the Smith-eigenvalue relation if and only if $\varphi \in \pm\{0,1\}^{n\times n}$.)*

(ii) *$\varphi + tI$ has the Smith-eigenvalue relation over $\mathbb{Z}[t]$ if and only if*

$$\varphi \in \{0,1\}^{n\times n} + \mathbb{Z}I.$$

Proof. Let $\{\lambda_i\}$ be distinct representatives for the diagonal entries of φ. Suppose all λ_i are nonzero. Then the claim follows by induction and

$$s_n = \mathrm{lcm}_i \lambda_i = \prod_i \lambda_i \Leftrightarrow \gcd_i \lambda_i = 1.$$

Suppose now that φ is singular. If $\varphi \in \{0,1\}^{n\times n}$, it is already in Smith form, and it is easy to see that

$$s_{n+1-i} = \prod_{\lambda \in E_i} \lambda.$$

On the other hand, if φ contains an entry strictly greater than 1, then the product of the nonzero products

$$\prod_{\lambda \in E_i} \lambda$$

is strictly less than the product of φ's nonzero Smith entries.

By the first part, setting t to the negative of a diagonal entry of φ gives the necessity in the second assertion. Sufficiency follows from

$$\begin{pmatrix} t+j+1 & \\ & t+j \end{pmatrix} \sim \begin{pmatrix} 1 & \\ & (t+j+1)(t+j) \end{pmatrix}$$

over $\mathbb{Z}[t]$ for $j \in \mathbb{Z}$, and that scalar matrices clearly possess the Smith-eigenvalue relation. □

Example 2.5. Consider $\varphi = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$. Then

(a) φ is nonsingular and has the Smith-eigenvalue relation.
(b) $\varphi + I$ is nonsingular and does not have the Smith-eigenvalue relation.
(c) $\varphi - 3I$ is singular and has the Smith-eigenvalue relation.
(d) $\varphi - I$ is singular and does not have the Smith-eigenvalue relation.

Because $\varphi + kI$ does not have the Smith-eigenvalue relation for some $k \in \mathbb{Z}$, we have that $\varphi + tI$ does not have the relation over $\mathbb{Z}[t]$.

With the above terminology, we are now in a position to state our main conjecture:

Conjecture 2.6. *Let $\mathbf{P}$ be an r-differential poset, and set $n \geq 0$. Then*

(i) *U_n has all Smith entries equal to 1;*
(ii) *$\Delta p_{n+1} \geq \Delta p_n$;*
(iii) *$DU_n + tI$ has the Smith-eigenvalue relation over $\mathbb{Z}[t]$.*

It should be remarked that Conjecture 2.6 is invariant under interchanging U's and D's; this follows from $D_n = U_{n-1}^t$ and the relation $DU_n = DU_n + rI$. The following is also an important observation:

Observation 2.7. Let $\mathbf{P}$ be an r-differential poset. Assume $p_1 < p_2 < p_3 < \cdots$[1], and parts (ii) and (iii) of Conjecture 2.6 hold in $\mathbf{P}$. Then we have the Smith entries of $DU_n + tI$ over $\mathbb{Z}[t]$ are given by

entry	multiplicity
$(n+t)^{-\delta_{r,1}}(n+1)!_{r,t}$	$\Delta^2 p_0 = 1$
$(i+1)!_{r,t}$	$\Delta^2 p_{n-i}$
1	p_{n-1}

where $0 \leq i \leq n-1$, we take $p_k = 0$ for k negative, and

$$\ell!_{r,t} = (r \cdot \ell + t)(r \cdot (\ell - 1) + t) \cdots (r \cdot 1 + t).$$

Another observation one should make is that we have the following proposition relating the parts of Conjecture 2.6:

Proposition 2.8. *Let $\mathbf{P}$ be an r-differential poset in which part (i) of Conjecture 2.6 holds. Then $DU_n - rI$ has the Smith-eigenvalue relation if and only if*

$$\Delta p_n \geq \Delta p_{n-1-\delta_{r,1}}, \Delta p_{n-2-\delta_{r,1}}, \ldots$$

for all n. In particular, (i) and (ii) of Conjecture 2.6 together imply the aforementioned special case of part (iii).

Proof. We begin by making a small observation: the assumption that (i) holds in $\mathbf{P}$ provides a unimodular matrix $\tilde{M}$ so that

$$\tilde{M}U = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

To see this, we start with unimodular matrices M and N such that

$$MUN = \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

From this, we have

$$MU = \begin{pmatrix} I \\ 0 \end{pmatrix} N^{-1} = \begin{pmatrix} N^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix}.$$

As

$$\begin{pmatrix} N^{-1} & 0 \\ 0 & I \end{pmatrix}$$

is surely unimodular, the assertion follows.

Now that $U^t = D$ and $DU_n + UD_n = rI$, the above observation shows

$$\tilde{M}(DU_n - rI)\tilde{M}^t = \tilde{M}UD_n\tilde{M}^t = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where the right hand side has rank p_{n-1} by the injectivity of U_{n-1}. That is,

$$\text{coker } (DU_n - rI) \cong \mathbb{Z}^{\Delta p_n}.$$

The result now follows from the fact that

$$\dim \ker UD_n = \Delta p_n,$$

i.e. $0 \in E_1, \ldots, E_{p_n - p_{n-1}}$, and that each of the remaining p_{n-1} sets E_i has product 1 if and only if $\Delta p_n \geq \Delta p_{n-1-\delta_{r,1}}, \Delta p_{n-2-\delta_{r,1}}, \ldots$. □

[1] It is an open problem to show this holds for all differential posets; it was originally posed in in [15].

3. Constructions

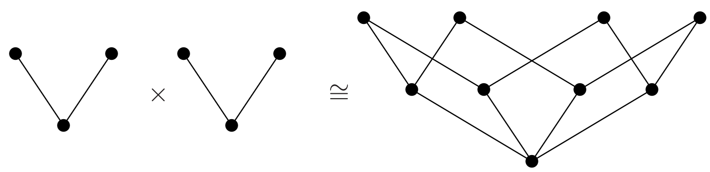
From existing differential posets, there are two natural ways one can construct new differential posets. The first way is by employing the Cartesian product; the second is known as Wagner's construction.

Definition 3.1. If $\mathbf{P}$ and $\mathbf{Q}$ are posets, we define their *Cartesian product* to be the poset $\mathbf{P} \times \mathbf{Q}$ on the set

$$\{(p, q) \ : \ p \in \mathbf{P} \text{ and } q \in \mathbf{Q}\}$$

such that $(p, q) \leq (p', q')$ if $p \leq p'$ in $\mathbf{P}$ and $q \leq q'$ in $\mathbf{Q}$.

Example 3.2. The following is a small example for the Cartesian product of two posets.



One then observes the following

Lemma 3.3. [15] *Assume that $\mathbf{P}$ and $\mathbf{Q}$ are r- and s-differential posets. Then $\mathbf{P} \times \mathbf{Q}$ is an $(r+s)$-differential poset.*

We now describe *Wagner's construction*, a method used to produce r-differential posets from *partial r-differential posets* of some finite rank (see [15, §6]).

