# CYCLIC AND DIHEDRAL SIEVING PHENOMENON FOR PLANE PARTITIONS

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Abstract. We conjecture that the action of jeu de taquin demotion on plane partitions exhibits the cyclic sieving phenomenon. In addition, we find that jeu de taquin demotion and complementation together generate a dihedral group, which seems to possess some other sieving phenomenon.

## 1. Introduction

In Reiner-Stanton-White's paper [3], the cyclic sieving phenomenon is defined, generalizing Stembridge's $q = -1$ phenomenon [7]. Briefly, a triple $(X, X(q), C)$ consisting of a set $X$, a polynomial $X(q) \in \mathbb{Z}[q]$, and a cyclic group $C$ permuting $X$ is said to exhibit the cyclic sieving phenomenon if for every $c \in C$, $|X^c| = |X(q)|_{q=\omega}$ where $\omega \in C$ is a root of unity of the same multiplicative order as $c$. It was proved [3, Theorem 1.1] that $q$-binomial coefficients, together with a suitable cyclic action on $k$-subsets or $k$-multisubsets of $[N] := \{1, 2, \ldots, N\}$, exhibit the cyclic sieving phenomenon.

It is well-known that $q$-binomial coefficients are generating functions of integer partitions fitting inside a rectangular box. More precisely,

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q = \sum_{\lambda \in S} q^{|\lambda|},$$

where $S$ is the set of all integer partitions with at most $k$ parts and with part size at most $n$, and $|\lambda|$ stands for the size of the partition $\lambda$.(See [1])

Considering its higher dimensional counterpart, we have the following conjecture.

**Conjecture 1.1.** *Let $a, b$ and $c$ be positive integers. Suppose $X$ denotes the set of all plane partitions fitting inside an $a \times b \times c$ box, and $C$ denotes the cyclic group action of order $N := b + c$ acting on $X$ generated by jeu de taquin demotion (explained in Section 2. Let*

$$X(q) := \sum_{\pi \in B(a,b,c)} q^{|\pi|} = \prod_{i=1}^{c} \frac{(q^{i+b})_a}{(q^i)_a} = \textit{MacMahon Box Formula},$$

*where $(x)_n := \prod_{j=0}^{n-1}(1 - q^j x)$ and $B(a, b, c)$ is the set of plane partitions inside an $a \times b \times c$ box.*

*Then, the triple $(X, X(q), C)$ exhibits the cyclic sieving phenomenon.*

In section 2 of this paper, we will briefly describe some basic definitions and relevant facts. In section 4, we will prove the cases of Conjecture 1.1 in which $a = 1$ or $b = 1$ or $c = 1$. In section 6, we give some possible directions for the cases where $b + c$ is prime, or where $a = 2$ and $b$ is even.

Let $j$ be jeu de taquin demotion, and let $c$ be the usual complementation inside an $a \times b \times c$ box. It turns out (see Proposition 2.12 below) that $cjc = j^{-1}$.

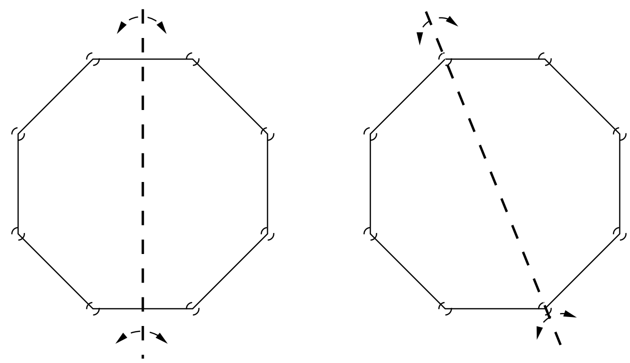
Hence, the group generated by $j$ and $c$ acting on the set of plane partitions inside a given box is a dihedral group of order $2N$, with $c$ acting as an involution and $j$ of order $N$. From representation theory, one understands this dihedral action completely if one knows the number of elements fixed by a representative of each conjugacy class. Conjecture 1.1 gives this for the powers of $j$. On the other hand, we know the number of fixed points of $c$ is $X(-1)$ from Stembridge's work on the

This work was carried out during an REU in summer 2005 at the University of Minnesota, supervised by V. Reiner and D. Stanton, and supported by their NSF grants, DMS-0245379 and DMS-0503660, respectively.

$q = -1$ phenomenon [7]. So, to understand this dihedral group, we are left to look at the fixed points of $jc$.

It is known (see Section 3 below) that the MacMahon box formula can be written as a specialization of a Schur function $s_\lambda(x_1, x_2, \ldots, x_N)$ through the hook-content formula. Hence, instead of substituting a value for $q$ into the MacMahon box formula, we can evaluate its corresponding Schur function at a suitable set of eigenvalues.

We now explain our motivation for the "suitable set" of eigenvalues. We can visualize the dihedral group acting as rotations and reflections on an $N$-gon. When $N$ is odd, we have only one conjugacy class of reflection containing both $c$ and $jc$. When $N$ is even, $c$ and $jc$ represent two conjugacy classes of reflection.



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Considering a reflection of the $N$-gon as an action on the $N$ vertices, we can find its eigenvalues by looking at its matrix in the permutation representation. When $N$ is odd, we have one more '+1's than '-1's. When $N$ is even, we have equal number of '+1's and '-1's for the reflection without fixed vertices, and we have 2 more '+1's than '-1's for the reflection with two fixed vertices. This motivates the following theorem.

**Theorem 1.2.** *Let $a, b$, and $c$ be positive integers where at least one of $b$ and $c$ is even. Let $\lambda$ be the integer partition with rectangular Ferrers diagram of $b$ rows and $a$ columns. Let $\mathcal{S}$ be the set of all plane partitions inside an $a \times b \times c$ box, and set $N := b + c$. Then we have*

$$\begin{aligned} |\{\pi \in \mathcal{S} : jc(\pi) = \pi\}| &= s_\lambda(+1, -1, +1, -1, \ldots, (-1)^{N-1}) \\ &= X(-1) \\ &= |\{\pi \in \mathcal{S} : c(\pi) = \pi\}| \,. \end{aligned}$$

Note that the right hand side of the equality simply means putting in the same number of '+1's and '-1's when $b + c$ is even and $b$ is even; and putting in one more '+1's than '-1's when $b + c$ is odd. In either case, it is the same as evaluating the MacMahon box formula at $q = -1$, which gives you the number of self-complementary plane partitions by Stembridge [7]. In section 7, we will prove Theorem 1.2 by explicitly giving a bijection between the set of fixed points of $jc$ and the set of fixed points of $c$.

For the remaining case, where $b$ and $c$ are both odd, it seems that the reflections $jc$ and $c$ can have different numbers of fixed points.

**Conjecture 1.3.** *Let $a, b$, and $c$ be positive integers with $b$ and $c$ both odd. Let $\lambda$ be the integer partition with rectangular Ferrers diagram of $b$ rows and $a$ columns. Let $\mathcal{S}$ be the set of all plane partitions inside an $a \times b \times c$ box, and set $N := b + c$. Then we have*

$$|\{\pi \in \mathcal{S} : jc(\pi) = \pi\}| = s_\lambda(+1, -1, +1, -1, \ldots, (-1)^{N-3}, (-1)^{N-2}, +1).$$

It turns out that there is a helpful formula that evaluates the Schur function in this conjecture effectively; see Lemma 8.2 below.

Thus, whenever we have closed formulae for the number of fixed points of $jc$, Conjecture 1.3 can be proved easily by comparing formulae. In section 8, we will prove the cases when $a = 1$ or $a = 2$ in Conjecture 1.3.

Note that Theorem 1.2 and Conjecture 1.3 reveal that we can find the number of fixed points of $jc$ by a similar method as in the cyclic sieving phenomenon. Therefore, there seems to be another sieving phenomenon when a dihedral group acts on a set.

We provide some background materials of the jeu de taquin action in section 5.

## 2. Definitions and Facts

First, we start with some basic terminologies on partitions.(See [5])

An *integer partition* $\lambda$ is a weakly decreasing infinite sequence of nonnegative integers $(\lambda_1, \lambda_2, \ldots)$ with finitely many non-zero entries. We say that $\lambda$ is a partition of $n$, denoted by $\lambda \vdash n$, if $\sum \lambda_i = n$.

A *plane partition* $\pi$ is a rectangular array of nonnegative integers $(\pi_{ij})_{i,j\geq 1}$ with finitely many nonzero entries, such that $\pi_{ij} \geq \max(\pi_{i,j+1}, \pi_{i+1,j})$, in other words, $\pi$ is weakly decreasing along rows and columns.

A *restricted plane partition* $\pi$ is a rectangular array of nonnegative integers with $a$ rows and $b$ columns such that it is weakly decreasing along rows and columns, and with $\pi_{ij} \leq c$ for all $1 \leq i \leq a$ and $1 \leq j \leq b$, where $a, b$, and $c$ are positive integers.We also say that the plane partition $\pi$ *fits inside an* $a \times b \times c$ *box.* An example of a plane partition inside an $4 \times 5 \times 7$ box is

**Example 2.1.**

$$\begin{matrix} 7 & 7 & 7 & 5 & 2 \\ 7 & 6 & 6 & 3 & 1 \\ 6 & 2 & 2 & 1 & 0 \\ 5 & 1 & 0 & 0 & 0 \end{matrix}.$$

From now on, we mean "restricted plane partition inside an $a \times b \times c$ box" whenever we use the word "plane partition".

Alternatively, we may identify a plane partition $\pi$ with the set of lattice points

$$\{(i, j, k) \in \mathbb{P}^3 : 1 \leq k \leq \pi_{ij}\},$$

where $\mathbb{P}$ stands for the set of positive integers. With this notation, plane partitions are just order ideals of the poset $\mathbb{P}^3$. Consider the mapping

$$(i, j, k) \mapsto (i, j, k)^c := (a + 1 - i, b + 1 - j, c + 1 - k).$$

We define the *complement* of a plane partition $\pi$ inside an $a \times b \times c$ box by setting $\pi^c := \{(i, j, k) : (i, j, k)^c \notin \pi\}$. An example is shown below:

**Example 2.2.**

$$\begin{matrix} 7 & 7 & 7 & 5 & 2 \\ 7 & 6 & 6 & 3 & 1 \\ 6 & 2 & 2 & 1 & 0 \\ 5 & 1 & 0 & 0 & 0 \end{matrix} \quad \longleftrightarrow \quad \begin{matrix} 7 & 7 & 7 & 6 & 2 \\ 7 & 6 & 5 & 5 & 1 \\ 6 & 4 & 1 & 1 & 0 \\ 5 & 2 & 0 & 0 & 0 \end{matrix}.$$

Let $\lambda$ be an integer partition. A *column strict tableau (CST) of shape* $\lambda$ is an array $T = (T_{ij})$ of positive integers of shape $\lambda$ that is weakly increasing in every row and strictly increasing in every column. An example of a CST of shape $(4, 4, 4, 4)$ is given by

**Example 2.3.**

$$\begin{matrix} 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 5 & 5 & 6 & 7 \\ 6 & 6 & 8 & 8 \end{matrix}.$$

In this paper, we are mainly interested in CSTs of rectangular shape. Rectangular CSTs are closely related to plane partitions as seen in the following proposition. From now on, we use $[N]$ to denote the set $\{1, 2, \ldots, N\}$ and $\langle a^b \rangle$ to denote the rectangular shape of $a$ columns and $b$ rows.

