G-PARKING FUNCTIONS AND THE TUTTE POLYNOMIAL

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Abstract. It was observed by Postnikov and Shapiro that the work of several authors implies the following relation between the Tutte Polynomial of a graph G and the set $\mathrm{Park}(G)$ of G-parking functions -

$$T_G(1,y) = \sum_{f \in \mathrm{Park}(G)} y^{|E|-|V|+1-\sum_{v\neq v_0} f(v)}$$

In this paper we give a quick inductive proof of this relationship, similar to one presented by Merino Lopez for an equivalent result on chip-firing games. Our proof avoids any mention of chip-firing games.

1. Introduction and background

Postnikov and Shapiro [1], building on work of Cori, Rossin and Salvy [2], defined G-parking functions of a graph, as we now explain. Consider an undirected, connected graph G on vertices $V = \{v_0, v_1, ..., v_n\}$ with edge set $|E|$. The v_0 vertex is also referred to as the *ground vertex.* A parking function is defined by $f := (f(v_1), ..., f(v_n))$, where $f(v)$ is a non-negative integer assigned to v with the following restrictions. For any non-empty subset A of $\{v_1, ..., v_n\}$, define $\mathrm{outdeg}_A(v)$ to be the number of edges having v as one endpoint and the other endpoint in V - A. Then f is a G-parking function if, for every subset A, at least one vertex $v \in A$ has $f(v) < \mathrm{outdeg}_A(v)$. A vertex with this property will be called a *witness* for A in f. A conjecture of Biggs on chip-firing games, proven by Merino Lopez [4], when combined with a result of Dhar [3], implies the following theorem; see Postnikov and Shapiro [1].

Theorem 1. *(Dhar* [3] *and Merino Lopez* [4]*)*

$$T_G(1,y) = \sum_{f \in \mathrm{Park}(G)} y^{\mathrm{wt}_G(f)}$$

where $\mathrm{wt}_G(f) := |E| - |V| + 1 - \sum_{v\neq v_0} f(v)$.

For the purposes of the proof of the theorem, let $GP(G, y)$ denote the right hand side, that is,

$$GP(G,y) := \sum_{f \in \mathrm{Park}(G)} y^{\mathrm{wt}_G(f)}.$$

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2. Proof by induction on $|E|$

To prove the theorem, we will show $GP(G, y)$ obeys the deletion-contraction recurrence and has the same base cases as $T_G(1, y)$. Starting with a graph G, we will assume a few things. First of all, the edge e picked for deletion-contraction must be a non-loop connected to the ground vertex v_0. One endpoint will be the ground, and the other we will call u. The case where e is a loop is easy and will be dealt with later.

We will prove that each G-parking function induces another in either G/e or $G - e$. We need to divide the G-parking functions up into two groups - ones that correspond to each of these two graphs. One group will contain all parking functions where $f(u) = 0$, and the other group will have what's left. We claim that the first group will biject to parking functions in G/e, and the second will biject to $G - e$.

Lemma 2. *Let e be a non-loop edge having endpoints v_0, u. Then restriction of f from $V - \{v_0\}$ to $V - \{u, v_0\}$ gives a bijection between*

$$\{f \in \mathrm{Park}(G) : f(u) = 0\} \textit{ and } \mathrm{Park}(G/e)$$

Furthermore, $\mathrm{wt}_G(f) = \mathrm{wt}_{G/e}(f|_{V-\{u,v_0\}})$.

Proof. We will begin by assuming $f \in \mathrm{Park}(G)$, and we want to show $f|_{V-\{u,v_0\}} \in \mathrm{Park}(G/e)$. Note that G/e has vertices given by $V - \{u, v_0\}$. We must exhibit, for each subset $A \subset V - \{u, v_0\}$, a witness for A in $f|_{V-\{u,v_0\}}$. As $\mathrm{outdeg}_A(v)$ is the same in G or G/e for $v \in V - \{u, v_0\}$, the witness v for A in f will do the trick.

Next we assume that $f|_{V-\{u,v_0\}} \in \mathrm{Park}(G/e)$ and we want to show $f \in \mathrm{Park}(G)$. Since $A \subset V - \{u, v_0\}$, we must exhibit a witness for A in f. There are 2 cases to consider:

Case 1: $u \in A$

In this case, since $f(u) = 0$ and $\mathrm{outdeg}_A(u) \geq 1$ because of the edge $\{v_0, u\}$, one can use u as a witness.

Case 2: $u \notin A$

The witness v for A in $f|_{V-\{u,v_0\}}$ will do the trick as in the previous paragraph

The reader is welcome to check that $\mathrm{wt}_G(f) = \mathrm{wt}_{G/e}(f|_{V-\{u,v_0\}})$. The important notes are that $|E(G)| = |E(G/e)| + 1$, and $|V(G)| = |V(G/e)| + 1$. Also, since $f(u) = 0$ in both cases, the sum of all $f(v)$ is the same in both graphs. □

Lemma 3. *Let e be a non-loop, non-isthmus edge having endpoints v_0, u. Then the map sending f to f^- defined by*

$$f^-(v) = \begin{cases} f(v) & \textit{if } v \neq u \\ f(u) - 1 & \textit{if } v = u, \end{cases}$$

gives a bijection between

$$\{f \in \mathrm{Park}(g) : f(u) > 0\} \textit{ and } \mathrm{Park}(G - e)$$

Furthermore, $\mathrm{wt}_G(f) = \mathrm{wt}_{G-e}(f^-)$.

Proof. We must show that $f \in \mathrm{Park}(G)$ if and only if $f^- \in \mathrm{Park}(G - e)$. Given a subset $A \subset V - \{v_0\}$, note that for any $v \neq u$ in A, $\mathrm{outdeg}_A(v)$ is the same in G as in $G - e$, and $f(v) = f^-(v)$. If u is in A, then $\mathrm{outdeg}_A(u)$ is one less in $G - e$ than in G, but also $f^-(u) = f(u) - 1$. Thus a witness for A in f will work as a witness for A in f^-, and vice versa.

It is easy to show how $\mathrm{wt}_G(f) = \mathrm{wt}_{G-e}(f^-)$ since $|E(G)| = |E(G-e)| + 1$, $|V(G)| = |V(G-e)|$, and $\sum_{v \neq v_0} f(v)$ will be one lower in $G - e$ by design. This is a very simple thing for the reader to check. □

Lemmas 2 and 3 imply that

$$GP(G, y) = GP(G/e, y) + GP(G - e, y)$$

if e is neither a loop nor an isthmus, and

$$GP(G, y) = GP(G/e, y)$$

if e is an isthmus. Since $T_G(1, y)$ is easily checked to satisfy the same recurrences, we are reduced to the special case where e is a loop.

The one special case is when there is a loop attached to the ground. In this case, the loop is in no way affecting $\mathrm{outdeg}_A(v)$ for any v, A, and thus can be removed without affecting the parking functions. However $\mathrm{wt}_G(f) = \mathrm{wt}_{G-e}(f) + 1$. Thus, when e is a loop,

$$GP(G, y) = yGP(G - e, y)$$

It is easy for the reader to check that $T_G(1, y) = yT_{G-e}(1, y)$ in this case.

Now we must check the base cases where $|E| = 1$, so G is a single loop or a single isthmus. There is only one parking function for the loop and the isthmus, and the reader is invited to check that $T_G(1, y) = GP(G, y)$ in these cases.

References

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