CHAPTER 5

The Plactic Monoid

5.0. Introduction

Young tableaux have had a long history since their introduction by A. Young at the turn of the century. It is only in the sixties that came to the fore a monoid structure on them, a structure taking into account most of their combinatorial properties, and having applications to the different fields in which Young tableaux were used.

Summarizing what had been his motivation to spend so much time on the plactic monoid, M.P. Schützenberger detached three reasons: (1) it allows to embed the ring of symmetric polynomials into a noncommutative ring; (2) it is the syntactic monoid of a function on words generalizing the maximal length of a nonincreasing subword; (3) it is a natural generalization to alphabets with more than two letters of the monoid of parentheses.

The starting point of the theory is an algorithm, due to C. Schensted, for the determination of the maximal length of a nondecreasing subword of a given word. The output of this algorithm is a tableau, and if one decides to identify the words leading to the same tableau, one arrives at the plactic monoid, whose defining relations were determined by D. Knuth.

The first significant application of the plactic monoid was to provide a complete proof of the Littlewood-Richardson rule, a combinatorial algorithm for multiplying Schur functions (or equivalently, to decompose tensor products of representations of unitary groups, a fundamental issue in many applications, e.g., in particle physics), which had been in use for almost 50 years before being fully understood. In fact, as will be shown in Section 5.4, the algebra of Schur functions can be lifted to the plactic algebra, and even to the free associative algebra. Once this crucial step is realized, all the proofs become straightforward.

Subsequent applications, also connected with group theory, physics and geometry, include a combinatorial description of the Kostka-Foulkes polynomials, which arise as entries of the character table of the finite linear groups $\mathrm{GL}_n(\mathbf{F}_q)$, as Poincaré polynomials of certain algebraic varieties, or in the solution of certain lattice models in statistical mechanics. One can also mention a noncommutative version of the Demazure character formula, and the construction of keys, leading to a better understanding of the standard bases of Lakshmibai and Seshadri, and to a combinatorial description of the Schubert polynomials.

Quite recently, the combinatorics of Young tableaux has been illuminated by the theory of quantum groups, and especially by Kashiwara's theory of crystal bases. Roughly speaking, quantum groups are deformations depending on a parameter q of certain algebras classically associated with a Lie group G, which give back the classical object for $q = 1$. With some care, it is possible to take the limit $q \to 0$ in certain formulas, and to recover in this way classical bijections such as the Robinson-Schensted correspondence.

From a group-theoretic point of view, the combinatorics of Young tableaux is associated with root systems of type A. By means of quantum groups, it is now possible to define plactic monoids for other root systems, and to use them for describing the corresponding Littlewood-Richardson rules. There is also a similar construction taking into account the combinatorics of quasi-symmetric functions (the hypoplactic monoid).

Conventions. In this chapter, A will denote a totally ordered alphabet of n letters $a_1 < a_2 < \ldots < a_n$. In the examples, we shall usually take $A = \{1, 2, \ldots, n\}$.

5.1. Schensted's algorithm

Consider the following problem: given a word $w \in A^*$ on the totally ordered alphabet A, find the length of the longest nondecreasing subwords of w.

C. Schensted has given an elegant algorithmic solution, which does not require the actual determination of a maximal nondecreasing subword. His method relies on the notion of *Young tableau*, a combinatorial structure issued from group theory.

A nondecreasing word $v \in A^*$ is called a *row*. Let $u = x_1 \cdots x_r$ and $v = y_1 \cdots y_s$ be two rows ($x_i, y_j \in A$). We say that u *dominates* v ($u \triangleright v$) if $r \leq s$ and for $i = 1, \ldots, r$, $x_i > y_i$. Clearly, every word w has a unique factorization $w = u_1 \cdots u_k$ as a product of rows of maximal length. A *tableau* is a word w such that $u_1 \triangleright u_2 \triangleright \ldots \triangleright u_k$. It is customary to think of tableaux as planar objects and to represent w as the left justified superposition of its rows. For instance, taking $A = \{1 < 2 < \ldots\}$,

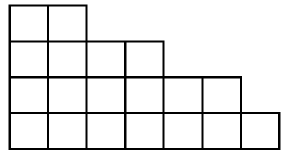
$$t = 68\ 4556\ 223357\ 1112444$$

is a tableau whose planar representation is

6	8					
4	5	5	6			
2	2	3	3	5	7	
1	1	1	2	4	4	4

Similarly, a strictly decreasing word is called a *column*. Reading from bottom to top the lengths of the rows of a tableau t, one obtains a nonincreasing sequence