**Proposition 2.4.** *Let $a, b$ and $c$ be positive integers. Then there is a bijection between the set of plane partitions inside an $a \times b \times c$ box and the set of CSTs of shape $\langle a^b \rangle$ and entries in $[b + c]$.*

*Proof.* Given a plane partition $\pi$ inside an $a \times b \times c$ box, we first add 1 to the last column, add 2 to the second last column,. . ., add $b$ in the first column. Then we rotate the whole tableau by 180° and then do the transpose. It is easy to check that this transformation is a bijection between the two sets. □

Each column of a CST forms a subset of $[N]$. Therefore, we can represent this subset with a 0,1-column of height $N$ whose $i^{\text{th}}$ entry is 1 if the subset contains $i$ and 0 otherwise.

**Definition 2.5.** Let $T$ be a CST of shape $\langle a^b \rangle$ with entries in $[N]$, and let $A_1, \ldots, A_a$ be the 0,1-columns corresponding to the columns of $T$. Define $M(T) := (A_1, \ldots, A_a)$ to be the matrix whose columns are $A_1, \ldots, A_a$. We will call $M(T)$ the tableau matrix of $T$.

**Example 2.6.** If $T$ is the tableau of Example 2.3, then $M(T)$ is

$$\begin{array}{cccc}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1.
\end{array}$$

Notice that we have grouped the first 1 in each column, the second 1 in each column, and so on. Since our tableaux are weakly increasing across the rows, these groups must move weakly downward as they are read from left to right.

For more information about plane partitions and CSTs, see [6].

There is a natural cyclic action acting on the set of CSTs called jeu de taquin demotion, which was first introduced by Schützenberger. We often abbreviate this action as $j$.

**Definition 2.7** (Jeu de taquin demotion on CSTs)**.** Given a CST, $T$,with b rows, a columns and entries in $[b + c]$, do the following steps:

Step 1: Decrement each entry in the tableau by 1.
Step 2: Turn every 0 entry into a star.
Step 3: If there is at least one star, do the following. If not, we are done.
Step 4: Starting from the right-most star, compare the entries immediately below and to the right of it, pick the smaller one and swap this entry with the star; if there is a tie, pick the one below it. Repeat this procedure until the star stacks in the lower right corner and cannot move anymore. Then, apply the same process to the second star, the third star, etc., until you exhaust all the stars.
Step 5: Turn all the stars back to $b + c$.

The resulting CST is called $j(T)$.

It is not hard to see that the process above is well-defined and the resulting tableau is actually a CST. An example is shown below with $a = b = c = 4$.

**Example 2.8.**

$$T = \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 5 & 5 & 6 & 7 \\ 6 & 6 & 8 & 8 \end{array} \longrightarrow \begin{array}{cccc} 0 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 4 & 4 & 5 & 6 \\ 5 & 5 & 7 & 7 \end{array} \longrightarrow \begin{array}{cccc} * & * & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 4 & 4 & 5 & 6 \\ 5 & 5 & 7 & 7 \end{array} \longrightarrow \begin{array}{cccc} * & 1 & * & 2 \\ 1 & 2 & 3 & 4 \\ 4 & 4 & 5 & 6 \\ 5 & 5 & 7 & 7 \end{array} \longrightarrow$$

$$\begin{array}{cccc} * & 1 & 2 & * \\ 1 & 2 & 3 & 4 \\ 4 & 4 & 5 & 6 \\ 5 & 5 & 7 & 7 \end{array} \longrightarrow \begin{array}{cccc} * & 1 & 2 & 4 \\ 1 & 2 & 3 & * \\ 4 & 4 & 5 & 6 \\ 5 & 5 & 7 & 7 \end{array} \longrightarrow \begin{array}{cccc} * & 1 & 2 & 4 \\ 1 & 2 & 3 & 6 \\ 4 & 4 & 5 & * \\ 5 & 5 & 7 & 7 \end{array} \longrightarrow \begin{array}{cccc} * & 1 & 2 & 4 \\ 1 & 2 & 3 & 6 \\ 4 & 4 & 5 & 7 \\ 5 & 5 & 7 & * \end{array} \longrightarrow$$

$$\begin{array}{cccc} 1 & 1 & 2 & 4 \\ * & 2 & 3 & 6 \\ 4 & 4 & 5 & 7 \\ 5 & 5 & 7 & * \end{array} \longrightarrow \begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & * & 3 & 6 \\ 4 & 4 & 5 & 7 \\ 5 & 5 & 7 & * \end{array} \longrightarrow \begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 3 & * & 6 \\ 4 & 4 & 5 & 7 \\ 5 & 5 & 7 & * \end{array} \longrightarrow \begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 3 & 5 & 6 \\ 4 & 4 & * & 7 \\ 5 & 5 & 7 & * \end{array} \longrightarrow$$

$$\begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 3 & 5 & 6 \\ 4 & 4 & 7 & 7 \\ 5 & 5 & * & * \end{array} \longrightarrow \begin{array}{cccc} 1 & 1 & 2 & 4 \\ 2 & 3 & 5 & 6 \\ 4 & 4 & 7 & 7 \\ 5 & 5 & 8 & 8 \end{array} = j(T)$$

Note that the procedures in jeu de taquin demotion are designed so that the rows are weakly increasing and columns are strictly increasing at ANY intermediate steps during the action.

It turns out that $j$ generates a cyclic group.

**Proposition 2.9** (D. White)**.** *For any positive integers $a, b$ and $c$, the group action acting on the set of CSTs with $b$ rows, $a$ columns and entries in $[b+c]$, generated by $j$ is a cyclic group of order $b+c$.*

The proof of Proposition 2.9 involves standardizing a CST into a standard Young tableau(SYT)and then showing that every tableau will return to itself after $ab$ steps of $j$ on its standardized tableau.

Moreover, the action of $j$ is in some sense symmetric in the parameters $b$ and $c$.

**Proposition 2.10.** *We define $B(a, b, c)$ to be the set of all plane partitions inside an $a \times b \times c$ box. Let $\sigma : B(a, b, c) \to B(a, c, b)$ be the mapping that takes a plane partition $\pi$ in $B(a, b, c)$ to a plane partition $\sigma(\pi)$ in $B(a, c, b)$ by switching the $b$ and $c$ axes.*

*Let $j_{abc} : B(a, b, c) \to B(a, b, c)$ be the jeu de taquin demotion defined in Definition 2.7. (Note that jdt is an action depending on the underlying dimensions of the plane partition.) Then, we have*

$$\sigma j_{acb} \sigma = j_{abc}^{-1}.$$

*Proof.* Using the notion of tableau matrices, the action of $\sigma$ on these tableau matrices is simply turning the whole matrix upside down and then interchanging ‘ 1’s and ‘0’s. Together with the equivalent action of $j$ on tableau matrices (see Proposition 5.3 below), the result follows easily. □

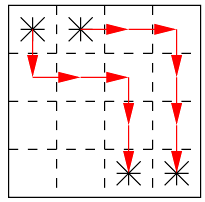
It turns out that it is quite useful to keep track of the path traced by the stars involved in jeu de taquin demotion. This motivates our definition of a path diagram.

**Definition 2.11.** The path diagram of a CST, $T$, is the directed paths traced out by the stars when we apply jeu de taquin demotion on $T$.

Consider the tableau in Example 2.8, the path diagram of the CST

$$T = \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 5 & 5 & 6 & 7 \\ 6 & 6 & 8 & 8 \end{array}$$

is given by



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We now prove the following important group relation on $c$ and $j$.

**Proposition 2.12.**

$$cjc = j^{-1}$$

*Proof of Proposition2.12.* Let $T$ be a CST. It is not hard to see that the path diagram we obtain when we apply $j$ to $c(T)$ differs only by a 180°-rotation from the path diagram we get by applying $j^{-1}$ on $T$. Once this fact is noted, the result follows easily. □

## 3. MacMahon Box Formula

Recall that the generating function for the number of plane partitions fitting inside an $a \times b \times c$ box is

$$\sum_{\pi \in B(a,b,c)} q^{|\pi|},$$

where $B(a,b,c)$ is the set of plane partitions fitting inside an $a \times b \times c$ box and $|\pi|$ stands for the size of the plane partition $\pi$. This generating function can be expressed as the *MacMahon's Box Formula*

$$mb(a,b,c;q) := \prod_{i=1}^{c} \frac{(q^{b+i})_a}{(q^i)_a},$$

where $(x)_n := \prod_{i=0}^{n-1}(1-xq^i)$.

Note that the *MacMahon Box Formula* reduces to the ordinary $q$-binomial coefficients when either $a$, $b$ or $c$ equals one.

We are going to use some notations for Schur functions. So we give a very brief description of Schur functions here. (See [4, 6])

**Definition 3.1.** For any CST, $T$, of shape $\lambda$ with entries in $[N]$, let $\mu(T) = (\mu_1, \mu_2, \ldots, \mu_N)$ where $\mu_k$ is the number of '$k$' in $T$. Then the Schur function of shape $\lambda$ is

$$s_\lambda(x_1, \ldots, x_N) = \sum_T x_1^{\mu_1} \cdots x_N^{\mu_N},$$

where the sum runs through all CSTs, $T$, of shape $\lambda$ with entries in $[N]$.

It is well-known that Schur functions are symmetric, and they have several nice determinantal formula, for example,the bialternant and Jacobi-Trudi formulae. More important here is the following famous principal specialization formula [6, Theorem 7.21.2].

**Theorem 3.2** (Hook-content formula)**.**

$$s_\lambda(1, q, q^2, \ldots, q^{N-1}) = q^{n(\lambda)} \prod_{cells\ x \in \lambda} \frac{[N + c(x)]_q}{[h(x)]_q},$$

*where*

$$\begin{aligned} [n]_q &:= 1+q+q^2+\cdots+q^{n-1}, \\ n(\lambda) &:= \min\{\text{sum of entries in } T | \text{all CSTs } T \text{of shape } \lambda \text{ ( allowing 0 as a part) }\}, \\ h(x) &:= \text{hook length of } \lambda \text{ at } x, \text{ and} \\ c(x) &:= \text{content of } \lambda \text{ at } x. \end{aligned}$$

From the hook-content formula, we can easily derive the MacMahon Box Formula. We have

$$mb(a,b,c;q) = q^{-n(\lambda)} s_\lambda(1,q,q^2,\ldots,q^{b+c-1}),$$

where $\lambda$ is of shape $\langle a^b \rangle$.