Let $\mathbf{P}$ be a finite graded poset of rank n, with $\hat{0}$. Furthermore, assume that

$$DU - UD = rI$$

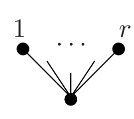
as operators on $P_0, P_1, \ldots, P_{n-1}$. We call $\mathbf{P}$ a *partial r-differential poset* of rank n. Let $\mathbf{P}^+$ be the poset of rank $n+1$ obtained from $\mathbf{P}$ in the following way: for each $v \in P_{n-1}$, add a vertex v^* of rank $n+1$ to $\mathbf{P}$ that covers exactly those $x \in P_n$ covering v. Finally, above each $x \in P_n$ we adjoin r new vertices in $\mathbf{P}^+$. We denote the resulting poset of rank $n+1$ by $E_r(\mathbf{P})$. Iterating this construction produces an r-differential poset:

Proposition 3.4. *Let $\mathbf{P}$ be a partial r-differential poset of rank n. Let*

$$W(\mathbf{P}) = \lim_{\ell \to \infty} E_r^{\ell}(\mathbf{P}).$$

Then $W(\mathbf{P})$ is r-differential. Moreover, $W(\mathbf{P})_{[0,n]} = \mathbf{P}$.

An important example of the construction is $Z(r)$, obtained by applying the Wagner construction to



As an example, see Figure 3 for $Z(2)$.

3.1. **Properties of constructions.**

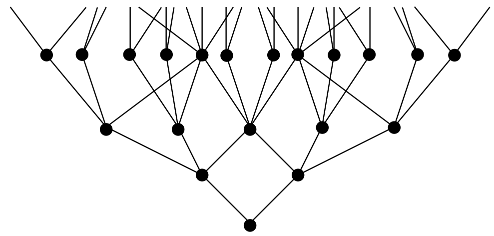


FIGURE 3. $Z(2)$.

3.1.1. *Conjecture 2.6(i).*

Proposition 3.5. *Let P and Q be differential posets in which Conjecture 2.6(i) holds. Then Conjecture 2.6(i) holds in $P \times Q$.*

Proof. We begin by noting that

$$U_{n-1}^{(P\times Q)} : \mathbb{Z}(P\times Q)_{n-1} \to \mathbb{Z}(P\times Q)_n$$

is given by

$$\begin{pmatrix} I_{q_0}\otimes U_{n-1}^{(P)} & & & & \\ U_0^{(Q)}\otimes I_{p_{n-1}} & I_{q_1}\otimes U_{n-2}^{(P)} & & & \\ & U_1^{(Q)}\otimes I_{p_{n-1}} & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_{q_{n-1}}\otimes U_0^{(P)} \\ & & & & U_{n-1}^{(Q)}\otimes I_{p_0} \end{pmatrix},$$

for a proper indexing. By our assumption on $U^{(P)}$ and $U^{(Q)}$, one can perform elementary row and column operations to arrive at a lower triangular matrix with factors

$$I_{q_i}\otimes \begin{pmatrix} I_{p_{n-1-i}} \\ 0 \end{pmatrix},$$

$0 \le i \le n-1$, along the diagonal. The assertion follows. $\square$

The analogous proposition for Wagner's construction is clear by construction:

Proposition 3.6. *Let $\tilde{\mathbf{P}}$ be a partial differential poset of rank n in which* coker U_i *is free for all $0 \le i \le n-1$. Then Conjecture 2.6(i) holds in the differential poset $\mathbf{P}$ obtained from $\tilde{\mathbf{P}}$ by Wagner's construction.*

Corollary 3.7. *Let $\mathbf{P}$ be a differential poset constructed using Wagner's construction and Cartesian products on differential posets and partial differential posets in which* coker U_i *is free for all applicable i. Then Conjecture 2.6(i) holds in $\mathbf{P}$.*

3.1.2. *Conjecture 2.6(ii).*

Proposition 3.8. *If $\mathbf{P}$ and $\mathbf{Q}$ are arbitrary differential posets, Conjecture 2.6(ii) holds in $\mathbf{P}\times\mathbf{Q}$. That is, Conjecture 2.6(ii) holds in all decomposable differential posets.*

Proof. Let $P(t) := \sum p_i t^i$, $Q(t) := \sum q_i t^i$, and $(P \times Q)(t) := \sum r_i t^i = P(t)Q(t)$. Then

$$(1-t)P(t),\ (1-t)Q(t) \in \mathbb{N}[[t]]$$

because U is injective. Thus,

$$(1-t)^2(P \times Q)(t) = (1-t)^2 P(t)Q(t) \in \mathbb{N}[[t]].$$

That is, $\Delta^2 r_n \geq 0$. □

Proposition 3.9. *Let $\overline{\mathbf{T}}$ be a partial r-differential poset of rank n in which Conjecture 2.6(ii) holds for available ranks, and let $\mathbf{T}$ be the differential poset obtained via Wagner's construction. If $r > 1$, Conjecture 2.6(ii) holds in $\mathbf{T}$. If $r = 1$, Conjecture 2.6(ii) holds in $\mathbf{T}$ if and only if $p_n \leq 2p_{n-1}$.*

Proof. First note

$$p_{n+1} = p_{n-1} + rp_n.$$

Suppose now $r > 1$. In this case we have

$$\Delta p_{n+1} \geq p_n \geq \Delta p_n.$$

For $r = 1$ we have

$$\Delta p_{n+1} = p_{n-1} \geq \Delta p_n \Leftrightarrow 2p_{n-1} \geq p_n.$$

The assertion now follows from the fact that

$$p_{n+1} = p_{n-1} + p_n \leq 2p_n$$

by the injectivity of U. □

3.1.3. *Conjecture 2.6(iii).* We begin with a special case of Conjecture 2.6(iii) that follows from the previous two sections.

Proposition 3.10. *Let $\mathbf{P}$ be an r-differential poset constructed using Wagner's construction and Cartesian products on differential posets and partial differential posets in which* coker U_i *is free for all applicable i. If $r > 1$, then $DU - rI$ possesses the Smith-eigenvalue relation. If $r = 1$, then $DU - rI$ possesses the Smith-eigenvalue relation if and only if*

$$\Delta p_j \geq \Delta p_{j-2}, \Delta p_{j-3}, \ldots$$

holds for all applicable j in the poset used to obtain $\mathbf{P}$.

Proof. The $r > 1$ case follows immediately from Corollary 3.7 and Propositions 3.8, 3.9, and 2.8.

Suppose $r = 1$. By Corollary 3.7, we know that coker U is free in $\mathbf{P}$. Thus, by Proposition 2.8 we have that $DU - rI$ possesses the Smith-eigenvalue relation if and only if

$$\Delta p_j \geq \Delta p_{j-2}, \Delta p_{j-3}, \ldots$$

in $\mathbf{P}$ for all j.

Because $r = 1$, it is clear that $\mathbf{P}$ was obtained from a single poset $\mathbf{P}'$. Moreover, it is clear that either $\mathbf{P} = \mathbf{P}'$ or $\mathbf{P} = W(\mathbf{P}')$. If $\mathbf{P} = \mathbf{P}'$, then we are done by our hypothesis. Assume that $\mathbf{P} = W(\mathbf{P}')$, where $\mathbf{P}'$ is a partial differential poset of rank $n-1$. We need to show that the above inequalities hold in $\mathbf{P}$ if and only if they hold in $\mathbf{P}'$. The necessity is clear, as $W(P)_{[0,n-1]} = P'$. To see the other direction, recall

$$p_{n+i} = p_{n+i-2} + rp_{n+i-1}$$

for all $i \geq 0$. Thus, with $r = 1$ we have

$$\Delta p_{n+i} = p_{n+i-2} \geq \Delta p_{n+i-2-j}$$

for all $i, j \geq 0$, by the injectivity of U. This finishes the proof. $\square$

Remark 3.11. Here we note that

$$DU^{(P\times Q)} = D^{(P)} \otimes U^{(Q)} \oplus (I \otimes D^{(Q)}U^{(Q)} + D^{(P)}U^{(Q)} \otimes I) \oplus U^{(P)} \otimes D^{(Q)}.$$

But it is not clear how to use this.