$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ which is called the *shape* of t. Such a sequence is called a *partition* of the integer $|\lambda| = \lambda_1 + \cdots + \lambda_k$. On our example, $\lambda = (7, 6, 4, 2)$. The graphical representation of a partition by a planar diagram of boxes is called its *Ferrers* (or *Young*) *diagram*. Thus, the Ferrers diagram of $(7, 6, 4, 2)$ is



The *conjugate partition* λ' of λ is obtained by reading the heights of the columns of the diagram of λ. For example, the conjugate partition of $(7, 6, 4, 2)$ is $(4, 4, 3, 3, 2, 2, 1)$.

Schensted's algorithm associates to each $w \in A^*$ a tableau $t = P(w)$. The elementary step of the algorithm consists in the insertion of a letter into a row. Given a row $v = y_1 \cdots y_s$ and a letter x, the insertion of x into v is $P(vx) = vx$ if vx is a row, and $P(vx) = y_i v'$ otherwise, where y_i is the leftmost letter of v which is strictly greater that x, and v' is obtained from v through replacing y_i by x. To insert a letter x into a tableau $t = v_1 \cdots v_k$, one first inserts x into the bottom row v_k. Then, if $v_k x$ is not a row, $P(v_k x) = y v'_k$ and one inserts y into v_{k-1}, and so on. The process terminates when one reaches the top row v_1, or when a letter has been inserted at the right end of a row. For example, the insertion of 3 in the tableau t above goes through the following steps:

$$\begin{aligned}
P(1112444 \cdot 3) &= 4 \cdot 1112344\,, \\
P(223357 \cdot 4) &= 5 \cdot 223347\,, \\
P(4556 \cdot 5) &= 6 \cdot 4555\,, \\
P(68 \cdot 6) &= 8 \cdot 66\,,
\end{aligned}$$

and the result is

$$P(t \cdot 3) = 8 \cdot 66 \cdot 4555 \cdot 223347 \cdot 1112344\,.$$

In a more formal way, the map P is defined recursively by

$$P(tx) = \begin{cases} tx & \text{if } v_k x \text{ is a row} \\ P(v_1 \cdots v_{k-1} y) v'_k & \text{if } P(v_k x) = y v'_k \end{cases}$$

for a tableau t with row decomposition $t = v_1 \cdots v_k$, and for an arbitrary word $w \in A^*$, $P(wx) = P(P(w)x)$.

As an example of the general case, the successive steps of the calculation of $P(132541)$ are

$$\begin{array}{|c|}\hline 1 \\ \hline\end{array}
\qquad
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}
\qquad
\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}
\qquad
\begin{array}{|c|cc}\cline{1-1} 3 & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\ \hline\end{array}
\qquad
\begin{array}{|c|c|c}\cline{1-2} 3 & 5 & \\ \hline 1 & 2 & \multicolumn{1}{c|}{4} \\ \hline\end{array}
\qquad
\begin{array}{|c|c|c}\cline{1-1} 3 & \multicolumn{1}{c}{} & \\ \cline{1-2} 2 & 5 & \\ \hline 1 & 1 & \multicolumn{1}{c|}{4} \\ \hline\end{array}$$

THEOREM 5.1.1. *The maximal length of a nondecreasing subword of w is equal to the length of the bottom row of $P(w)$.*

Similarly, the maximal length of a decreasing subword of w is equal to the height of the first column of $P(w)$.

For example, the maximal nondecreasing subwords of $w = 132541$ are 125, 124, 135 and 134. Note that 114, the bottom row of $P(w)$ is not a subword of w.

Schensted's theorem will be proved in the forthcoming section. Actually, we will prove a more general result due to C. Greene, which gives an interpretation of the lengths of all rows and the heights of all columns of $P(w)$.

5.2. Greene's invariants and the plactic monoid

For $w \in A^*$, let $l_k(w)$ be the maximum of the sum of the lengths of k disjoint nondecreasing subwords of w. Similarly, let $l'_k(w)$ be the maximum of the sum of the lengths of k decreasing subwords of w.

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the shape of $P(w)$, and let $\lambda' = (\lambda'_1, \ldots, \lambda'_s)$ be the conjugate partition.