Thus, we can evaluate the MacMahon Box Formula at various roots of unity using Schur functions, which is shown in the following theorem [3, Theorem 4.3].

**Theorem 3.3.** *Let $d \mid N$, and let $q$ be a primitive $d^{th}$ root of unity. Then $s_\lambda(1,q,\ldots,q^{N-1})$ is zero unless the $d$-core of $\lambda$ is empty, in which case*

$$s_\lambda(1,q,\ldots,q^{N-1}) = \operatorname{sgn}(\chi^\lambda(d^k)) \prod_{i=0}^{d-1} s_{\lambda^{(i)}}(\underbrace{1,1,\ldots,1}_{\frac{N}{d}}),$$

*where $k = \frac{|\lambda|}{d}$, the $d$-quotient of $\lambda$ is $(\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(d-1)})$, and $\chi^\lambda$ is the irreducible character of the symmetric group $\mathfrak{S}_{kd}$ indexed by $\lambda$.*

Using Theorem 3.3, we can evaluate the MacMahon box formula at roots of unity easily.

**Proposition 3.4.** *Let $a,b,c$ be positive integers, and $d \mid b+c$. Let $q$ be a primitive $d^{th}$ root of unity.*

(i) *If $d \mid a$, then*

$$mb(a,b,c;q) = q^{-n(\lambda)} s_\lambda(1,q,\ldots,q^{b+c-1}) = \prod_{k=0}^{d-1} B\left(\frac{a}{d}, \left\lceil \frac{b-k}{d} \right\rceil, \frac{b+c}{d} - \left\lceil \frac{b-k}{d} \right\rceil\right).$$

(ii) *If $d \mid b$, then*

$$mb(a,b,c;q) = q^{-n(\lambda)} s_\lambda(1,q,\ldots,q^{b+c-1}) = \prod_{k=0}^{d-1} B\left(\left\lceil \frac{a-k}{d} \right\rceil, \frac{b}{d}, \frac{c}{d}\right).$$

(iii) *Otherwise,*

$$mb(a,b,c;q) = 0.$$

*Proof.* Note that $\lambda$ is rectangular with $b$ rows and $a$ columns. For any $d \mid b+c$, the $d$-core is empty if and only if either $d \mid a$ or $d \mid b$.

In the case $d \mid a$, the $d$-quotient of $\lambda$ is $(\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(d-1)})$, where $\lambda^{(k)}$ is rectangular with $\frac{a}{d}$ columns and $\lceil \frac{b-k}{d} \rceil$ rows. We treat each $\lambda^{(k)}$ as a tableau with entries in $[\frac{b+c}{d}]$ when evaluating the Schur function on the right hand side of the equality in Theorem 3.3. This proves the equation in (i).

For case (ii), when $d \mid b$, the $d$-quotient of $\lambda$ is $(\lambda^{(0)},\lambda^{(1)},\ldots,\lambda^{(d-1)})$, where $\lambda^{(k)}$ is rectangular with $\lceil \frac{a-k}{d} \rceil$ columns and $\frac{b}{d}$ rows. Similarly, we treat each $\lambda^{(k)}$ as a tableau with entries in $[\frac{b+c}{d}]$. □

When $a=b=c$, the formula is particularly nice.

**Proposition 3.5.** *Let $\omega$ be a primitive $d$-th root of unity.*

(i) *When $d \mid a$, we have*

$$mb(a,a,a;q)|_{q=\omega} = \prod_{j=1}^{k} \left(\frac{k+j}{j}\right)^{jd} \prod_{j=k+1}^{2k-1} \left(\frac{k+j}{j}\right)^{(2k-j)d},$$

*where $a = kd$ and $k > 1$.*

(ii) *When $d = a$, we have*
$$mb(a, a, a; q)|_{q=\omega} = 2^a.$$
(iii) *When $d = 2a$, we have*
$$mb(a, a, a; q)|_{q=\omega} = 0.$$
(iv) *When $d = \frac{2a}{t}$ where $t$ is a positive odd divisor of $a$, then*
$$mb(a, a, a; q)|_{q=\omega} = 0.$$

## 4. Proof of some cases in Conjecture 1.1

In this section, we prove some cases in Conjecture 1.1.

Up to now, we can only prove these cases:

(i) $a = 1$.
(ii) $b = 1$, and
(iii) $c = 1$.

For case (i), we have $a = 1$. That means we are working on CSTs with 1 column only. By column strictness, we can establish a 1-1 correspondence between the set of 1-column CSTs and the set of $b$-subsets of $[b + c]$, simply collecting all the entries. The action of $j$ is just cyclically permuting elements in the $b$-subset. For example, when $b = 4, c = 6$, $j(\{1, 3, 5, 6\}) = \{2, 4, 5, 10\}$. Also, notice that the MacMahon Box Formula is just
$$X(q) = \begin{bmatrix} b + c \\ b \end{bmatrix}_q.$$
So, this is exactly the case in [3, Theorem 1.1(b)].

For case (ii), we have $b = 1$. Now we are working on 1-row CSTs. Note that the set of 1-row CSTs can be put into a bijection with the set of $a$-multisubsets of $[c + 1]$. Again,$j$ cyclically permutes elements in the multisubset. The MacMahon Box Formula in this case is:
$$X(q) = \begin{bmatrix} a + c \\ a \end{bmatrix}_q,$$
which corresponds to [3, Theorem 1.1(a)].

For case (iii), using Proposition 2.10, it is easy to see that this case is exactly the same as case (ii) with $j$ replaced by $j^{-1}$.

## 5. Properties of jeu de taquin

In this section, we will discuss some important properties of jeu de taquin.

There is an equivalent formulation of jeu de taquin demotion in terms of tableau matrices.

**Definition 5.1** (Jeu de taquin demotion on tableau matrices)**.** Let $T$ be a CST of rectangular shape and let $M = M(T)$ be the corresponding tableau matrix. Perform the following steps on $M$:

Step 1: Move the top row of $M$ to the bottom.

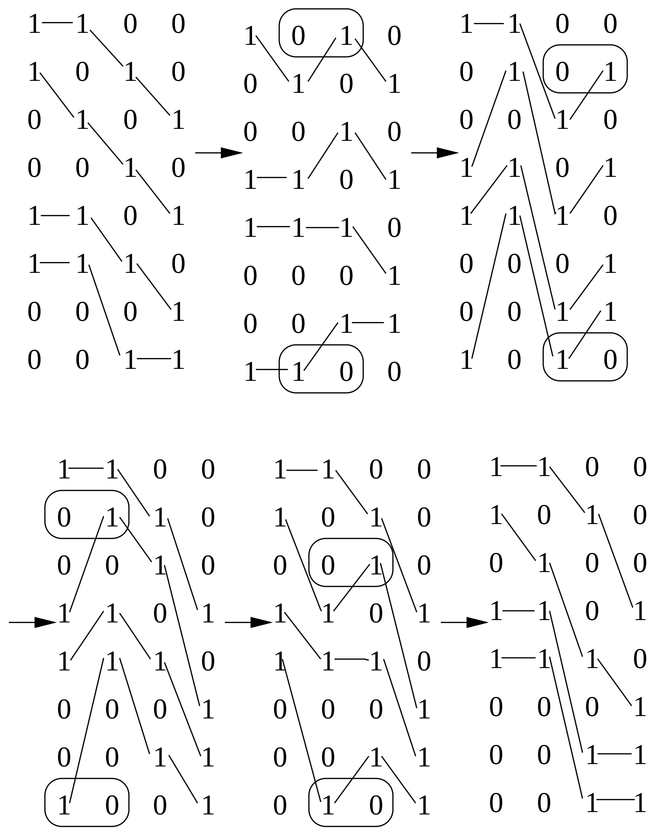
Step 2: Group the 1s by the order they appear in their columns as in Example 2.6. If none of these groupings move upward reading from left to right then this corresponds to a valid CST and we are done. Otherwise,

Step 3: Find the rightmost pair of adjacent columns which contains an upward move in some grouping of the 1s. Find the highest row and the lowest row which are involved in an upward move. The highest will contain 0 1 and the lowest will contain 1 0. Switch these to 1 0 and 0 1 respectively without changing the rest of the matrix.

Step 4: Repeat from Step 2 until finished.

The resulting matrix is called $j(M)$.

**Example 5.2.** Using the same tableau $T$ as in Example 2.6:



The last matrix corresponds to $j(T)$.

Note that in step 3 of the definition, the lower row chosen for swapping is always the bottom row of the matrix. It is easy to see that this will always be the case, since the matrix came from a valid tableau matrix by moving one row to the bottom.

Now we want to see that this is equivalent to our previous jeu de taquin demotion.

**Proposition 5.3.** *Let $T$ be a CST of rectangular shape and let $M(T)$ be its corresponding tableau matrix. Then $M(j(T)) = j(M(T))$.*

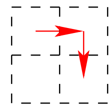
*Proof.* We want to see that swapping of 1s in step 3 of jeu de taquin demotion on tableau matrices is equivalent to moving a star down a column some number of steps and then swapping it to the right once in the corresponding tableau.

Suppose we have a pair of columns whose bottommost row is 1 0. If we let the 1 in the bottom row stand for a star somewhere in the column rather than a $N$ at the bottom of the column, then the swapping in step 3 does become moving a star to the right. If we start the star at the top of the column and have it move down by jeu de taquin, it will always move to the right at the highest possible point to give a valid tableau above the place where it moves right. This will correspond to swapping at the row chosen in step 3. $\square$

Although the paths in a path digram of a CST may intersect each other, they satisfy a certain non-crossing criterion.

**Proposition 5.4.** *During a jeu de taquin demotion on a CST, $T$, once an entry has been swapped up by a star, it stays there until the end of the whole jeu de taquin process.*

*In other words, we are not allowed to have part of a path diagram like this:*

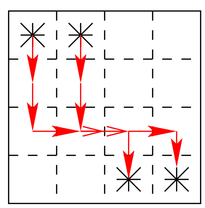
,

*where the vertical arrow belongs to a path preceding the path containing the right arrow.*

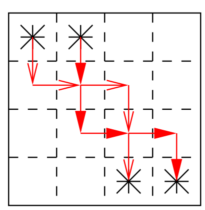
*Proof.* We call those entries that satisfy our assumption "good" entries, otherwise, we call them "bad" entries. Assume that there exists a "bad" entry. First choose the highest row that contains such an entry, then pick the left-most "bad" entry in that row. Note that every entry that is weakly above and to the left of it is "good".