We now develop a tremendously useful lemma, though unattractively technical, for detecting the Smith-eigenvalue relation for $DU + tI$ over $\mathbb{Z}[t]$ in differential posets obtained through Wagner's construction. It will follow as a simple corollary that Conjecture 2.6(iii) holds for $\mathbf{Y}F$. For the remainder of this section, we take $r = 1$.

From the description of Wagner's construction, we know that

$$U_n = \begin{pmatrix} D_n \\ I \end{pmatrix} \quad \text{and} \quad D_{n+1} = \begin{pmatrix} U_{n-1} & I \end{pmatrix} \tag{3.1}$$

for $n \geq \ell$. This gives a useful recursive description of DU_n, for $n \geq \ell + 2$:

$$DU_n = \begin{pmatrix} DU_{n-2} + I_{p_{n-2}} & U_{n-3} & I_{p_{n-2}} \\ D_{n-2} & 2I_{p_{n-3}} & \\ I_{p_{n-2}} & & 2I_{p_{n-2}} \end{pmatrix}. \tag{3.2}$$

Next, we note a pivotal lemma.

Lemma 3.12. Let $\mathbf{P}$ be a (1-)differential poset obtained from a partial differential poset of rank ℓ. We have

$$\begin{pmatrix} aI_{p_n} & -(a-1)D_{n+1} \\ -aU_n & -bI_{p_{n+1}} \end{pmatrix} \sim \begin{pmatrix} I_{p_n} & & \\ & bI_{p_{n-1}} & -a(a-1)D_n \\ & -bU_{n-1} & -a(a+b-1)I_{p_n} \end{pmatrix}$$

for $n \geq \ell$.

Proof. We simply use row and column operations together with (3.1). We have

$$\begin{aligned}
\begin{pmatrix} aI_{p_n} & -(a-1)D_{n+1} \\ -aU_n & -bI_{p_{n+1}} \end{pmatrix} &\sim \begin{pmatrix} aI_{p_n} & -(a-1)U_{n-1} & -(a-1)I_{p_n} \\ -aD_n & -bI_{p_{n-1}} & \\ -aI_{p_n} & & -bI_{p_n} \end{pmatrix} \\
&\sim \begin{pmatrix} I_{p_n} & -(a-1)U_{n-1} & -(a-1)I_{p_n} \\ -aD_n & -bI_{f_{n-1}} & \\ -(a+b)I_{p_n} & & -bI_{p_n} \end{pmatrix} \\
&\sim \begin{pmatrix} I_{p_n} & & \\ & -a(a-1)DU_{n-1} - bI_{p_{n-1}} & -a(a-1)D_n \\ & -(a+b)(a-1)U_{n-1} & -(a+b)(a-1)I_{p_n} - bI_{p_n} \end{pmatrix} \\
&\sim \begin{pmatrix} I_{p_n} & & \\ & bI_{p_{n-1}} & -a(a-1)D_n \\ & -bU_{n-1} & -a(a+b-1)I_{p_n} \end{pmatrix}.
\end{aligned}$$

$\square$

For the next lemma, we introduce the following notation: for positive integer n, let $(t)_n = t(t-1)\cdots(t-n+1)$.

Lemma 3.13. *Let* $\mathbf{P}$ *be as in Lemma 3.12, and let* $n \geq \ell + 3$. *Then*

$$DU_n + tI \sim I_{p_{n-2}} \oplus I_{p_{n-3}} \oplus (t+1)_1 I_{p_{n-4}} \oplus (t+2)_2 I_{p_{n-5}} \oplus (t+3)_3 I_{p_{n-6}} \oplus \cdots \oplus (t+m-4)_{m-4} I_{p_{n-m+1}}$$
$$\oplus (t+m-3)_{m-3} \begin{pmatrix} I_{p_{n-m}} & & \\ & (t+m)(t+m-2)I_{p_{n-m-1}} & -(t+m-1)(t+m-2)D_{n-m} \\ & -(t+m)(t+m-2)U_{n-m-1} & -(t+m+1)(t+m-1)(t+m-2)I_{p_{n-m}} \end{pmatrix},$$

for $3 \leq m \leq n - \ell$.

Proof. We simply induct on m. Working to show the base case holds, we have

$$DU_n + tI = \begin{pmatrix} DU_{n-2} + (t+1)I_{p_{n-2}} & U_{n-3} & I_{p_{n-2}} \\ D_{n-2} & (t+2)I_{p_{n-3}} & \\ I_{p_{n-2}} & & (t+2)I_{p_{n-2}} \end{pmatrix}$$
$$\sim \begin{pmatrix} 0 & 0 & I_{p_{n-2}} \\ D_{n-2} & (t+2)I_{p_{n-3}} & \\ -(t+2)(DU_{n-2} + (t+1)I) + I & -(t+2)U_{n-3} & (t+2)I_{p_{n-2}} \end{pmatrix}$$
$$\sim \begin{pmatrix} I_{p_{n-2}} & & \\ & -(t+2)DU_{n-2} - (t+2)(t+1)I + I & -(t+2)U_{n-3} \\ & D_{n-2} & (t+2)I_{p_{n-3}} \end{pmatrix}$$
$$\sim \begin{pmatrix} I_{p_{n-2}} & & \\ & [-(t+2) - (t+2)(t+1) + 1]I_{p_{n-2}} & -(t+2)U_{n-3} \\ & -(t+1)D_{n-2} & (t+2)I_{p_{n-3}} \end{pmatrix}$$
$$\sim \begin{pmatrix} I_{p_{n-2}} & & \\ & (t+2)I_{p_{n-3}} & -(t+1)D_{n-2} \\ & -(t+2)U_{n-3} & -(t+3)(t+1)I_{p_{n-2}} \end{pmatrix}$$

Now a single application of Lemma 3.12 yields the base case:

$$DU_n + tI \sim \begin{pmatrix} I_{p_{n-2}} & & \\ & (t+2)I_{p_{n-3}} & -(t+1)D_{n-2} \\ & -(t+2)U_{n-3} & -(t+3)(t+1)I_{p_{n-2}} \end{pmatrix}$$
$$\sim I_{p_{n-2}} \oplus \begin{pmatrix} I_{p_{n-3}} & & \\ & (t+3)(t+1)I_{p_{n-4}} & -(t+2)(t+1)D_{n-3} \\ & -(t+3)(t+1)U_{n-4} & -(t+4)(t+2)(t+1)I_{p_{n-3}} \end{pmatrix}.$$

For the induction step, assume $3 \leq m \leq n - \ell - 1$. Then Lemma 3.12 says

$$\begin{pmatrix} (t+m)(t+m-2)I_{p_{n-m-1}} & -(t+m-1)(t+m-2)D_{n-m} \\ -(t+m)(t+m-2)U_{n-m-1} & -(t+m+1)(t+m-1)(t+m-2)I_{p_{n-m}} \end{pmatrix}$$
$$\sim (t+m-2) \begin{pmatrix} I_{p_{n-m-1}} & & \\ & (t+m+1)(t+m-1)I_{p_{n-m-2}} & -(t+m)(t+m-1)D_{n-m-1} \\ & -(t+m+1)(t+m-1)U_{n-m-2} & -(t+m+2)(t+m)(t+m-1)I_{p_{n-m-1}} \end{pmatrix},$$

completing the induction. $\square$

3.2. Proof for $\mathbf{Y}F$.

Theorem 3.14. *Conjecture 2.6 holds for $\mathbf{Y}F$ the Young-Fibonacci lattice.*

Proof. Parts (i) and (ii) of Conjecture 2.6 are completely clear. We need only show (iii) holds.