THEOREM 5.2.1. *For $k = 1, \ldots, r$, $\lambda_k = l_k(w) - l_{k-1}(w)$, and for $k = 1, \ldots, s$, $\lambda'_k = l'_k(w) - l'_{k-1}(w)$ (where $l_0(w) = l'_0(w) = 0$).*

To prove this theorem, it is natural to investigate the relationship between two words having the same Schensted tableau. Therefore, we introduce an equivalence relation $\sim$ on A^* defined by

$$u \sim v \Longleftrightarrow P(u) = P(v)\,.$$

For words of length ≤ 2, one has $u \sim v \Leftrightarrow u = v$, since each such word is either a row or a column. The first nontrivial relations occur in length 3, and come from the tableaux of shape $(2,1)$. With three letters $x < y < z$ we have four non monotonic words whose P-symbols are

$$P(xzy) = P(zxy) = \begin{array}{|c|c|}\cline{1-1} z & \multicolumn{1}{c}{} \\ \hline x & y \\ \hline \end{array}\,, \quad P(yzx) = P(yxz) = \begin{array}{|c|c|}\cline{1-1} y & \multicolumn{1}{c}{} \\ \hline x & z \\ \hline \end{array}\,, \tag{5.2.1}$$

and similarly, with two distinct letters $x < y$

$$P(xyx) = P(yxx) = \begin{array}{|c|c|}\cline{1-1} y & \multicolumn{1}{c}{} \\ \hline x & x \\ \hline \end{array}\,, \quad P(yxy) = P(yyx) = \begin{array}{|c|c|}\cline{1-1} y & \multicolumn{1}{c}{} \\ \hline x & y \\ \hline \end{array}\,. \tag{5.2.2}$$

We will prove in the sequel that $\sim$ is in fact the congruence on A^* generated by the relations implied by (5.2.1), (5.2.2). It is the quotient of the free monoid by these relations that will be the main object of this chapter.

DEFINITION 5.2.2. The *plactic monoid* on the alphabet A is the quotient $\mathrm{Pl}\,(A) = A^*/\equiv$, where $\equiv$ is the congruence generated by the *Knuth relations*

$$xzy \equiv zxy \quad (x \le y < z)\,, \tag{5.2.3}$$
$$yxz \equiv yzx \quad (x < y \le z)\,. \tag{5.2.4}$$

The first step in proving Greene's theorem is

PROPOSITION 5.2.3. *Every word is congruent to its Schensted tableau, that is,*

$$w \equiv P(w)\,.$$

Proof. By definition of $\equiv$, the proposition is true for $|w| \le 3$. We proceed by induction on $|w|$. Assume that for a word w we have $P(w) \equiv w$, and let x be a letter. We have to show that $P(wx) \equiv wx$, or equivalently $P(wx) \equiv P(w) \cdot x$. The definition of the map P allows us to reduce this verification to the case where w is a row. Assuming this, if wx is a row then $P(wx) = wx$, and otherwise, $P(wx) = yw'$ where y is the leftmost letter in w which is $> x$, and w' is obtained from w by replacing y by x. Then, writing $w = uyv$, we have $wx \equiv uyxv$ by a sequence of applications of (5.2.4), and $uyxv \equiv yuxv$ by a sequence of applications of (5.2.3). ∎

Next, we show that

PROPOSITION 5.2.4. *If $w \equiv w'$, then $l_k(w) = l_k(w')$ for all k.*

Proof. We can assume that w' is obtained from w by a single Knuth transformation. Let us write, for instance,

$$w = uxzyv\,, \qquad w' = uzxyv \qquad (x \le y < z)\,.$$

Clearly, all nondecreasing subwords of w' are also subwords of w. Hence, $l_k(w) \ge l_k(w')$. Conversely, let $(w_1, \ldots, w_k)$ be a k-tuple of disjoint nondecreasing subwords of w. Then, w_i is also a subword of w', unless $w_i = u'xzv'$, where u' and v' are subwords of u and v. If y does not occur in any of the remaining w_j, then w_i can be replaced by $w_i' = u'xyv'$, which is a nondecreasing subword of w'. Otherwise, if some $w_j = u''yv''$, then, one replaces the pair (w_i, w_j) by $w_i' = u'xyv''$ and $w_j' = u''zv'$. The case of a Knuth transformation of type (5.2.4) is similar. Therefore, we have $l_k(w) \le l_k(w')$. ∎

Thus the integers $l_k(w)$ are not modified by Knuth's transformations (5.2.3) (5.2.4). They are called *Greene's plactic invariants*. Two other important plactic invariants, the charge and cocharge, will be studied in Section 5.6.