Suppose this "bad" entry is swapped up by the $i^{\text{th}}$ star. For any $j > i$, the $j^{\text{th}}$ star cannot arrive right above our "bad" entry, because the star is blocked by those "good" entries that are swapped up by the $i^{\text{th}}$ path. Also, when the $j^{\text{th}}$ star arrives on the left of our "bad" entry, then it must go down (since our "bad" entry is originally on the right of the entry below). So, our "bad" entry is indeed a "good" entry. We are done. $\square$

As a direct consequence of the Proposition 5.4, the paths in a path diagram can only touch, but not cross, each other. Two paths can share the same horizontal arrows but not a vertical arrow. In other words, we may have situation like this:

,

where the double arrow represents where the two paths overlap. However, we cannot have situation like this:

.

Moreover, it follows that all the stars will end up in the last row. Otherwise, some paths are going to cross each other.

## 6. Possible Directions on Conjecture 1.1

In this section, we will give some possible directions to prove Conjecture 1.1 in the case (i) $b+c$ is prime, and (ii) $a = 2$ and $b$ is even.

When $b + c = p$ is prime, there are only two possible stabilizer orders: 1 and $p$. We can use Proposition 3.4 to evaluate the polynomial at a $p^{\text{th}}$ root of unity, giving:

$$X(q) = \begin{cases} 1 & \text{if } p \mid a, \\ 0 & \text{otherwise.} \end{cases}$$

(There is no case $p \mid b$ because $p = b + c > b$.)

Now we need to count the number of fixed points of $j$. What we would like is the following:

**Conjecture 6.1.** *Let $a$, $b$, and $c$ be positive integers, and consider CST of shape $\langle a^b \rangle$ with entries in $[b+c]$.*

(i) *If* $b+c \mid a$*, then there is exactly one fixed point of* $j$*.*
(ii) *If* $b+c \nmid a$*, then there are no fixed points of* $j$*.*

This conjecture would prove cyclic sieving whenever $b+c$ is prime. Unfortunately, we do not have a complete proof of this. However, we can make the following observations to prove some cases.

Note that a CST may be recorded as an $ab$-multisubset of $[b+c]$ by simply collecting the entries of the tableau. Note that $j$ has an induced action on these multisubsets by cyclically permuting the set $[b+c]$. So a necessary condition for a fixed point of $j$ is that the multisubset is invariant under cyclic permutation. This implies immediately that a fixed point must have the same number of 1s, 2s, $\ldots$, $(b+c)$s in the tableau, so there must be exactly $\frac{ab}{b+c}$ of each in the tableau. So $b+c \mid ab$.

Also note that the path diagram for a fixed point must have a path passing through every square, since if a square does not have a path passing through it, $j$ decrements it in place and so it is not fixed by $j$. Because of the limitations on how paths can cross (Proposition 5.4), there must be at least $\min\{a,b\}$ paths. So for a fixed point, $\frac{ab}{b+c} \geq \min\{a,b\}$.

Once the path diagram is chosen, there will be at most one way to construct a CST with that path diagram which is fixed by $j$. By considering path diagrams we have the following:

**Proposition 6.2.** *When* $b+c=a$*,* $j$ *has exactly one fixed point.*

(This proves cyclic sieving when $a=b+c$ and $a$ is prime.)

*Proof.* When $b+c=a$, $j$ will generate $\frac{ab}{b+c}=b$ paths. These paths must pass through every square of the tableau. Since a later path cannot be higher than an earlier path, the first path must move all the way to the right before moving down. Similarly, the last path must move all the way down before moving to the right. The second path must move down once, then as far to the right as possible, then down. The second to last path must move down to the second row from the bottom, then right, then down once more. By continuing this process, we see that there is only one possible path diagram that will pass through every square of the tableau, so there is only one fixed point of $j$. □

The fixed point of Proposition 6.2 all have a similar form. When written as tableau matrices, this form is

$$\begin{array}{cccccccccc}
1 & 1 & \ldots & 1 & 1 & 0 & 0 & \ldots & 0 \\
1 & 1 & \ldots & 1 & 0 & 1 & 0 & \ldots & 0 \\
1 & 1 & \ldots & 0 & 1 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & & & & & & & \vdots \\
0 & 0 & \ldots & 1 & 0 & 1 & 0 & \ldots & 1 \\
0 & 0 & \ldots & 0 & 1 & 0 & 1 & \ldots & 1 \\
0 & 0 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1
\end{array}$$

where there are $b$ 1s in each row.

**Example 6.3.** When $a=5$, $b=3$, and $c=2$, the only fixed point of $j$ is

$$\begin{array}{ccccc}
1 & 1 & 1 & 2 & 3 \\
2 & 2 & 3 & 4 & 4 \\
3 & 4 & 5 & 5 & 5
\end{array}$$

The tableau matrix of this is

$$\begin{array}{ccccc}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1
\end{array}$$

It seems that the general fixed points of $j$ have a similar form to this, but with each column of the tableau matrix replaced by $\frac{a}{b+c}$ identical columns.

**Conjecture 6.4.** *The fixed points of $j$ have tableau matrices of the form*

$$\begin{array}{ccccccccc}
\overbrace{1}^{\frac{a}{b+c}} & \overbrace{1}^{\frac{a}{b+c}} & \dots & \overbrace{1}^{\frac{a}{b+c}} & \overbrace{1}^{\frac{a}{b+c}} & \overbrace{0}^{\frac{a}{b+c}} & \overbrace{0}^{\frac{a}{b+c}} & \dots & \overbrace{0}^{\frac{a}{b+c}} \\
1 & 1 & \dots & 1 & 0 & 1 & 0 & \dots & 0 \\
1 & 1 & \dots & 0 & 1 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & & & & & & & \vdots \\
0 & 0 & \dots & 1 & 0 & 1 & 0 & \dots & 1 \\
0 & 0 & \dots & 0 & 1 & 0 & 1 & \dots & 1 \\
0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1
\end{array}$$

Note that this conjecture immediately implies Conjecture 6.1.

It is more difficult to handle other non-trivial cases in Conjecture 1.1 since the orbits become much harder to count. In this section, we suggest a possible argument that may work for the case $a = 2$ and $b$ is even.

First, note that jeu de taquin behaves nicely when the tableau can be reduced to smaller tableaux.

**Lemma 6.5.** *Let $T$ be a CST of shape $\langle 2^b \rangle$ with parts less than or equal to $n$ with corresponding matrix $M(T)$. Let $M$ be a $m \times 2$ matrix, where $m > n$, corresponding to some CST such that $M(T)$ appears as a block of $M$ that is not at the top:*

$$M = \begin{array}{|c|}\hline A \\ \hline M(T) \\ \hline B \\ \hline \end{array}$$

*Then*

$$j(M) = \begin{array}{|c|}\hline A' \\ \hline M(T) \\ \hline B' \\ \hline \end{array},$$

*where*

$$j\left(\begin{array}{|c|}\hline A \\ \hline B \\ \hline \end{array}\right) = \begin{array}{|c|}\hline A' \\ \hline B' \\ \hline \end{array},$$

*and if $A$ has $N_A$ rows then $A'$ has $N_A - 1$ rows.*

*Proof.* If the top row of $M$ is 0 0 or 1 1, then this is obviously true. When the top row of $M$ is 1 0, we need to find the row that will be swapped by jeu de taquin. If it is not found in the $A$ block, then $A$ has more 1s in the first column than the second. Since $M(T)$ gives a valid CST, at every row of $M(T)$ there will be more 1s above that row in the first column than in the second. So the row to be swapped will be found in $B$. Since there are as many 1s in the second row of $M(T)$ as in the first, the row of $B$ that will be swapped will be the same as that in $\begin{array}{|c|}\hline A \\ \hline B \\ \hline \end{array}$. □

When $a > 2$, jeu de taquin consists of successive swaps on adjacent columns of the matrix to correct the bottom row. These swaps are independent of the other columns of the matrix. Thus we have the following proposition.

**Proposition 6.6.** *Let $T$ be a CST of shape $\langle a^b \rangle$ with parts less than or equal to $n$ and with corresponding matrix $M(T)$. Let $M$ be an $m \times a$ matrix corresponding to some CST,*

$$M = \begin{array}{|c|}\hline A \\ \hline M(T) \\ \hline B \\ \hline \end{array}.$$

*Then*

$$j(M) = \begin{array}{|c|}\hline A' \\ \hline M(T) \\ \hline B' \\ \hline \end{array},$$

*where $A'$ and $B'$ are defined similarly as in Lemma6.5.*

Now, for the case $a = 2$ in Conjecture 1.1, the polynomial becomes

$$X(q) = q^{-b(b+1)} s_{\langle 2^b \rangle}(1, q, \ldots, q^{b+c-1}) = \frac{1}{[b+c+1]_q} \begin{bmatrix} b+c+1 \\ b \end{bmatrix}_q \begin{bmatrix} b+c+1 \\ b+1 \end{bmatrix}_q .$$

Note that the expression on the right is the $q$-Narayana number

$$N_q(n,k) := \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q q^{k(k+1)}.$$

When $q = 1$, $N_q(n,k)$ reduces to the Narayana number, which has many combinatorial interpretations [2]. Moreover, we know that

$$\sum_{k=0}^{n} N_q(n,k) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q ,$$

where the right hand side is the well-known $q$-Catalan number. In [3], it was shown that $N_q(n,k)$ with the action of rotation on non-crossing partitions of $n$-gon exhibits the cyclic sieving phenomenon. Now, the case $a = 1$ in Conjecture 1.1 corresponds to a different cyclic action with order $n-1$, rather than $n$, with the same polynomial. This may give another natural cyclic action on these non-crossing partitions.

Returning to the point, when $q$ is a $d^{\text{th}}$ root of unity, where $d \mid b+c$, we have

$$X(q) = \begin{cases} \binom{\frac{b+c}{d}}{\frac{b}{d}}^2 & \text{if } b \mid d, \\ \binom{\frac{b+c}{2}}{\frac{b-1}{2}} \binom{\frac{b+c}{2}}{\frac{b+1}{2}} & \text{if } d = 2 \text{ and } b \text{ odd}, \\ 0 & \text{otherwise}. \end{cases}$$

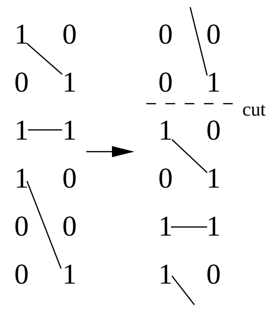
We can account for the first case by finding a bijection $\phi$ between pairs of $\frac{b}{d}$-subsets of $\left[\frac{b+c}{d}\right]$ and CST fixed by $j^{\frac{b+c}{d}}$.

First, $\frac{b}{d}$-subsets of $\left[\frac{b+c}{d}\right]$ correspond to $b$-subsets of $[b+c]$ which are fixed by $d$ cyclic demotions of the elements. Now we will map these pairs of sets into tableaux.