In particular, Lemma 3.13 says that for $n \geq 4$ we have

$$\begin{aligned} DU_n + tI \sim I_{f_{n-2}} &\oplus I_{f_{n-3}} \oplus (t+1)_1 I_{f_{n-4}} \oplus (t+2)_2 I_{f_{n-5}} \\ &\oplus (t+3)_3 I_{f_{n-6}} \oplus \cdots \oplus (t+n-4)_{n-4} I_{f_1} \\ &\oplus (t+n-3)_{n-3} \begin{pmatrix} (t+n-1) & (t+n-2) \\ -(t+n-1) & (t+n)(t+n-2) \end{pmatrix}. \end{aligned}$$

Since

$$\begin{pmatrix} (t+n-1) & (t+n-2) \\ -(t+n-1) & (t+n)(t+n-2) \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & (t+n+1)(t+n-1)(t+n-2) \end{pmatrix},$$

we have

$$\begin{aligned} DU_n + tI \sim & I_{f_{n-2}} \oplus I_{f_{n-3}} \oplus (t+1)_1 I_{f_{n-4}} \oplus (t+2)_2 I_{f_{n-5}} \oplus (t+3)_3 I_{f_{n-6}} \\ &\oplus \cdots \oplus (t+n-4)_{n-4} I_{f_1} \oplus (t+n-3)_{n-3} \oplus (t+n-1)_{n-1}(t+n+1), \end{aligned}$$

which one can check matches the conjectured Smith normal form of $DU_n + tI$ in $\mathbf{Y}F$. The verifications for $n = 1, 2, 3$ are straightforward, and we leave them to the reader. $\square$

We expect similar arguments to give the general version of Lemma 3.13, and thus give the previous result for all $Z(r)$.

4. Young's lattice $\mathbf{Y}$

In this section, we prove Conjecture 2.6(i) holds in $\mathbf{Y}$, DU has at least as many 1's in its Smith form as conjectured, and the largest invariant factor of DU is as predicted.

We will assume basic knowledge of the representation theory of $\mathfrak{S}_n$, and its language in terms of symmetric functions. To brush up on such material, see [17], [5] or [9]. Here we will also warn the reader of some notational abuses. Unfortunately, we use s's for Smith entries and Schur polynomials, and p's for ranks, partitions, and power sums. However, in most cases the meaning should be clear from the context. One additional notation we will mention is that we bracket linear maps when emphasizing we are thinking in matrix form.

It turns out that differential posets in general are associated with towers of algebras [7]; in this setting, induction and restriction play the roles of U and D, respectively. In Young's lattice, our towers are very nice:

$$\mathbb{C}\mathfrak{S}_0 \subset \mathbb{C}\mathfrak{S}_1 \subset \mathbb{C}\mathfrak{S}_2 \subset \cdots.$$

In this situation, our poset $\mathbf{Y}$ consists of the irreducible characters of symmetric groups. Instead of $\lambda \in Y$, as in the previous section, we have the irreducible character χ^λ of the Specht module S^λ indexed by λ. Moreover, our covering relation is defined by

$$\chi^\lambda \succ \chi^{\tilde{\lambda}}$$

if S^λ is a summand of $\mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} S^{\tilde{\lambda}}$. That is, our *up* and *down* operators are induction and restriction. By *the branching rule*, it is obvious that our two descriptions of $\mathbf{Y}$ agree.

To help us realize the power of looking at $\mathbf{Y}$ through this representation theoretic lens, we use symmetric functions.

Let $R^n_{\mathbb{C}}$ be the space of complex class functions on $\mathfrak{S}_n$, let

$$\Lambda^n_{\mathbb{C}} = \mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n}$$

be the space of symmetric polynomials of degree n, and

$$R_{\mathbb{C}} = \bigoplus_{n \geq 0} R^n_{\mathbb{C}} \qquad \text{and} \qquad \Lambda_{\mathbb{C}} = \bigoplus_{n \geq 0} \Lambda^n_{\mathbb{C}}$$

be corresponding graded rings. Likewise, we let R^n be the $\mathbb{Z}$-module generated by the irreducible characters of $\mathfrak{S}_n$ and

$$\Lambda^n = \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$$

be the symmetric polynomials of degree n, with corresponding graded rings defined analogously:

$$R = \bigoplus_{n \geq 0} R^n \qquad \text{and} \qquad \Lambda = \bigoplus_{n \geq 0} \Lambda^n.$$

We now recall two important symmetric functions. First, let h_k be defined to be the sum of all monomials of degree k. Then $\{h_\lambda\}_{\lambda \vdash n}$ are the *complete homogeneous symmetric functions* of degree n, where $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$. Second, let

$$p_k = \sum x_i^k$$

and set $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$. Then $\{p_\lambda\}_{\lambda \vdash n}$ are the *power sum symmetric functions* of degree n.

One can then endow R with a scalar product, which we omit, to obtain the following result.

Lemma 4.1. [9] *The* characteristic *map* $\mathrm{ch} : R_{\mathbb{C}} \to \Lambda_{\mathbb{C}}$*, defined by*

$$\mathrm{ch}(f) = \frac{1}{|\mathfrak{S}_n|} \sum_{\pi \in \mathfrak{S}_n} f(\pi) p_\pi,$$

where $p_\pi = p_\mu$ and $\mu \vdash n$ is the cycle type of π, is an isometric isomorphism of graded algebras. Moreover, it restricts to an isometric isomorphism of R onto Λ.

It is then known [15] that the up and down maps in R, given by induction and restriction, correspond to multiplication by the 1*st power sum symmetric function* p_1 and applying the linear operator $\partial/\partial p_1$, respectively, in Λ.

Proposition 4.2. *With $U : \mathbb{Z}\mathbf{Y}_n \to \mathbb{Z}\mathbf{Y}_{n+1}$,* coker U *is free for all n.*

Proof. From the above description of U, we have $U(h_\lambda) = h_\lambda p_1 = h_{(\lambda,1)}$, noting that $h_1 = p_1$. The result now follows from the fact that $\{h_\lambda\}_{\lambda \vdash n}$ form a $\mathbb{Z}$-basis of Λ^n. □

The problem becomes much more difficult for $UD + I = DU$. In what follows we prove DU has

(1) at least as many 1's in its Smith form as conjectured, and
(2) the largest invariant factor is as conjectured.

We begin by analyzing DU when put in the h-basis, observing that in this basis DU is lower triangular, with respect to a simple indexing scheme. Letting $\lambda = (\lambda, \ldots, \lambda_\ell) \vdash n$, with ℓ the length of λ, we have

$$\begin{aligned} DU(h_\lambda) &= \frac{\partial}{\partial p_1} p_1 h_\lambda \\ &= \frac{\partial}{\partial p_1} h_{(\lambda,1)} \\ &= h_\lambda + h_1 \sum_{i=1}^{\ell} h_{(\lambda_1,\ldots,\lambda_i-1,\ldots,\lambda_\ell)} \\ &= h_\lambda + \sum_{i=1}^{\ell} h_{(\lambda_1,\ldots,\lambda_i-1,\ldots,\lambda_\ell,1)}, \end{aligned} \tag{4.1}$$

where in the last step we are using $p_1 = h_1$ and $\partial/\partial p_1 h_n = h_{n-1}$.