Proof of Theorem 5.2.1. Using Propositions 5.2.3 and 5.2.4, the only thing to prove is that for a tableau t of shape λ, $l_k(t) = \lambda_1 + \cdots + \lambda_k$. Taking for $w_1, \ldots, w_k$ the k longest rows of t, we see that $l_k(t) \ge \lambda_1 + \cdots + \lambda_k$. Conversely, a nondecreasing subword w of t uses at most one letter from each column of the planar representation of t, therefore k disjoint nondecreasing subwords can use at most $\lambda_1 + \cdots + \lambda_k$ letters of t. ∎

We are now in a position to prove the cross-section theorem:

THEOREM 5.2.5. *The equivalence $\sim$ coincides with the plactic congruence. In particular, each plactic class contains exactly one tableau.*

Proof. Let us assume that $w \sim w'$. Then, by Proposition 5.2.3,

$$w \equiv P(w) = P(w') \equiv w' .$$

Conversely, suppose that $w \equiv w'$. Then, from Proposition 5.2.4 and Theorem 5.2.1 we see that $P(w)$ and $P(w')$ have the same shape. Now, let z be the greatest letter of w and w', and write $w = uzv$, $w' = u'zv'$, where z does not occur neither in v nor in v'. Then, we claim that $uv \equiv u'v'$. Indeed, we can assume that w and w' differ by a single Knuth transformation. If z is not involved in this transformation, then either $u \equiv u'$ and $v = v'$, or $u = u'$ and $v \equiv v'$. And if z is involved, erasing z in (5.2.3) or (5.2.4) leaves us with $xy = xy$ or $yx = yx$, so that $uv = u'v'$.

By induction on the length of w, we can assume that $P(uv) = P(u'v')$. From the description of Schensted's algorithm, since z is the greatest letter, it is clear that after erasing z in $P(uzv)$, one is left with $P(uv)$. Therefore, $P(w)$ is obtained from $P(uv)$ by adding a box z at a place imposed by the shape of $P(w)$, and since the same is true for w', we conclude that $P(w) = P(w')$. ∎

5.3. The Robinson-Schensted-Knuth correspondence

We have seen in the preceding section that the set $\mathrm{Tab}\,(A)$ of all tableaux over the alphabet A is a cross-section of the canonical projection $\pi: A^* \to \mathrm{Pl}\,(A) = A^*/\equiv$. It is now a natural question to investigate the structure of the plactic classes $\pi^{-1}(t)$, $t \in \mathrm{Tab}\,(A)$. As we will see, the elements of $\pi^{-1}(t)$ are also parametrized by certain tableaux.

Let us say that a tableau is *standard* if its entries are the integers $1, 2, \ldots, n$, each of them occurring exactly once. The set of standard tableaux is denoted by STab. For a partition λ, we denote by $\mathrm{Tab}\,(\lambda, A)$ (resp. $\mathrm{STab}\,(\lambda)$) the set of tableaux over A (resp. of standard tableaux) of shape λ.

By keeping track of the successive steps of the insertion algorithm, one can define a map $Q: A^* \to \mathrm{STab}$ such that $w \mapsto (P(w), Q(w))$ is one-to-one. More precisely, let $w = y_1 \cdots y_m$. Observe that a standard tableau t is nothing but a chain of partitions $\lambda^{(1)} \subset \lambda^{(2)} \subset \ldots \subset \lambda^{(m)}$ such that the diagram of $\lambda^{(i+1)}$ is obtained from that of $\lambda^{(i)}$ by adding one box, which is the one labelled $i+1$ in t. Now, $Q(w)$ is by definition the standard tableau encoding the chain of shapes of $P(y_1), P(y_1y_2), \ldots, P(w)$. For example, the chain of insertions seen above gives

$$Q(132541) = \begin{array}{|c|c|c|}\cline{1-1} 6 \\ \cline{1-2} 3 & 5 \\ \hline 1 & 2 & 4 \\ \hline\end{array}\,.$$

Clearly, $Q(w)$ has the same shape as $P(w)$.