It is convenient to write this map in terms of the tableau matrices. Each subset in the pair corresponds to a single 0,1-column. We place these columns side-by-side to get a matrix. This matrix may correspond to a valid CST. If so, that tableau is the result of the map. If not, we apply the following lemma:

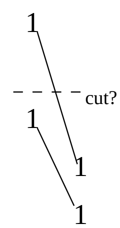
**Lemma 6.7.** *If $A$, $B$ are 0,1-columns corresponding to a pair of $b$ subsets of $[b+c]$ and $[A,B]$ is the matrix formed by placing $A$ and $B$ side-by-side, then there is some $n$ such that if $\widetilde{[A,B]}$ is the matrix formed by taking the last $n$ rows of $[A,B]$ and moving them to the top, then $\widetilde{[A,B]}$ corresponds to a valid CST.*

*Proof.* If a 2-column matrix corresponds to a valid CST, we can connect the first 1 in each column, the second 1 in each column, and so on. All of the connections move down, and none of them cross. When some rows are moved from the top to the bottom, this will still be true, except that some of the connections will move down cyclically; that is, they will move down off the bottom of the matrix and wrap around to the top. There will also be a place between two rows where the matrix may be cut to restore the original. No connections will cross this cutting line. Observe the diagram:




The matrix on the left corresponds to a CST and the pairings are shown. The matrix on the right is the same matrix with the top four rows moved to the bottom. The connections and the cutting line are show. Note that the bottommost 1 in the first column connects to the topmost 1 in the second column, but that connection still moves cyclically downward.

It is now sufficient to show that for any pair of columns with the same number of 1s, there is such a cyclically downward pairing with a cutting line. It is easy to find some cyclically downward pairing, since there are the same number of 1s in each column. Suppose that we have such a pairing that does not admit a cutting line. Then for any potential cutting line, the situation will look like this:




The second 1 in the first column here could be validly connected to the first 1 in the second column, if partners could be found for the other 1s in the diagram. However, if no potential cutting line works, then every 1 in the first column can be paired with a 1 in the second column higher than the one it is currently paired with. This reconnection cannot be repeated forever, because eventually a 1 would need to connect cyclically upward. The repetition will only stop at a pairing that admits a cutting line. $\square$

Thus we define $f(A,B) = j^n(\widetilde{[A,B]})$, where $n$ is a number satisfying the condition in Lemma 6.7.

**Proposition 6.8.** *This map $f$ is well-defined.*

*Proof.* It suffices to show that if $A$, $B$, $A'$, $B'$ are such that $[A,B]$ is a valid tableau matrix and $\widetilde{[A,B]} = [A',B']$, then $j^n([A',B']) = [A,B]$, where $\widetilde{[A,B]}$ and $n$ are as in Lemma 6.7.

When this occurs, a pairing of 1s in $[A,B]$ will admit two cutting lines: one at the very top or bottom of the matrix and the other giving $\widetilde{[A,B]} = [A',B']$. These will divide $[A,B]$ into two valid tableau matrices stacked on top of each other:

$$[A,B] = \begin{array}{|c|}\hline M_1 \\ \hline M_2 \\ \hline \end{array},$$

such that

$$[A',B'] = \begin{array}{|c|}\hline M_2 \\ \hline M_1 \\ \hline \end{array},$$

where $M_2$ has height $n$. Then by Proposition 6.6

$$j^n([A',B']) = [A,B].$$

$\square$

**Proposition 6.9.** $j^{\frac{b+c}{d}}(f(A,B)) = f(A,B)$

*Proof.* Some element of the jeu-de-taquin orbit of $f(A,B)$ will have the form $[A',B']$. This matrix will consist of a stack of $d$ copies of a block $K$ of height $\frac{b+c}{d}$ which corresponds to a valid tableau of shape $\left\langle 2^{\frac{b}{d}}\right\rangle$. Then the proposition follows from Proposition 6.6. $\square$

Now it would be nice to see that all fixed point of $j^{\frac{b+c}{d}}$ have this form, proving that $f$ is a bijection between pairs of subsets and fixed points and showing that $(X, X(q), C)$ exhibits the cyclic sieving phenomenon when $a=2$ and $b$ is even.

## 7. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2 with an explicit bijection. Sometimes, we say that a CST is $j$-self-complementary if it is a fixed point of $jc$. Before we can prove the bijection we need some notation. Formally, a tableau $T$ of shape $b\times a$ with entries in $[b+c]$ is a map

$$T:[b]\times[a]\to[b+c].$$

We will use $T[l,k]$ to represent the image of $(l,k)$ under this map. When working with the actions of $j$ and $c$ on tableaux it is useful to think in terms of two separate actions, one 'permuting the numbers' and one 'permuting the boxes'. These actions are realized as actions on the domain and range of $T$ such that applying both at once gives you the action on the tableau itself.

**Definition 7.1.** In cycle notation, define

$$\mathbf{j} := (b+c,\ b+c-1,\ \dots\ 2,\ 1),\ \text{and}$$

$$\mathbf{c} := (1,\ b+c)(2,\ b+c-1)\cdots.$$

$\mathbf{j}$ and $\mathbf{c}$ are the actions we think of as 'permuting the numbers'. It is easy to see that

$$\mathbf{jc} = (1,\ b+c-1)(2,\ b+c-2)\cdots(\lfloor\tfrac{b+c}{2}\rfloor,\ \lceil\tfrac{b+c}{2}\rceil)(b+c),$$

and hence

$$\begin{aligned}\mathbf{c}^2 &= 1,\\ \mathbf{j}^{b+c} &= 1,\ \text{and}\\ (\mathbf{jc})^2 &= 1.\end{aligned}$$

$j$ and $c$ also induce an actions on $[b]\times[a]$ which we think of as 'permuting the boxes'.

**Definition 7.2.** Define a map C

$$C:[b]\times[a]\to[b]\times[a],$$

which sends

$$(l,k)\longmapsto(a+1-l,\ b+1-k).$$

It is clear that $C^2=1$. More importantly the operations $\mathbf{c}$ and $C$ satisfy

$$(cT)[C(l,k)] = \mathbf{c}\ T[l,k].$$

While $c$ always acts the same way on the boxes of every tableaux, $j$ does not. This makes the corresponding action for $j$ on $[b]\times[a]$ more cumbersome to specify since we must indicate the tableau $T$ which the action was derived from. Exactly how will define this action is more clear with an example.

**Example 7.1.**

Consider the following tableau with $a=4$, $b=3$ and $c=5$:

$$T = \begin{matrix} 1 & 2 & 4 & 4 \\ 3 & 3 & 5 & 7 \\ 6 & 6 & 7 & 8 \end{matrix} \xrightarrow{-1} \begin{matrix} * & \mathbf{1} & 3 & 3 \\ 2 & \mathbf{2} & \mathbf{4} & 6 \\ 5 & 5 & \mathbf{6} & \mathbf{7} \end{matrix} \xrightarrow{slide} \begin{matrix} 1 & 2 & 3 & 3 \\ 2 & 4 & 6 & 7 \\ 5 & 5 & 6 & 8 \end{matrix} = jT$$

The corresponding box permutation, $J_T$, is:

$$J_T : 11 \to 34 \to 33 \to 23 \to 22 \to 12 \to 11,$$

leaving other boxes fixed.

With this definition we have

$$(jT)[J_T(l,k)] = \mathbf{j}\ T[l,k] \tag{1}$$

Similarly we can define $(J^{-1})_T$ as the permutation given by applying $j^{-1}$ to this tableau:

$$T = \begin{matrix} 1 & 2 & 4 & 4 \\ 3 & 3 & 5 & 7 \\ 6 & 6 & 7 & 8 \end{matrix} \xrightarrow{+1} \begin{matrix} \mathbf{2} & \mathbf{3} & \mathbf{5} & 5 \\ 4 & 4 & \mathbf{6} & \mathbf{8} \\ 7 & 7 & 8 & * \end{matrix} \xrightarrow{slide} \begin{matrix} 1 & 2 & 3 & 5 \\ 4 & 4 & 5 & 6 \\ 7 & 7 & 8 & 8 \end{matrix} = j^{-1}T$$

The corresponding box permutation, $(J^{-1})_T$, is:

$$(J^{-1})_T : 11 \to 12 \to 13 \to 23 \to 24 \to 34 \to 11,$$

leaving other boxes fixed.

Note that

$$(J_T)^{-1} \neq (J^{-1})_T!$$

However, we do have the following relationship,

$$(J_T)^{-1} = (J^{-1})_{jT}. \tag{2}$$

This should be no suprise since $(J_T)^{-1}$ permutes the boxes of $jT$ back to $T$ and $j^{-1}$ brings the tableau $jT$ back to $T$.

**Definition 7.3.** Define $J_T$ to be the permutation of the boxes indicated by the path diagram of $j$ on the tableau $T$, and similarly for $(J^{-1})_T$ It is easy to see that Equations 1 and 2 hold for all $T$.

**Proposition 7.4.** For all tableaux $T$ and for all $x \in [b] \times [a]$,
1) $(cT)[C(x)] = \mathbf{c}\ T[x]$,
2) $(jT)[J_T(x)] = \mathbf{j}\ T[x]$,
3) $(J^{-1})_{jT}(x) = (J_T)^{-1}(x), and$
4) $J_{cT}C(x) = C(J^{-1})_T(x).$

*Proof.* The first 3 identities have already been discussed. Identity 4 essentially follows from the identity $cj^{-1} = jc$. It should be fairly clear why 4 is true without proof; $J_{cT}$ is exactly what we need for performing jeu de taquin on $cT$. We cannot however prove this identity by just taking $cj^{-1}T = jcT$ and applying identities 1 and 2. The problem is that

$$T[J_{cT}C(x)] = T[C(J^{-1})_T(x)] \ \ \forall\ x$$

does not imply

$$J_{cT}C(x) = C(J^{-1})_T(x) \ \ \forall\ x,$$

as tableaux can have repeated entries. For a rigorous proof we must convolute things slightly by converting to standard tableau. Define $S_T$ to be the standard tableaux corresponding to $T$. Starting with an identity,

$$\begin{aligned} cjT &= j^{-1}cT, \Rightarrow \\ cj^nS_T &= j^{-n}cS_T, \end{aligned}$$

where n is the number of 1's in $T =$ the number of $b + c$'s in $cT$.

$$\begin{aligned}
\Rightarrow \mathbf{j}^n\ (cS_T)[(J_T)^{-1}(x)] &= \mathbf{c}\ (j^{-n}S_T)[C(x)] \ \ \forall\ x,\ \Rightarrow \qquad (3)\\
\mathbf{j}^n\mathbf{c}\ (S_T)[C(J_T)^{-1}(x)] &= \mathbf{c}\mathbf{j}^{-n}\ (S_T)[J_{cT}C(x)] \ \ \forall\ x,\ \Rightarrow \qquad (4)\\
S_T[C(J_T)^{-1}(x)] &= S_T[J_{cT}C(x)] \ \ \forall\ x \Rightarrow\\
J_{cT}C &= C(J^{-1})_T.
\end{aligned}$$