For $\lambda_1, \lambda_2 \vdash n$, we define $\lambda_1 <_h \lambda_2$ if

$$\#\{\text{1-parts in } \lambda_1\} < \#\{\text{1-parts in } \lambda_2\}$$

or

$$\#\{\text{1-parts in } \lambda_1\} = \#\{\text{1-parts in } \lambda_2\} \quad \text{and} \quad \lambda_1 > \lambda_2 \text{ in lexicographic order.}$$

Order the indexing λ's of $[DU_n]_h$ from left to right, top to bottom, in increasing order with $<_h$. One can then see $[DU_n]_h$ is given by

$$\begin{bmatrix} I_{p(n)-p(n-1)} & 0 \\ * & [DU_{n-1}] + I \end{bmatrix}. \tag{4.2}$$

Example 4.3. Consider the map $DU_5 : \mathbb{Z}\mathbf{Y}_5 \to \mathbb{Z}\mathbf{Y}_5$. We have $[DU_5]_s$ and $[DU_5]_h$ are given by

$$\begin{array}{c|c} & \begin{array}{ccccccc} 5 & 41 & 32 & 31^2 & 2^21 & 21^3 & 1^5 \end{array} \\ \begin{array}{c} 5 \\ 41 \\ 32 \\ 31^2 \\ 2^21 \\ 21^3 \\ 1^5 \end{array} & \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c|c} & \begin{array}{ccccccc} 5 & 32 & 41 & 2^21 & 31^2 & 21^3 & 1^5 \end{array} \\ \begin{array}{c} 5 \\ 32 \\ 41 \\ 2^21 \\ 31^2 \\ 21^3 \\ 1^5 \end{array} & \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 1 & 0 & 2 & & & & \\ 0 & 1 & 0 & 2 & & & \\ 0 & 1 & 1 & 0 & 3 & & \\ 0 & 0 & 0 & 2 & 1 & 4 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{pmatrix} \end{array},$$

respectively.

The assertion on the number of 1's now follows:

Proposition 4.4. *There are at least $p_n - p_{n-1} + p_{n-2}$ invariant factors of DU : $\mathbb{Z}\mathbf{Y}_n \to \mathbb{Z}\mathbf{Y}_n$ equal to* 1.

Proof. From above, it suffices to show there are at least p_{n-2} invariant factors of $DU_{n-1} + I$ equal to 1.

Let $[DU_{n-1}+I]_s$ be in the usual basis, indexed by the partitions from left to right, top to bottom, in decreasing lexicographical order. Consider now the submatrix of $[DU_{n-1} + I]_s$ whose columns are indexed by

$$\{\lambda \vdash n-1 \mid \lambda \text{ has a part of size } 1\},$$

and whose rows are indexed by the conjugate set

$$\{\lambda \vdash n-1 \mid \lambda \text{ has exactly 1 largest part}\}.$$

One can then see this submatrix is lower triangular, with 1's down its diagonal. $\square$

Before going on to show the largest invariant factor is as predicted, we recall the following result.

Theorem 4.5. [15] *The eigenvectors for $DU : \mathbb{C}Y_n \to \mathbb{C}Y_n$ are given by*

$$\left\{ X_\mu = \sum_{\lambda \vdash n} \chi^\lambda(\mu)\lambda \right\}_{\mu \vdash n},$$

with X_μ belonging to the eigenvalue $\#\{$parts of μ equal to $1\} + 1$.

Theorem 4.6. *The largest Smith entry of $DU : \mathbb{Z}\mathbf{Y}_n \to \mathbb{Z}\mathbf{Y}_n$ is $(n-1)!(n+1)$.*

Proof. We wish to apply Proposition 1.5, so we start by using Theorem 4.5 to compute the entries of $[DU]^{-1}$. Let $\lambda_1, \ldots, \lambda_{p(n)}$ be the partitions of n, and define

$$d(\mu) = 1 + \#\{\text{1-parts of } \mu\}.$$

Letting M be the matrix $(\chi^{\lambda_i}(\lambda_j))$, Theorem 4.5 says

$$(M^{-1}[DU_n]_s M)_{ij} = \begin{cases} 0 & \text{if } i \neq j; \\ d(\lambda_i) & \text{if } i = j. \end{cases}$$

Since

$$s_\lambda = \sum_\mu \frac{1}{z(\mu)} \chi^\lambda(\mu) p_\mu,$$

we know

$$M^{-1} = \left(\frac{\chi^{\lambda_j}(\lambda_i)}{z(\lambda_i)} \right).$$

Abusing notation, we let D denote the diagonal matrix of $[DU_n]$. Multiplying the above out, we have

$$\begin{aligned}
([DU_n]_s^{-1})_{ij} &= MD^{-1}M^{-1} \\
&= \sum_{i_1} \chi^{\lambda_i}(\lambda_{i_1}) \frac{1}{z(\lambda_{i_1})} \chi^{\lambda_j}(\lambda_{i_1}) d_{i_1 i_1}^{-1} \\
&= \sum_\mu \frac{1}{z(\mu) d(\mu)} \chi^{\lambda_i}(\mu) \chi^{\lambda_j}(\mu) \\
&= \frac{1}{|\mathfrak{S}_n|} \sum_{\pi \in \mathfrak{S}_n} \chi^{\lambda_i}(\pi) \chi^{\lambda_j}(\pi) \frac{1}{d(\pi)} \\
&= \left\langle \chi^{\lambda_i} \chi^{\lambda_j}, \frac{1}{d(\cdot)} \right\rangle_{\mathfrak{S}_n},
\end{aligned}$$

where $d(\pi) = d(\text{cycle type of } \pi)$.

Let f be the class function on $\mathfrak{S}_n$ defined by

$$f(\pi) = \frac{(n-1)!(n+1)}{d(\pi)}.$$

By Proposition 1.5, for an upper bound on Smith entry $s_{p(n)}$ we want to show

$$\langle \chi^\lambda \chi^{\tilde{\lambda}}, f \rangle_{\mathfrak{S}_n}$$

is an integer for all $\lambda, \tilde{\lambda} \vdash n$. Thus, we want to show that f is a virtual character of $\mathfrak{S}_n$, i.e. $\mathrm{ch}(f)$ is an element of Λ.

We have

$$\begin{aligned}
\mathrm{ch}(f) &= \frac{(n-1)!(n+1)}{n!} \sum_{\pi \in \mathfrak{S}_n} \frac{p_\pi}{d(\pi)} \\
&= \frac{n+1}{n} \sum_{\pi \in \mathfrak{S}_n} \left(\frac{1}{p_1} \int p_\pi \; dp_1 \right) \\
&= \frac{(n-1)!(n+1)}{p_1} \int h_n \; dp_1.
\end{aligned}$$

Letting $H(t) = \sum_{r \geq 0} h_r t^r$, it is straightforward to see

$$\sum_{r \geq 1} p_r \frac{t^r}{r} = \log H(t).$$

Thus,

$$\begin{aligned}
\int H(t) \, dp_q &= \int e^{\sum_{r \geq 1} p_r \frac{t^r}{r}} \, dp_1 \\
&= \frac{H(t)}{t} + C.
\end{aligned}$$