THEOREM 5.3.1. *The map*

$$\begin{aligned}\rho : A^* &\longrightarrow \textstyle\coprod_\lambda \operatorname{Tab}(\lambda, A) \times \operatorname{STab}(\lambda) \\ w &\longmapsto \qquad (P(w)\,,\,Q(w))\end{aligned}$$

is a bijection, called the Robinson-Schensted correspondence.

Proof. The inverse map ρ^{-1} can be explicitly constructed. The idea is that, given a row v and a letter y, there exists a unique row v' and letter x such that $yv \equiv v'x$. This shows that the insertion process described in Section 5.1 can be reversed, provided that one specifies the box to be erased. Given a pair $(t, t') \in \operatorname{Tab}(\lambda, A) \times \operatorname{STab}(\lambda)$, one constructs $w = \rho^{-1}(t, t')$ by deleting successively in t the boxes labelled $n, n-1, \ldots, 1$ in t'. ∎

COROLLARY 5.3.2. *Q induces a bijection between the plactic class of each tableau t and* $\operatorname{STab}(\lambda)$, *where λ is the shape of t. In particular, the cardinality of the class of t is equal to*

$$f_\lambda := |\operatorname{STab}(\lambda)|\,.$$

Restricting ρ to the set of standard words on $A = \{1, 2, \ldots, n\}$, which can be identified with the symmetric group $\mathfrak{S}_n$, one obtains a bijection

$$\mathfrak{S}_n \longleftrightarrow \coprod_\lambda \operatorname{STab}(\lambda) \times \operatorname{STab}(\lambda)\,. \tag{5.3.1}$$

It provides in particular a bijective proof of an identity of Frobenius:

$$n! = \sum_{|\lambda|=n} f_\lambda^2\,,$$

a special case of the fact that the cardinality of a finite group is equal to the sum of the squares of the dimensions of its irreducible representations (over $\mathbb{C}$).

As shown by the next theorem, there is some compatibility between the Robinson-Schensted map and the group structure of $\mathfrak{S}_n$.

THEOREM 5.3.3. *For $\sigma \in \mathfrak{S}_n$, $Q(\sigma) = P(\sigma^{-1})$.*

The original proof of Schützenberger proceeded by induction on n. We give below a simple derivation based on Greene's theorem.

To this aim, it will be convenient to represent a permutation σ by a *biword* (or word in *biletters*, that is, pairs of letters $(a, b) \in A \times B$ in the product of two alphabets, denoted here for convenience by $\begin{bmatrix} a \\ b \end{bmatrix}$).

$$\sigma \leftrightarrow \begin{bmatrix} i_1 & \dots & i_n \\ j_1 & \dots & j_n \end{bmatrix}$$

where each $j_k = \sigma(i_k)$. Among the biwords representing σ, we have two distinguished ones $\begin{bmatrix} \mathrm{id} \\ \sigma \end{bmatrix}$ and $\begin{bmatrix} \sigma^{-1} \\ \mathrm{id} \end{bmatrix}$, which are obtained by sorting one of them using the lexicographic order on biletters with priority on the top or bottom row.

More generally, for a biword $\begin{bmatrix} u \\ v \end{bmatrix}$ where $u, v \in A^*$ are not necessarily standard, we denote by $\begin{bmatrix} u' \\ v' \end{bmatrix}$ the nondecreasing rearrangement of $\begin{bmatrix} u \\ v \end{bmatrix}$ for the lexicographic order with priority on the top row, and by $\begin{bmatrix} u'' \\ v'' \end{bmatrix}$ the nondecreasing rearrangement for the lexicographic order with priority on the bottom row. Thus, for

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 21335424 \\ 13652414 \end{bmatrix},$$

we have

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 12233445 \\ 31156442 \end{bmatrix} \text{ and } \begin{bmatrix} u'' \\ v'' \end{bmatrix} = \begin{bmatrix} 22514433 \\ 11234456 \end{bmatrix}.$$

The crucial property is the following:

LEMMA 5.3.4. *For any biword* $\begin{bmatrix} u \\ v \end{bmatrix}$, *the tableaux* $P(v')$ *and* $P(u'')$ *have the same shape.*