Equations (3) and (4) use identities 1, 2 and $(J^n)_{S_T} = J_T$. □

**Proposition 7.5.** *$jc\ T = T \Rightarrow (J^{-1})_T = (J_{cT})^{-1} \Leftrightarrow (J_{cT}C)^2 = 1$.*

*Proof.* Assume $jc\ T = T$ and start with Identity 3 of Proposition 7.4:

$$\begin{aligned}
(J^{-1})_{jT} &= (J_T)^{-1}, \Rightarrow\\
(J^{-1})_{jcT} &= (J_{cT})^{-1}, \Rightarrow\\
(J^{-1})_T &= (J_{cT})^{-1}. \qquad (5)
\end{aligned}$$

Using identity 4 from Proposition 7.4 and Equation 5,

$$\begin{aligned}
(J_{cT}C)^2 &= C(J^{-1})_T\ J_{cT}C, \Rightarrow\\
(J_{cT}C)^2 &= C(J_{cT})^{-1}\ J_{cT}C, \Rightarrow\\
(J_{cT}C)^2 &= 1.
\end{aligned}$$

This proves $jc\ T = T \Rightarrow (J^{-1})_T = (J_{cT})^{-1} \Rightarrow (J_{cT}C)^2 = 1$. Assuming $(J_{cT}C)^2 = 1$ we have,

$$\begin{aligned}
1 = (J_{cT}C)^2 &= J_{cT}C\ C(J^{-1})_T\\
1 = &= J_{cT}(J^{-1})_T
\end{aligned}$$

Which proves $(J_{cT}C)^2 = 1 \Rightarrow (J^{-1})_T = (J_{cT})^{-1}$. □

Note $(J_{cT}C)^2 = 1$ is equivalent to saying the path diagram of $J_{cT}$ is symmetric.

**Definition 7.6.** Let $b$ be even. By "$(b/2,\ k)$ is a crossing point of $T \xrightarrow{j} jT$" we mean in the path diagram of jeu de taquin applied to $T$ there is a path which crosses from the top half to the bottom half in column $k$. Similiarly, By "$(b/2+1,\ k)$ is a crossing point of $T \xrightarrow{j^{-1}} j^{-1}T$" we mean in the path diagram of jeu de taquin inverse applied to $T$ there is a path which crosses from the bottom half to the top half in column $k$.

**Proposition 7.7.** If $jc\ T = T$, the following is an algorithm for finding the path diagram of $T \xrightarrow{j^{-1}} j^{-1}T$:

(i) Add 1 to each entry in the top half of $T$ and put a star one space above each crossing point of $T \xrightarrow{j^{-1}} j^{-1}T$.
(ii) Starting from the left, slide each star (using the rules of jeu de taquin inverse) to the upper left then turn it into a 1. This creates a path diagram on the top half of the tableau.
(iii) Create the bottom half of the path diagram by taking the complement of the top half.

For step (iii) we are using the fact that the path diagram of $T \xrightarrow{j^{-1}} j^{-1}T$ is symmetric when $jc\ T = T$. It is clear why this algorithm works.

We are ready to prove the bijection. Take note that when we say 'tableau' it is not necessarily column strict.

**Definition 7.8.** Define a map $\psi$ on tableaux

$$\psi : \left\{ \begin{array}{c} b \times a \textit{ tableaux with } b \textit{ even} \\ \textit{and entries in } [b+c] \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} b \times a \textit{ tableaux with } b \textit{ even} \\ \textit{and entries in } [b+c] \end{array} \right\},$$

where the top half of $\psi(T)$ is $T$ and the bottom half is $j^{-1}T$. Using $(jT)[J_T(x)] = \mathbf{j}\, T[x]$ (Identity 2 from Proposition 7.4), $\psi(T)$ is explicitly given by

$$\psi(T)[x] = \left\{ \begin{array}{cl} T[x] & \text{if } x \text{ is in the top half} \\ \mathbf{j}^{-1}\ T[J_{cT}(x)] & \text{if } x \text{ is in the bottom half.} \end{array} \right.$$

**Lemma 7.9.** *If $T$ is a column strict tableau of shape $b \times a$ then*

*1) $(b/2+1,\ k)$ is a crossing point of $T \overset{j^{-1}}{\to} j^{-1}T$ if and only if $\psi(T)[b/2+1,\ k] = \psi(T)[b/2,\ k]+1$, and*
*2) $\psi(T)$ is a column strict.*

*Proof.* Fix $k$. To prove the forward direction of 1 let $(b/2+1,\ k)$ be a crossing point and let $P$ be the intermediate tableau between $T$ and $j^{-1}T$ in which the last star that travels through $(b/2+1,\ k)$ has been evaluated. The properties of jeu de taquin paths (see Proposition 5.4) imply no earlier star has traveled through $(b/2,\ k)$, so

$$\begin{aligned} P[b/2+1,\ k] = T[b/2,\ k]+1,\ \Rightarrow \\ \psi(T)[b/2+1,\ k] = \psi(T)[b/2,\ k]+1. \end{aligned} \tag{6}$$

To prove the reverse direction of 1, assume $(b/2+1,\ k)$ is not a crossing point and let $P$ be as before.[1] Since $P$ is column strict,

$$P[b/2,\ k] < P[b/2+1,\ k] = \psi(T)[b/2+1,\ k].$$

Since no star has traveled through $(b/2,\ k)$,

$$\begin{aligned} P[b/2,\ k] = T[b/2,\ k]+1,\ \Rightarrow \\ P[b/2+1,\ k] > T[b/2,\ k]+1,\ \Rightarrow \\ \psi(T)[b/2+1,\ k] > \psi(T)[b/2,\ k]+1. \end{aligned} \tag{7}$$

This proves 1. $T$ being column strict implies $j^{-1}T$ is column strict. Therefore to prove 2 it is sufficient to prove

$$\psi(T)[b/2+1,\ k] \quad > \quad \psi(T)[b/2,\ k] \ \ \forall\ k.$$

Which is true by Equations 6 and 7. This completes the proof of the Lemma.

□

**Lemma 7.10.** *$(J_{cT}C)^2 = 1$ and $c\ \psi(T) = \psi(T)$ if and only if $jc\ T = T$. In other words,*

$$(J_{cT}C)^2 = 1 \textit{ and } \psi(T)[C(x)] = \mathbf{c}\ \psi(T)[x] \ \ \forall\ x \quad \Leftrightarrow \quad T[J_{cT}C(x)] = \mathbf{jc}\ T[x] \ \ \forall\ x.$$

[1] If no star travels through $(b/2+1,\ k)$ then $P$ is the intermediate tableau between $T$ and $j^{-1}T$ before any star has been moved, but after each entry has been incremented by 1.

*Proof.* By Proposition 7.5, $jc\ T = T$ implies $(J_{cT}C)^2 = 1$. To prove the $\Leftarrow$ direction it is enough to show $\psi(T)[C(x)] = \mathbf{c}\ \psi(T)[x]$ for all $x$ in the top half.[2] If $x$ is the top half then

$$\psi(T)[C(x)] = \mathbf{j}^{-1}\ T[J_{cT}C(x)] = \mathbf{j}^{-1}(\mathbf{jc})\ T[x] = \mathbf{c}\ \psi(T)[x]. \tag{8}$$

This proves the $\Leftarrow$ direction. Since $J_{cT}$ represents the action of a set of NW to SE paths, $x$ being in the bottom half implies that $(J_{cT}C(x))$ is either in the top half or in the last row. To prove the $\Rightarrow$ direction, it is enough to show

$$T[J_{cT}C(x)] = \mathbf{jc}\ T[x] \tag{9}$$

for all $x$ in the top half and for all $x$ in the bottom half such that $J_{cT}C(x)$ is in the last row.[3] In the latter case, $T[x] = b + c$ by the rules of jeu de taquin. But $J_{cT}C(x)$ is in the bottom half and $(J_{cT}C)^2(x) = x$ is in the bottom half, hence the bottom row. Therefore $J_{cT}C(x) = b + c$ too. Since $\mathbf{jc}(b + c) = b + c$, Equation 9 holds.

In the former case, $x$ is in the top half. If $J_{cT}C(x)$ is in the bottom half go back to Equation 8. This time we know $\psi(T)[C(x)] = \mathbf{c}\ \psi(T)[x]$, so we may deduce the middle equality giving us Equation 9. If $J_{cT}C(x)$ is in the top half then $C(x)$ must be a crossing point of $T$ because $J_{cT}$ can only move $C(x)$ up by at most 1 space. Using part 1 of Lemma 7.9 we have,

$$\begin{aligned}
\psi(T)[J_{cT}C(x)] &= \psi(T)[C(x)] - 1 \\
&= \mathbf{j}\ \psi(T)[C(x)],\ \Rightarrow \\
T[J_{cT}C(x)] &= \mathbf{jc}\ \psi(T)[x],\ \Rightarrow \\
T[J_{cT}C(x)] &= \mathbf{jc}\ T[x].
\end{aligned}$$

This proves the Lemma. ☐

**Lemma 7.11.** *Let $S$ be a tableau of shape $b \times a$. There is at most one tableau $T$ such that $\psi(T) = S$ and $jc\ T = T$.*

*Proof.* We will prove this Lemma 2 steps:

*A. If $\psi(T) = S$ and $jc\ T = T$ then $J_{cT}$ is determined by $S$.*
*B. If $\psi(T) = S$ and $jc\ T = T$ then $T$ is determined by $S$.*

The path diagram of $T \xrightarrow{j^{-1}} j^{-1}T$ can be found using Proposition 7.7. Recall $jc\ T = T$ imples $J_{cT} = ((J^{-1})_T)^{-1}$ (Proposition 7.5). $((J^{-1})_T)^{-1}$ is obtained by reversing the arrows of this path diagram. In step (i) of Proposition 7.7, the location of each crossing point is determined by $S$ (part 1 of Lemma 7.9). Therefore $J_{cT}$ is determined by $S$, proving A. Define $H_S := J_{cT}$. By A, $jc(T) = T$ is equivalent to

$$\mathbf{jc}\ T[H_S C(x)] = T[x] \ \ \forall\ x. \tag{10}$$

The top half of $T$ is determined by $S$. Let $x$ be in the bottom half. Since $H_S$ represents the action of a NW to SE path diagram, $H_S C(x)$ is either in the top half or in the last row. In the latter case, $T[H_S C(x)]$ is $b + c$ by the rules of jeu de taquin. By Equation 10,

$$T[x] = \mathbf{jc}\ T[H_S C(x)] = \mathbf{jc}(b + c) = b + c.$$

In the former case $T[H_S C(x)]$ is in the top half. By Equation 10,

$$T[x] = \mathbf{jc}\ T[H_S C(x)] = \mathbf{jc}\ S[H_S C(x)].$$

[2] To recover the other half of the statment, let $y$ be in the bottom half and substitute $x := C(y)$ into $\psi(T)[C(x)] = \mathbf{c}\ \psi(T)[x]$.