Plugging in 0 for p_1 to determine the constant of integration C, we find

$$C = -\frac{H(t)}{te^{p_1 t}}.$$

This gives, since $h_1 = p_1$,

$$\begin{aligned}
\sum_{n \geq 0} \left(\int h_n \, dp_1 \right) t^n &= \frac{H(t)}{t}(1 - e^{-h_1 t}) \\
&= (1 + h_1 t + h_2 t^2 + \ldots)(h_1 - \frac{h_1^2}{2!} t + \frac{h_1^3}{3!} t^2 - \ldots),
\end{aligned}$$

so we have

$$\begin{aligned}
\int h_n \, dp_1 &= \frac{h_n h_1}{1} - \frac{h_{n-1} h_1^2}{2!} + \frac{h_{n-2} h_1^3}{3!} - \ldots \pm \frac{h_2 h_1^{n-1}}{(n-1)!} \mp \frac{h_1 h_1^n}{n!} \pm \frac{h_1^{n+1}}{(n+1)!} \\
&= \frac{h_n h_1}{1} - \frac{h_{n-1} h_1^2}{2!} + \frac{h_{n-2} h_1^3}{3!} - \ldots \pm \frac{h_2 h_1^{n-1}}{(n-1)!} \mp h_1^{n+1} \frac{n}{(n+1)!}.
\end{aligned}$$

Thus, we see

$$\begin{aligned}
\frac{(n-1)!(n+1)}{p_1} \int h_n \, dp_1 =& \frac{(n-1)!(n+1)}{1!} h_n - \frac{(n-1)!(n+1)}{2!} h_{n-1} h_1 \\
&+ \ldots \pm (n+1) h_2 h_1^{n-2} \mp h_1^n,
\end{aligned} \tag{4.3}$$

which is surely in Λ.

Having proved $(n-1)!(n+1)[DU_n]^{-1}$ is an integral matrix, if we can point out an entry that is ± 1, then $(n-1)!(n+1)$ is indeed the largest Smith invariant, as claimed.

We will use (4.3) for the image of our virtual character f under the characteristic map. Recall the formula

$$h_\mu = \sum_{\lambda \vdash n} K_{\lambda\mu} s_\lambda,$$

where $K_{\lambda\mu}$ is the Kostka number of SSYT of shape λ and content μ. This shows that when $h_k \cdot h_1^\ell$ is written as a sum of s_λ's, for $k > 1$, that $s_{(1^{k+\ell})}$ does not appear. Moreover, $s_{(1^\ell)}$ occurs only once in such an expansion of h_1^ℓ. Thus, by (4.3), we see $s_{(1^n)}$ occurs in $\mathrm{ch}(f)$ with coefficient ± 1. That is, the irreducible character of the alternating representation $\chi^{(1^n)}$ appears in our virtual character f with coefficient ± 1. Therefore, the entry of

$$(n-1)!(n+1)([DU]_s^{-1})$$

indexed by (1^n) and (n) is

$$\langle \chi^{(1^n)} \chi^{(n)}, f \rangle_{\mathfrak{S}_n} = \pm 1,$$

since $\chi^{(n)}$ is the trivial character, finishing the proof. $\square$

5. Dual graded graphs

The aim of this section is to present the notion of duality between graded graphs, and to analyze our previous work in this generalized setting.

5.1. **Duality of graded graphs.** In [3], S. Fomin developed the notion of duality between graded graphs.

Definition 5.1. A *graded graph* G is a triple (P, ρ, E), where

(i) P is a set of vertices,
(ii) E is a multiset of edges $(x, y) \in P^2$,
(iii) ρ is a rank function such that if $(x, y) \in E$ then $\rho(y) = \rho(x) + 1$.

As in differential posets, we can define *up* and *down* operators.

Definition 5.2. Let $G = (P, \rho, E)$ be a graded graph. We define linear operators $U, D : \mathbb{Z}P \to \mathbb{Z}P$ by

$$Ux = \sum_{(x,y)\in E} m(x, y)y$$

and

$$Dy = \sum_{(x,y)\in E} m(x, y)x,$$

where $m(x, y)$ is the multiplicity of the edge (x, y) in E.

Definition 5.3 (Fomin)**.** Let $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ be two graded graphs with a common set of vertices and a common rank function. The *oriented graded graph* $G = (G_1, G_2) = (P, \rho, E_1, E_2)$ is then the directed graded graph on P, with edges in E_1 directed upwards and edges in E_2 directed downwards. Naturally, the up and down operators in G are defined by

$$Ux = \sum_{(x,y)\in E_1} m_1(x, y)y$$

and

$$Dy = \sum_{(x,y)\in E_2} m_2(x, y)x,$$

where $m_i(x, y)$ denotes the multiplicity of (x, y) in E_i.

Definition 5.4. Let (G_1, G_2) be an oriented graded graph such that

(i) it has a zero $\hat{0}$, and
(ii) each rank has a finite number of elements.

Let r be a positive integer. Then G_1 and G_2 are said to be *r-dual* if

$$DU - UD = rI,$$

as operators in $G = (G_1, G_2)$.

If G_1 and G_2 are r-dual graded graphs, we call the pair (G_1, G_2) an *r-dual graded graph*. Moreover, we will freely pass from one to the other, via their clear identification with each other. As we will not consider non-graded creatures, we will often omit "graded."

5.2. **Examples.**

5.2.1. *Shifted shapes.* The graph of *shifted shapes* $S\mathbf{Y} = (G_1, G_2)$ is an important example in the land of dual graded graphs. Let Sh_n denote the set of shifted Young diagrams of n having strictly decreasing row lengths. The set Sh_n indexes those vertices of P having rank n. There are no multiple edges in G_1, where elements x and y are adjacent if $|y| = |x| + 1$ and $x \subset y$ as diagrams. On the other hand, one obtains G_2 from G_1 by changing an edge to a double edge if x is obtained from y by removing a non-diagonal element; see Figure 4.

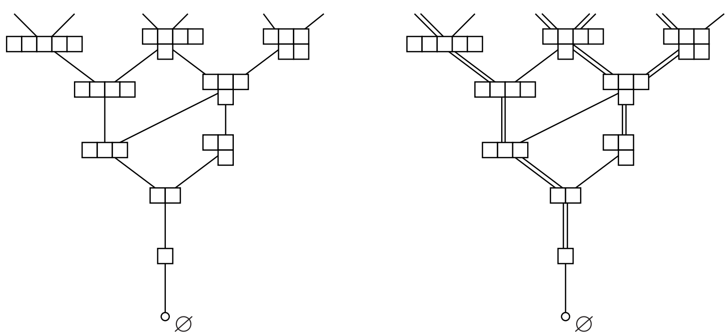


FIGURE 4. The graph of shifted shapes $S\mathbf{Y}$.

5.2.2. *The lifted binary tree and Binword.* [2] Here we define a dual graded graph whose components are *the lifted binary tree* and *Binword*, each being multiplicity free. Let W_ℓ be the set of all words of length ℓ in alphabet $\{0, 1\}$ that begin with 1. The set of vertices of rank n in this dual pair is defined to be W_n. In the lifted binary tree, y covers x if it is obtained by adjoining a single 0 or 1 to the end of x. On the other hand, x is covered by y in Binword if it is obtained from y by removing a single letter; see Figure 5. One can then check that the lifted binary tree and Binword are dual [3].

[2] These are the graphs associated to the dual Hopf algebras **NSym** (noncommutative symmetric functions) and **QSym** (quasi-symmetric functions) by multiplication of ribbon Schur functions and of quasi-ribbon functions, respectively.

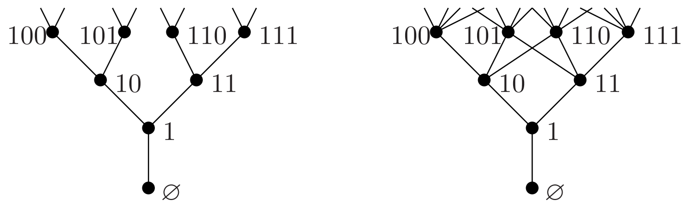


FIGURE 5. The lifted binary tree and Binword.