Proof. Let $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_1 \cdots u_m \\ v_1 \cdots v_m \end{bmatrix}$ and consider a nondecreasing subword $\beta = v_{i_1} \cdots v_{i_r}$ of v'. Then, by definition of $\begin{bmatrix} u' \\ v' \end{bmatrix}$, $\alpha = u_{i_1} \cdots u_{i_r}$ is also nondecreasing, and

$$\begin{bmatrix} u_{i_1} \\ v_{i_1} \end{bmatrix} \leq \dots \leq \begin{bmatrix} u_{i_r} \\ v_{i_r} \end{bmatrix}$$

for *both* lexicographic orders. Therefore, α is also a nondecreasing subword of u''. From this remark, we see that there is a bijection between the k-tuples of disjoint nondecreasing subwords of v' and those of u''. By Theorem 5.2.1 the conclusion follows. ∎

Proof of Theorem 5.3.3. Let $\sigma \in \mathfrak{S}_n$ and $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \mathrm{id} \\ \sigma \end{bmatrix}$, $\begin{bmatrix} u'' \\ v'' \end{bmatrix} = \begin{bmatrix} \sigma^{-1} \\ \mathrm{id} \end{bmatrix}$. The left factors of σ are encoded by the biwords

$$\begin{bmatrix} u(k)' \\ v(k)' \end{bmatrix} = \begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma_1 & \sigma_2 & \cdots & \sigma_k \end{bmatrix}$$

for which we have

$$\begin{bmatrix} u(k)'' \\ v(k)'' \end{bmatrix} = \begin{bmatrix} \sigma^{-1}|_{[1,k]} \\ (\sigma_1 \cdots \sigma_k) \uparrow \end{bmatrix}$$

where $(\sigma_1 \cdots \sigma_k) \uparrow$ is the increasing rearrangement of the left factor $\sigma_1 \cdots \sigma_k$, and for a word $w \in A^*$ and a subset B of A, $w|_B$ denotes the subword of w obtained by erasing the letters which are not in B. From Lemma 5.3.4, at each step of the insertion algorithm, we have that $P(\sigma_1 \cdots \sigma_k)$ and $P(\sigma^{-1}|_{[1,k]})$ have the same shape. So at the end, $P(\sigma^{-1}) = Q(\sigma)$. ■

In fact, Theorem 5.3.3 can be readily generalized to give a similar result for the insertion tableau $Q(w)$ of an arbitrary word $w \in A^*$. To do this, we need the notion of *standardization*.

Let $x_1 < x_2 < \ldots < x_r$ be the letters occurring in w, with respective multiplicities $m_1, \ldots, m_r$. By labelling from 1 to m_1 the occurrences of x_1, reading from left to right, then from $m_1 + 1$ to $m_1 + m_2$ the occurrences of x_2, and so on, we get a standard word, denoted by $\mathrm{std}\,(w)$. For example

$$\mathrm{std}\,(31156442) = 41278563\,.$$

This defines in particular the standardization of a tableau. It is immediate to check from Knuth's relations that

LEMMA 5.3.5. *If* $w \equiv w'$, *then* $\mathrm{std}\,(w) \equiv \mathrm{std}\,(w')$. *In particular,* $P(\mathrm{std}\,(w)) = \mathrm{std}\,(P(w))$. ■

It is also clear from the description of the Robinson-Schensted algorithm that

LEMMA 5.3.6. $Q(w) = Q(\mathrm{std}\,(w))$. ■

We can now state:

COROLLARY 5.3.7. *For any* $w \in A^*$, $Q(w) = P(\mathrm{std}\,(w)^{-1})$.

Proof. By Theorem 5.3.3, $P(\mathrm{std}\,(w)^{-1}) = Q(\mathrm{std}\,(w))$, which is equal to $Q(w)$ by Lemma 5.3.6. ■

In the Robinson-Schensted correspondence for non standard words, there is a dissymmetry between the left tableau $P(w)$ and the right tableau $Q(w)$. Lemma 5.3.4 shows the way to restore the symmetry, by extending the correspondence to commutative classes of biwords, *i.e.* monomials in commutative biletters

$\binom{x}{y}$. Given two words $u = u_1 \dots u_m$ and $v = v_1 \dots v_m$ of the same length, we denote by $\binom{u}{v} = \binom{u_1}{v_1} \cdots \binom{u_m}{v_m}$ the associated monomial in commutative biletters (not to be confused with the biword $\begin{bmatrix} u \\ v \end{bmatrix}$).