[3] To recover the rest of the statement let $y$ be in the bottom half such that $J_{cT}C(x)$ is in the top half and substitute $x := J_{cT}C(y)$ into $T[J_{cT}C(x)] = \mathbf{jc}\ T[x]$.

All the entries of $T$ are then determined by $S$. This proves the Lemma. We can continue simplifying the previous equation. Since $(H_S C)^2 = 1$, [4]

$$T[x] = \mathbf{jc}\ S[(H_S C)^{-1}(x)] = \mathbf{jc}\ S[C(H_S)^{-1}(x)].$$

$cS = S$ by the $\Leftarrow$ direction of Lemma 7.10. Therefore,

$$T[x] = \mathbf{j}\ S[(H_S)^{-1}(x)].$$

□

**Definition 7.12.** Define a map $\phi$ on tableaux

$$\phi : \left\{ \begin{array}{c} b \times a \textit{ tableaux with } b \textit{ even} \\ \textit{and entries in } [b+c] \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} b \times a \textit{ tableaux with } b \textit{ even} \\ \textit{and entries in } [b+c] \end{array} \right\},$$

where $\phi(S)$ is the tableau $T$ determined by $S$ in part B of the previous Lemma. $\phi(S)$ is explicitly given by:

$$\phi(S)[x] = \begin{cases} S[x] & \text{if } x \text{ is in the top half} \\ \mathbf{j}\ S[(H_S)^{-1}(x)] & \text{if } x \text{ is in the bottom half and } H_S C(x) \text{ is in the top half} \\ b+c & \text{if } x \text{ is in the bottom half and } H_S C(x) \text{ is in the last row} \end{cases}$$

**Lemma 7.13.** *If $S$ is column strict and $cS = S$ then*

*1) $\phi(S)$ is column strict,*
*and*
*2) $J_{c\phi(S)} = H_S$.*

*Proof.* Define a tableaux $R$

$$R[x] = \begin{cases} \mathbf{j}^{-1} S[H_S(x)] & \text{if } x \text{ is in the top half} \\ S[x] & \text{if } x \text{ is in the bottom half.} \end{cases}$$

In other words, $R$ is given by adding 1 to each entry in the top half of $S$, replacing entries in row $b/2$ with stars where column strictness fails, then sliding each star (using the rules of jeu de taquin inverse) to the upper left. Since the top and bottom halves of $R$ are column strict, it is enough to show

$$R[b/2,\ k] < R[b/2+1, k]\ \ \forall\ k \tag{11}$$

to prove $R$ itself is column strict. Let $P$ be the intermediate tableau between $S$ and $R$ just after the star starting at $(b/2,\ k)$ has been evaluated. Since $S$ is column strict,

$$\begin{array}{rcl} P[b/2,\ k] & \leq & S[b/2,\ k] + 1 \leq S[b/2+1,\ k],\ \Rightarrow \\ P[b/2,\ k] & \leq & S[b/2+1,\ k]. \end{array}$$

Choose this star so that

$$P[b/2,\ k] = S[b/2+1,\ k] \textit{ and } P[b/2,\ m] < S[b/2+1,\ m]\ \ \forall\ m < k. \tag{12}$$

Since $P[b/2,\ k]$ was directly above $S[b/2+1,\ k-1]$ before this star was evaluated it follows that

$$P[b/2,\ k] \leq S[b/2+1,\ k-1] \leq S[b/2+1,\ k].$$

[4] $H_S$ is the action of a symmetric path diagram, therefore $H_S C$ is an involution on $[b] \times [a]$ (see Proposition 7.5)

Applying the first part of Equation 12 to the above relation we get

$$P[b/2,\ k] = S[b/2+1,\ k-1],$$

which contradicts the second part of Equation 12. Therefore,

$$\begin{aligned} P[b/2,\ k] &< S[b/2+1,\ k] = R[b/2+1,\ k] \ \ \forall\, k,\ \Rightarrow \\ R[b/2,\ k] &< R[b/2+1,\ k] \ \ \forall\, k. \end{aligned}$$

This proves Equation 11 holds when $(b/2,\ k)$ starts as star, and Equation 11 is trivial when $(b/2,\ k)$ does not start as a star. This proves $R$ is column strict. Using $cS = S$, it is straight forward to check $\phi(S) = cR$, therefore $\phi(S)$ is column strict. This proves 1.

It follows from the definitions of $R$ and $H_S$ that the path diagram of $R \xrightarrow{j} jR$ strictly above row $b/2$ is the path diagram corresponding to $H_S$. Let $(b/2,\ k)$ be a crossing point of $R \xrightarrow{j} jR$ and let $P$ be the intermediate tableau between $R$ and $jR$ in which the last star to travel through $(b/2,\ k)$ has been evaluated. Since no earlier star has traveled through $(b/2+1,\ k)$,

$$\begin{aligned} P[b/2,\ k] &= R[b/2+1,\ k]-1,\ \Rightarrow \\ (jR)[b/2,\ k] &= S[b/2+1,\ k]-1,\ \Rightarrow \\ S[b/2,\ k] &= S[b/2+1,\ k]-1. \end{aligned} \tag{13}$$

Assume $(b/2,\ k)$ is not a crossing point of $R \xrightarrow{j} jR$ and let $P$ be as before.[5] Since $P$ is column strict,

$$P[b/2,\ k] < P[b/2+1,\ k].$$

Since no star has traveled through $(b/2+1,\ k)$,

$$\begin{aligned} P[b/2+1,\ k] &= R[b/2+1,\ k]-1,\ \Rightarrow \\ P[b/2,\ k] &< R[b/2+1,\ k]-1,\ \Rightarrow \\ (jR)[b/2,\ k] &< S[b/2+1,\ k]-1,\ \Rightarrow \\ S[b/2,\ k] &< S[b/2+1,\ k]-1. \end{aligned} \tag{14}$$

Relations 13 and 14 say the crossing points of $R \xrightarrow{j} jR$ are precisely those of $H_S$. Since the bottom half of $R$ is $S$, it follows the entire path diagram of $R \xrightarrow{j} jR$ is $H_S$. With $c\ \phi(S) = R$, this is equivalent to $J_{c\phi(S)} = H_S$. This completes the Lemma. □

**Theorem 7.14.** *$\psi$ is a bijection between column strict fixed points of $jc$ and column strict fixed points of $c$.*

*Proof.* By the $\Leftarrow$ direction of Lemma 7.10 and part 2 of Lemma 7.9, $\psi$ is a map between these two sets. $\psi$ is injective by Lemma 7.11.

Let $S$ be column strict and $cS = S$. $\phi(S)$ is column strict by part 1 of Lemma 7.13. Using $J_{c\phi(S)} = H_S$ (part 2 of Lemma 7.13) it is straight forward to check $\psi(\phi(S)) = S$. Therefore by the $\Rightarrow$ direction of Lemma 7.10, $jc\ \phi(S) = \phi(S)$. Putting all this together, $\phi(S)$ is a column strict fixed point of $jc$ which gets mapped to $S$ by $\psi$, i.e. $\psi$ is surjective. □

[5] If no star travels through $(b/2,\ k)$ then $P$ is $R$ before any star has been moved, but after each entry has been decremented by 1.

## 8. Proof of some cases of Conjecture 1.3

In this section, we will prove the cases (i) $a = 1$, (ii) $b = 1$, (iii) $c = 1$, and (iv) $a = 2$ in Conjecture 1.3. The following proposition gives closed formulae for the number of $j$-self-complementary tableaux in these cases.

**Proposition 8.1.** *Let $\mathcal{S}$ be the set of CSTs with $b$ rows, $a$ columns and entries in $[b+c]$.*

(i) *Assume $a = 1$, $b$ and $c$ are odd, then*

$$|\{T \in \mathcal{S} : jc(T) = T\}| = \begin{bmatrix} b+c-1 \\ b-1 \end{bmatrix}_{q=-1} + \begin{bmatrix} b+c-1 \\ b \end{bmatrix}_{q=-1}.$$

(ii) *Assume $b = 1$, $a$ is a positive integer and $c$ is odd, then*

$$|\{T \in \mathcal{S} : jc(T) = T\}| = \sum_{k=0}^{a} \begin{bmatrix} k+c-1 \\ k \end{bmatrix}_{q=-1}.$$

(iii) *Assume $c = 1$, $a$ is a positive integer and $b$ is odd, then*

$$|\{T \in \mathcal{S} : jc(T) = T\}| = \sum_{k=0}^{a} \begin{bmatrix} k+b-1 \\ k \end{bmatrix}_{q=-1}.$$

(iv) *Assume $a = 2$, $b$, and $c$ are odd, then*

$$|\{T \in \mathcal{S} : jc(T) = T\}| = mb(2, b-1, c; q)|_{q=-1} + mb(2, b, c-1; q)|_{q=-1} + mb\left(2, \frac{b-1}{2}, \frac{c-1}{2}; q\right)\Big|_{q=1}.$$

*Proof.* (i) Let $T$ be a CST with 1 column, $b$ rows and entries in $[b+c]$. Column-strictness implies that either there is
(1) one '$b+c$', or
(2) no '$b+c$' in $T$.

Note that in this case, the set of CSTs is in 1-1 correspondence with the set of $b$-subsets of $[b+c]$, simply collecting all the entries in $T$. Hence, $jc$ induces an action on these $b$-subsets by (in cycle notation):

$$jc = (1, b+c-1)(2, b+c-2) \cdots \left(\frac{b+c}{2} - 1, \frac{b+c}{2} + 1\right)\left(\frac{b+c}{2}\right)(b+c).$$

For (1), it is easy to see that $T$ is a fixed point of $jc$ if and only if the first $(b-1)$ rows of $T$ is a self-complementary tableau of 1 column, $(b-1)$ rows and entries in $[b+c-1]$. This gives us the first term on the right.

For (2), $T$ is a fixed point of $jc$ if and only if $T$ is a self-complementary tableau of 1 column, $b$ rows with entries in $[b+c-1]$. This gives us the second term.

(ii) Observe that the set of CSTs with 1 row, $a$ columns and entries in $[c+1]$, is in 1-1 correspondence with the set of $a$-multisubsets of $[c+1]$. The action induced by $jc$ is

$$jc = (1, c)(2, c-1)(3, c-2) \cdots \left(\frac{c-1}{2}, \frac{c+3}{2}\right)\left(\frac{c+1}{2}\right)(c+1).$$

Suppose $T$ has $k$ '$c+1$'s, where $0 \le k \le a$. Then, $T$ is a fixed point of $jc$ if and only if the first $(a-k)$ columns form a self-complementary tableau with entries in $[c]$. The result follows by summing over $k$.