5.2.3. *The lattice of binary trees and the bracket tree.* [3] The *lattice of binary trees* and the *bracket tree* form another important example of duality between graded graphs. The vertices of rank n are given by the binary trees with n nodes. The covering relations are:

(1) In the lattice of binary trees, a tree T covers exactly those trees obtained from it by removing a single leaf.
(2) In the bracket tree, a tree T covers *the* tree obtained by deleting and contracting the edge, if any, below the leftmost node; see Figure 6.

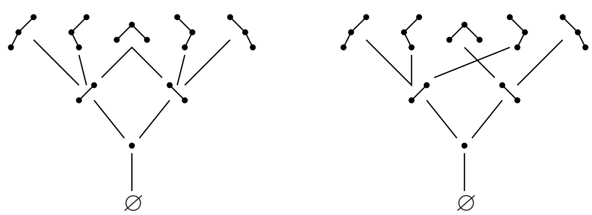


FIGURE 6. The lattice of binary trees and the bracket tree.

5.3. **Eigenvalues.** As was the case in differential posets, the eigenvalues of DU and UD are simple to write down. In fact, Proposition 1.1 holds for r-dual graphs [3]:

Proposition 5.5. *Assume that G_1 and G_2 are r-dual. Then*

$$\mathrm{Ch}(DU_n, \lambda) = \prod_{i=0}^{n} (\lambda - r(i+1))^{\Delta p_{n-i}}$$

and

$$\mathrm{Ch}(UD_n, \lambda) = \prod_{i=0}^{n} (\lambda - ri)^{\Delta p_{n-i}}.$$

It should be remarked that this is only a special case in [3], where one can find a generalized version that depends on a function relating DU and UD.

One should note that Proposition 5.5 gives the eigenvalues of DU_n and UD_n in (G_1, G_2) and in (G_2, G_1), as a matrix has the same characteristic polynomial as its transpose.

[3] This pair is associated to the Hopf algebra of planar binary trees, studied by Loday and Ronco in [8], and its dual.

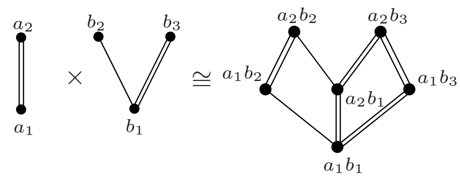
5.4. **Constructions for dual graded graphs.** This section generalizes Section 3.

Definition 5.6. If $G = (P_G, \rho_G, E_G)$ and $H = (P_H, \rho_H, E_H)$ are graded graphs, we define their *Cartesian product* to be the graded graph $G \times H$ given by the triple $(P_G \times P_H, \rho, E)$, where

$$\rho(g, h) = \rho_G(g) + \rho_H(h)$$

and edges $((g, h),(g', h))$ and $((g, h),(g, h'))$ occur in E with multiplicities $m(g, g')$ and $m(h, h')$, respectively.

Example 5.7. The following is a small example for the Cartesian product of two graded graphs.



One then observes the following

Lemma 5.8. [3] *Assume that G_1 and G_2 are r-dual, and that H_1 and H_2 are s-dual. Then $G_1 \times H_1$ and $G_2 \times H_2$ are $(r + s)$-dual.*

As we saw in Section 3, Wagner's construction is a second method used to produce r-differential posets from partial r-differential posets of some finite rank. In what follows, we present a generalization of this construction for dual graded graphs. As far as we know, the generalization of Wagner's construction is new, and it looks to be useful in current explorations (e.g. [13]).

Let $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ be two finite graded graphs of rank n with $\hat{0}$'s, sharing a common vertex set and rank function. Furthermore, assume that

$$DU - UD = rI$$

as operators on $P_0, P_1, \ldots, P_{n-1}$. We call the pair a *partial r-dual graph* of rank n. Let G_1^+ be the graph of rank $n+1$ obtained from G_2 in the following way: for each $v \in P_{n-1}$, add a vertex v^* of rank $n+1$ to G_1 that is $m_2(v, x)$-adjacent to each $x \in P_n$. We build G_2^+ in the same way, but with the roles swapped: for each $v \in P_{n-1}$, add a rank $n+1$ vertex v^* to G_2 that is $m_1(v, x)$-adjacent to each $x \in P_n$. Finally, above each $x \in P_n$ we adjoin r new vertices in both G_1^+ and G_2^+. We denote the resulting pair of rank $n+1$ graded graphs by $E_r(G_1, G_2)$. Iterating this construction produces an r-dual pair:

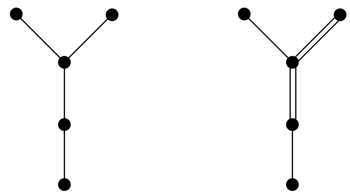
Proposition 5.9. *Let (G_1, G_2) be a partial r-dual graph of rank n. Let*

$$(W(G_1), W(G_2)) = \lim_{\ell \to \infty} E_r^{\ell}(G_1, G_2).$$

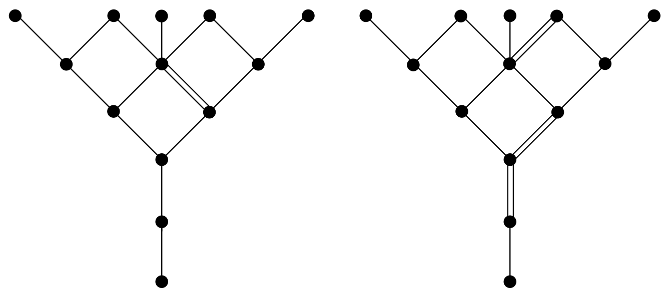
Then $W(G_1)$ and $W(G_2)$ are r-dual. Moreover, $W(G_i)_{[0,n]} = G_i$.

Indeed, one can easily see that this construction agrees with Wagner's original construction when applied to an r-differential poset.

Example 5.10. Let (G_1, G_2) be the pictured partial dual pair.



Then $E_r^2(G_1, G_2)$ is the following



5.5. **Notes on Conjecture 2.6 and dual graded graphs.** Conjecture 2.6 only deals with differential posets, but this is simply because it is not clear to us what the general setting should be. We only have an inkling that our target setting will be contained in dual graded graphs, and contain differential posets. To elaborate, when we expand to graded dual graphs, we find that the obvious generalization of Conjecture 2.6 seems to hold sometimes and fail other times. The aim of this section is to present specific examples in which Conjecture 2.6 holds, and also to present some in which the conjecture fails.

Observation 5.11. Conjecture 2.6 is not generally true for dual graphs. For example, one can easily see that the pair in Figure 7 is a counterexample for part

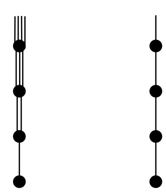


FIGURE 7. Counterexample for general U conjecture.

(i). For part (ii), in $S\mathbf{Y}$ the graph of shifted shapes (Example 5.2.1) we have $\Delta p_4 = 0$, while $\Delta p_3 = 1$. As for part (iii), consider DU_7 in $S\mathbf{Y}$.

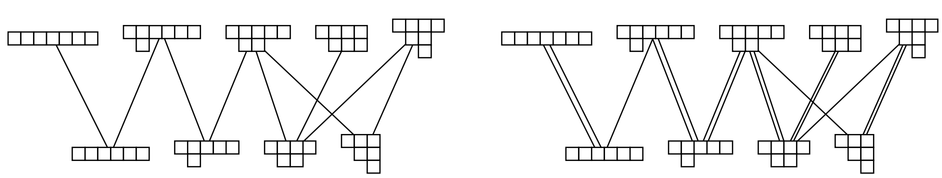


FIGURE 8. Ranks 6 and 7 of $S\mathbf{Y}$.