Definition 5.3.8. Let $\binom{u}{v}$ be a monomial, and $\begin{bmatrix} u' \\ v' \end{bmatrix}$, $\begin{bmatrix} u'' \\ v'' \end{bmatrix}$ be the two biwords associated as above to the biword $\begin{bmatrix} u \\ v \end{bmatrix}$. The *Knuth correspondence* κ is defined by

$$\kappa \binom{u}{v} = (P(v'), P(u'')) .$$

By corollary 5.3.7, we recover the Robinson-Schensted correspondence by encoding $w = y_1 \cdots y_m$ as the monomial $\binom{1}{y_1} \cdots \binom{m}{y_m}$. By Lemma 5.3.4, we know that $P(v')$ and $P(u'')$ have the same shape. It will follow from the alternative description given below that κ is a bijection between monomials in biletters and pairs of tableaux of the same shape. Recall that the *evaluation* of a word is the vector $\mathrm{ev}\,(w) = (|w|_{a_1}, |w|_{a_2}, \dots, |w|_{a_n})$, where $A = \{a_1, \dots, a_n\}$.

Proposition 5.3.9. *$P(u'')$ is the unique tableau of evaluation $\mathrm{ev}\,(u'')$ such that $\mathrm{std}\,(P(u'')) = Q(v')$.*

Proof. By lexicographic sorting of $\begin{bmatrix} \mathrm{std}\,(u) \\ \mathrm{std}\,(v) \end{bmatrix}$ we have $(\mathrm{std}\,(v)')^{-1} = \mathrm{std}\,(u)''$. Since lexicographic sorting obviously commutes with standardization, it follows that $(\mathrm{std}\,(v'))^{-1} = \mathrm{std}\,(u'')$. Hence,

$$\begin{aligned} Q(v') &= P((\mathrm{std}\,(v')^{-1}) \quad \text{(Corollary 5.3.7)} \\ &= P(\mathrm{std}\,(u'')) \\ &= \mathrm{std}\,(P(u'')) \quad \text{(Lemma 5.3.5)} . \end{aligned}$$

∎

Therefore, to compute the inverse image of a pair of tableaux (t, t') under the Knuth correspondence, we can apply the inverse Robinson-Schensted map to $(t, \mathrm{std}\,(t'))$ to get $v' = \rho^{-1}(t, \mathrm{std}\,(t'))$. Then, $\kappa^{-1}(t, t') = \binom{t' \uparrow}{v'}$.

Note that the symmetry

$$\kappa \binom{u}{v} = (t, t') \Longleftrightarrow \kappa \binom{v}{u} = (t', t) ,$$

which generalizes Theorem 5.3.3 is incorporated in the definition of κ. In particular, taking $t' = t$, κ establishes a bijection between $\mathrm{Tab}\,(A)$ and the set of *symmetric* monomials in biletters, *i.e.* those such that $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u \end{pmatrix}$ (which amounts to say that for any $x, y \in A$, $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} y \\ x \end{pmatrix}$ occur with the same multiplicity). As an immediate consequence of this observation, we can compute the generating series of the numbers

$$d_\alpha := |\{t \in \mathrm{Tab}\,(A) \mid \mathrm{ev}\,(t) = \alpha\}| \quad (\alpha \in \mathbb{N}^A)$$

which are the cardinalities of the multihomogeneous components of the plactic monoid.

THEOREM 5.3.10. *Let $\xi_1, \xi_2, \ldots$ be commuting indeterminates. Then,*

$$\sum_{\alpha \in \mathbb{N}^A} d_\alpha \xi^\alpha = \prod_i \frac{1}{1 - \xi_i} \prod_{i<j} \frac{1}{1 - \xi_i \xi_j}\,.$$

Proof. The commutative image $\underline{\underline{t}}$ of a tableau t under $a_i \mapsto \xi_i$ is obtained from $\begin{pmatrix} u \\ v \end{pmatrix} = \kappa^{-1}(t, t)$ by mapping each biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ to $(\xi_i \xi_j)^{1/2}$. Now, the generating series of all symmetric monomials in biletters is clearly

$$\prod_i \frac{1}{1 - \begin{pmatrix} i \\ i \end{pmatrix}} \prod_{i<j} \frac{1}{1 - \begin{pmatrix} i \\ j \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix}}\,.$$

■

COROLLARY 5.3.11. *For $|A| = n$, the cardinality of the homogeneous component of degree k of $\mathrm{Pl}\,(A)$ is equal to the coefficient of z^k in*

$$\frac{1}{(1-z)^n} \cdot \frac{1}{(1-z^2)^{n(n-1)/2}}\,.$$

■

5.4. Schur functions and the Littlewood-Richardson rule

Let $\xi_1, \xi_2, \ldots, \xi_n$ be commuting indeterminates as in the preceding section, and retain the notation $w \mapsto \underline{\underline{w}}$ for the commutative image $a_i \mapsto \xi_i$ of a word $w \in A^*$.