(iii) By Proposition 2.10, this reduces to case (ii) since

$$\sigma(jc)\sigma = (\sigma j \sigma)c = j^{-1}c,$$

where we use the fact that $\sigma$ and $c$ commute. Our argument is completed by noting the number of fixed points of $j^{-1}c$ is the number of fixed points of $jc$ (complementation is a bijection between the sets).

(iv) Let $T$ be a CST with 2 columns, $b$ rows and entries in $[b+c]$. We consider three cases: there are
(1)two '$b+c$'s,
(2)no '$b+c$', and
(3)one '$b+c$' in $T$.

For (1), it is not hard to see that $T$ is a fixed point of $jc$ if and only if the first $(b-1)$ rows of $T$ is a self-complementary tableau of 2 columns, $b-1$ rows and entries in $[b+c-1]$.

For (2), we observe that $T$ is a fixed point of $jc$ if and only if $T$ is a self-complementary tableau of 2 columns and $b$ rows with entries in $[b+c-1]$.

For (3), it is more convenient to work on the tableau matrices. In this case, the tableau matrix looks like

$$T = \begin{array}{|cc|} \hline * & * \\ \vdots & \vdots \\ * & * \\ \hline 0 & 1 \\ \hline \end{array}.$$

Recall that $j$ cyclically permutes the rows of these tableau matrices. Complementation on these matrices is the same as turning it upside down and interchanging the two columns. So,

$$c(T) = \begin{array}{|cc|} \hline 1 & 0 \\ \hline * & * \\ \vdots & \vdots \\ * & * \\ \hline \end{array},$$

and hence

$$jc(T) = \begin{array}{|cc|} \hline * & * \\ \vdots & \vdots \\ \hline 1 & 0 \\ \hline \vdots & \vdots \\ * & * \\ \hline 0 & 1 \\ \hline \end{array}.$$

If $T$ is fixed by $jc$, it is not hard to see that the middle '1 0' row (which is the upper row changed in $j$) shown here corresponds to the $(\frac{b+c}{2}-1)^{\text{th}}$ row in $T$. Let us write $T$ as

$$T = \begin{array}{|c|} \hline \pi \\ \hline 1\ 0 \\ \hline \sigma \\ \hline 0\ 1 \\ \hline \end{array}.$$

Using $jc(T) = T$, we know that $\pi$ itself is a valid CST of 2 columns, $\frac{b-1}{2}$ rows and entries in $[\frac{b+c-2}{2}]$(otherwise, we are not going to have the $(\frac{b+c}{2}-1)^{\text{th}}$ row flipped). Moreover, $\sigma = c(\pi)$. So, given any valid CST of 2 columns, $\frac{b-1}{2}$ rows and entries in $[\frac{b+c-2}{2}]$, we can construct a unique fixed point of $jc$, and these are the only possible fixed points of $jc$.

□

**Lemma 8.2.** *Let $b$ and $c$ be any positive odd integers, $\lambda$ a rectangular shape with $b$ rows and $a$ columns.*

(i) *When $a$ is a positive even integer,*

$$s_\lambda(1, q, q^2, \ldots, q^{b+c-3}, q^{b+c-2}, q^{b+c})\big|_{q=-1} = \frac{2a+b+c}{b+c} mb(a,b,c;q)\bigg|_{q=-1}.$$

(ii) *When $a$ is a positive odd integer,*

$$s_\lambda(1, q, q^2, \ldots, q^{b+c-3}, q^{b+c-2}, q^{b+c})\big|_{q=-1} = 2 \times \frac{mb(a,b,c;q)}{[b+c]_q}\bigg|_{q=-1},$$

*where* $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$.

*Proof.* In [6, Exercise 7.32(b)],

$$s_\lambda(1, q, q^2, \ldots, q^{n-3}, q^{n-2}, q^n) = \frac{\sum_{i=1}^n q^{\lambda_i + n - i}}{[n]_q} s_\lambda(1, q, \ldots, q^{n-1}).$$

The result follows easily from this formula. □

With the above formulae, it is not hard to prove those cases in Conjecture 1.3.
*Proof of some cases in Conjecture 1.3:*
In this proof, we will use this notation:

$$s(-1) := s_\lambda(1, q, q^2, \ldots, q^{b+c-3}, q^{b+c-2}, q^{b+c})\big|_{q=-1}.$$

(i) Assume $a = 1$, $b$ and $c$ are odd.
Using Lemma 8.2 (ii),

$$\begin{aligned} s(-1) &= \frac{2}{[b+c]_q} \begin{bmatrix} b+c \\ b \end{bmatrix}_{q=-1} \\ &= 2 \left. \frac{[b+c-1]_q[b+c-2]_q \cdots [c+2]_q[c+1]_q}{[b]_q[b-1]_q \cdots [2]_q[1]_q} \right|_{q=-1} \\ &= \frac{1}{[b]_q} \begin{bmatrix} b+c-1 \\ b-1 \end{bmatrix}_{q=-1} + \frac{1}{[c]_q} \begin{bmatrix} b+c-1 \\ b \end{bmatrix}_{q=-1} \\ &= \begin{bmatrix} b+c-1 \\ b-1 \end{bmatrix}_{q=-1} + \begin{bmatrix} b+c-1 \\ b \end{bmatrix}_{q=-1}, \end{aligned}$$

which is the same as the formula given in case (i) of Proposition 8.1.

(ii) Assume $b = 1$, $a$ and $c$ are positive integers with $c$ odd.
We prove this by induction on $a$. We have to divide into two cases: (1) $a$ is odd and (2) $a$ is even.
When $a$ is odd, using Lemma 8.2 (ii),

$$\begin{aligned} s(-1) &= \frac{2}{[c+1]} \begin{bmatrix} a+c \\ c \end{bmatrix}_{q=-1} \\ &= 2 \left. \frac{[c+1]_q[c+a-1]_q \cdots [a+3]_q[a+2]_q}{[c+1]_q[c]_q \cdots [3]_q[2]_q} \right|_{q=-1} \\ &= 2 \left( \frac{c+a}{c+1} \right) \left( \frac{c+a-2}{c-1} \right) \cdots \left( \frac{a+1}{2} \right) \\ &= 2 \binom{\frac{c+a}{2}}{\frac{c+1}{2}}. \end{aligned}$$

So, when $a = 1$,

$$\begin{aligned} \sum_{k=0}^{1} \begin{bmatrix} k+c-1 \\ k \end{bmatrix}_{q=-1} &= \begin{bmatrix} c-1 \\ 0 \end{bmatrix}_{q=-1} + \begin{bmatrix} c \\ 1 \end{bmatrix}_{q=-1} \\ &= 2 \binom{\frac{c+1}{2}}{\frac{c+1}{2}}. \end{aligned}$$

Carrying out induction on odd $a$, we have

$$\begin{aligned}\sum_{k=0}^{a+2}\begin{bmatrix}k+c-1\\ k\end{bmatrix}_{q=-1} &= 2\binom{\frac{c+a}{2}}{\frac{c+1}{2}}+\begin{bmatrix}a+c\\ a+1\end{bmatrix}_{q=-1}+\begin{bmatrix}a+c+1\\ a+2\end{bmatrix}_{q=-1}\\ &= 2\binom{\frac{c+a}{2}}{\frac{c+1}{2}}+\binom{\frac{a+c}{2}}{\frac{c-1}{2}}+\binom{\frac{a+c}{2}}{\frac{c-1}{2}}\\ &= 2\binom{\frac{c+a+2}{2}}{\frac{c+1}{2}}.\end{aligned}$$

This finishes our induction on odd $a$.

When $a$ is even, we use Lemma 8.2 (i),

$$\begin{aligned}s(-1) &= \frac{2a+c+1}{c+1}\begin{bmatrix}a+c\\ c\end{bmatrix}_{q=-1}\\ &= \frac{2a+c+1}{c+1}\binom{\frac{a+c-1}{2}}{\frac{c-1}{2}}.\end{aligned}$$

So, when $a=2$,

$$\begin{aligned}\sum_{k=0}^{2}\begin{bmatrix}k+c-1\\ k\end{bmatrix} &= \sum_{k=0}^{2}\begin{bmatrix}k+c-1\\ c-1\end{bmatrix}_{q=-1}\\ &= \begin{bmatrix}c-1\\ c-1\end{bmatrix}_{q=-1}+\begin{bmatrix}c\\ c-1\end{bmatrix}_{q=-1}+\begin{bmatrix}c+1\\ c-1\end{bmatrix}_{q=-1}\\ &= \frac{c+5}{c+1}\binom{\frac{c+1}{2}}{1}.\end{aligned}$$

Carrying out induction on even $a$, we have

$$\begin{aligned}\sum_{k=0}^{a+2}\begin{bmatrix}k+c-1\\ k\end{bmatrix}_{q=-1} &= \frac{2a+c+1}{c+1}\binom{\frac{a+c-1}{2}}{\frac{c-1}{2}}+\begin{bmatrix}a+c\\ c-1\end{bmatrix}_{q=-1}+\begin{bmatrix}a+c+1\\ c-1\end{bmatrix}_{q=-1}\\ &= \frac{2a+c+1}{c+1}\binom{\frac{a+c-1}{2}}{\frac{c-1}{2}}+\binom{\frac{a+c-1}{2}}{\frac{c-1}{2}}+\binom{\frac{a+c+1}{2}}{\frac{c-1}{2}}\\ &= \frac{2a+c+3}{c+1}\binom{\frac{a+c+1}{2}}{\frac{c-1}{2}}.\end{aligned}$$

(iii) Assuming $c=1$, the proof is the same with the role of $b$ and $c$ reversed (whose symmetry is guaranteed by Proposition 2.10).

(iv) Assume $a=2$, where $b$ and $c$ are odd.

By Lemma 8.2 (i), we have

$$\begin{aligned}s(-1) &= \frac{b+c+4}{b+c}mb(2,b,c;q)\Big|_{q=-1}\\ &= \frac{b+c+4}{b+c}\binom{\frac{b+c}{2}}{\frac{b-1}{2}}\binom{\frac{b+c}{2}}{\frac{b+1}{2}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&mb(2,b-1,c;q)|_{q=-1}+mb(2,b,c-1;q)|_{q=-1}+mb\left(2,\frac{b-1}{2},\frac{c-1}{2}\right)\Big|_{q=1}\\ =\;& \binom{\frac{b+c-2}{2}}{\frac{b-1}{2}}\binom{\frac{b+c}{2}}{\frac{b-1}{2}}+\binom{\frac{b+c-2}{2}}{\frac{b-1}{2}}\binom{\frac{b+c}{2}}{\frac{b+1}{2}}+\binom{\frac{b+c-2}{2}}{\frac{b-1}{2}}\binom{\frac{b+c}{2}}{\frac{b+1}{2}}\\ =\;& \frac{b+c+4}{b+c}\binom{\frac{b+c}{2}}{\frac{b-1}{2}}\binom{\frac{b+c}{2}}{\frac{b+1}{2}}.\end{aligned}$$

Thus, we have finished our proof.

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