Computing, we have

$$DU_7 = \begin{pmatrix} 4 & 2 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \\ 1 & 2 & 5 & 2 & 0 \\ 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 2 & \\ & & & & 120 \end{pmatrix},$$

while its eigenvalues are $1, 2, 3, 5, 8$.

On the other hand, there are a number of examples in which at least one of the parts of the conjecture holds true.

Observations 5.12. We start by noting in which of our examples (Sections 5.2 and 1.1) we have coker U_n free for all n.

Proposition 5.13. *In the following graphs we have* coker U_n *is free for all* n*:*

(1) Young's lattice $\mathbf{Y}$*,*
(2) $Z(r)$ *for all* r*,*
(3) the lifted binary tree,
(4) the bracket tree,
(5) Binword,
(6) the lattice of binary trees, and
(7) G_1 *in the description of* $S\mathbf{Y} = (G_1, G_2)$ *(see 5.2.1).*

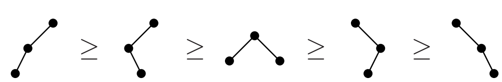
Proof. First, we have already seen that (1) and (2) are true. Furthermore, (3) and (4) are clear because they are trees. It therefore remains to show that (5)-(7) are true.

Going in order, we start with Binword. Let $w = w_1 w_2 \cdots w_n$ be a binary word of rank n in Binword. Let v be obtained from w by inserting a 0 into the second position:

$$v = w_1 0 w_2 \cdots w_n.$$

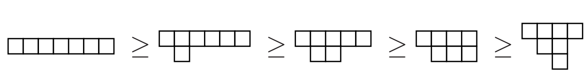
This is the smallest number covering w in Binword. Furthermore, this insertion operation injects P_n into P_{n+1} and is order preserving (where the order is simply $\leq$ on the numbers that the binary words read). The claim for Binword now follows.

For the lattice of binary trees we start by introducing a simple ordering on the ranks. We will say $T \geq T'$ if the leftmost vertex in which they differ is in T. For example, we have



Given a tree T, let T^* be obtained by adding a left leaf in the leftmost available position. It is clear that T^* is the largest tree covering T in the lattice of binary trees. Furthermore, it is also clear that this operations preserves order. The assertion for the lattice of binary trees now follows.

Lastly, we consider $S\mathbf{Y}$. Our ordering will be the natural lexicographic ordering. For example,



Now given a shifted shape T of rank n, let T' be obtained from it by removing the bottom, rightmost box. The shifted shape T' is the largest shape covered by T. Furthermore, it is clear that this operation preserves order, as defined above. The claim for G_1 follows. $\square$

We also remark that the following is clear by construction:

Proposition 5.14. *If U has free cokernel in a partial dual pair, then U has free cokernel in the dual pair obtained from the previous one by applying Wagner's construction.*

Our next example is a dual graded graph that is not self-dual, but in which $DU + tI$ possesses the Smith-eigenvalue relation over $\mathbb{Z}[t]$. In particular, it shows that Conjecture 2.6(iii) holds for graded graphs that are not differential posets. However, we first need the observation that Lemma 3.13 holds for dual graded graphs. That is, we have

Lemma 5.15. *Let (G, H) be a (1-)dual graded graph obtained from a partial dual graded graph of rank ℓ, and let $n \geq \ell + 3$. Then*

$$DU_n + tI \sim I_{p_{n-2}} \oplus I_{p_{n-3}} \oplus (t+1)_1 I_{p_{n-4}} \oplus (t+2)_2 I_{p_{n-5}} \oplus (t+3)_3 I_{p_{n-6}} \oplus \cdots \oplus (t+m-4)_{m-4} I_{p_{n-m+1}}$$
$$\oplus (t+m-3)_{m-3} \begin{pmatrix} I_{p_{n-m}} & & \\ & (t+m)(t+m-2) I_{p_{n-m-1}} & -(t+m-1)(t+m-2) D_{n-m} \\ & -(t+m)(t+m-2) U_{n-m-1} & -(t+m+1)(t+m-1)(t+m-2) I_{p_{n-m}} \end{pmatrix},$$

for $3 \leq m \leq n - \ell$.

Proposition 5.16. *Let G_1 and G_2 be the partial dual graded graphs of Example 5.10, and let (G, H) be the dual graded graph resulting from the Wagner construction applied to (G_1, G_2). Then Conjecture 2.6(iii) holds for (G, H).*

Proof. By Lemma 5.15 with $m = n - 3$, together with the fact

$$\begin{pmatrix} (t+n-3) I_{p_2} & -(t+n-4) D_3 \\ -(t+n-3) U_2 & -(t+n-2)(t+n-4) I_{p_3} \end{pmatrix} \sim \begin{pmatrix} 1 & & \\ & t+n-4 & \\ & & (t+n-4)(t+n-3)(t+n-2)(t+n+1) \end{pmatrix},$$

we have

$$DU_n + tI \sim I_{p_{n-2}} \oplus I_{p_{n-3}} \oplus \left(\bigoplus_{i=4}^{n-1} (t+i-3)_{i-3} I_{p_{n-i}} \right) \oplus (t+n-2)_{n-2}(t+n+1),$$

which one can check matches the conjectured Smith normal form of $DU_n + tI$ in (G, H) for $n \geq 6$. One can check that the Smith-eigenvalue relation holds for cases $n = 1, \ldots, 5$. $\square$

6. Remarks and questions

1. Given a self-dual graded graph we may construct a natural sequence

$$\mathscr{A}_0 \subseteq \mathscr{A}_1 \subseteq \cdots$$

of semisimple algebras whose relationship to the graph is analogous to the relationship between the tower

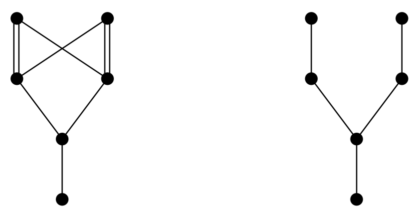
$$\mathbb{C}\mathfrak{S}_0 \subseteq \mathbb{C}\mathfrak{S}_1 \subseteq \cdots$$

and $\mathbf{Y}$. However, these algebras do not have a natural combinatorial description. In [14], S. Okada gave a natural combinatorial description of the algebras associated to $Z(r)$, leading to a natural analogue of Λ, and in particular an analogue of the h-basis. This together with (i) of Conjecture 2.6 leads one to ask the question of when can we expect an analogue of the h-basis to exist in a self-dual graded graph. This is related to Nzeutchap's conjecture in [13], that to each pair of dual graded graphs there is a pair of dual Hopf algebras, and vice versa.

2. Another question is how does Conjecture 2.6 (i) generalize for dual graded graphs? We have noticed that in a large number of examples one has

$$\left(\prod_{\substack{(x,y)\in E_1\\ \rho(x)=n}} m_1(x,y)\right)^k \mathrm{Tor}(\mathrm{coker}\ U_n) = 0, \tag{6.1}$$

for some exponent k. However, though (6.1) holds for many dual graded graphs, not all satisfy the equation: consider the following partial dual graded graph



Here,

$$\mathrm{coker}\ U_2 \cong \mathbb{Z}/3\mathbb{Z}.$$

Thus, (6.1) does not hold in the dual graded graph obtained from the above partial pair by Wagner's construction.

3. With Wagner's construction, Cartesian products, and transposition, can one find a simple set of dual graphs that generates all dual graded graphs?

4. Given a dual graded graph (G_1, G_2), is it true that coker U is always free in either (G_1, G_2) or (G_2, G_1)?

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