DEFINITION 5.4.1. Let λ be a partition. The generating function

$$s_\lambda(\xi_1, \ldots, \xi_n) = \sum_{t \in \mathrm{Tab}\,(\lambda, A)} \underline{\underline{t}}$$

is called a *Schur function.*

Although not obvious from this definition, s_λ is a symmetric polynomial in $\xi_1, \ldots, \xi_n$ (this will be proved in Section 5.6). Most of the combinatorial constructions of Section 5.3 imply interesting and classical Schur function identities. For example, Schur's identity 5.3.10 can be rewritten as

$$\sum_\lambda s_\lambda(\xi_1, \ldots, \xi_n) = \prod_i \frac{1}{1-\xi_i} \prod_{i<j} \frac{1}{1-\xi_i\xi_j} \, .$$

From Theorem 5.3.1 we get

$$\frac{1}{1-(\xi_1 + \cdots + \xi_n)} = \sum_\lambda f_\lambda s_\lambda(\xi_1, \ldots, \xi_n) \, .$$

Indeed, the left-hand side is clearly the generating function of A^*.

Finally, from the bijectivity of Knuth's correspondence, we obtain a classical and fundamental identity which can be tracked back to Cauchy. To state it, we need a second set $\eta_1, \ldots, \eta_n$ of commuting variables. Sending the biletter $\begin{pmatrix} a_i \\ a_j \end{pmatrix}$ onto $\xi_i\eta_j$ and the pair (t, t') to the product of the commutative image of t in the variables ξ and of t' in the variables η, we get

Theorem 5.4.2.

$$\prod_{i,j} \frac{1}{1-\xi_i\eta_j} = \sum_\lambda s_\lambda(\xi) s_\lambda(\eta) \ .$$

Group theoretical arguments show that a product of Schur functions is equal to a positive sum of Schur functions:

$$s_\lambda(\xi) s_\mu(\xi) = \sum_\nu c^\nu_{\lambda\mu} s_\nu(\xi) \tag{5.4.1}$$

where $c^\nu_{\lambda\mu} \in \mathbb{N}$. The calculation of the coefficients $c^\nu_{\lambda\mu}$ is of interest in many fields. A combinatorial interpretation of these numbers implying an efficient algorithm for their computation has been given without proof by Littlewood and Richardson.

The most illuminating proof of this rule proceeds by lifting the calculus of Schur functions to the algebra $\mathbb{Z}[\mathrm{Pl}(A)]$ of the plactic monoid, introducing the *plactic Schur function*

$$S_\lambda(A) = \sum_{t \in \mathrm{Tab}\,(\lambda, A)} t \ ,$$

where tableaux are evaluated in the plactic monoid. This plactic Schur function can be seen as the projection in $\mathbb{Z}[\mathrm{Pl}(A)]$ of anyone of the *free Schur functions*

$$\mathbf{S}_t(A) = \sum_{Q(w)=t} w \ \in \mathbb{Z}\langle A \rangle$$

indexed by $t \in \mathrm{STab}\,(\lambda)$. In fact the Littlewood-Richardson rule will be deduced from a statement in the free algebra $\mathbb{Z}\langle A \rangle$.

THEOREM 5.4.3. *Let A' and A'' be two subalphabets such that $a' < a''$, for all $a \in A'$, $a'' \in A''$. For $t' \in \mathrm{Tab}\,(A')$ and $t'' \in \mathrm{Tab}\,(A'')$ we have*

$$\left(\sum_{P(w')=t'} w'\right) \sqcup\!\sqcup \left(\sum_{P(w'')=t''} w''\right) = \sum_{t\in \mathrm{Sh}\,(t',t'')} \ \sum_{P(w)=t} w$$

where $\mathrm{Sh}\,(t',t'')$ is the set of all tableaux t such that $t|_{A'} = t'$ and $P(t|_{A''}) = t''$, that is, of all tableaux t occurring in the shuffle product of t' and a word in the plactic class of t''.

Thus the shuffle of a plactic class of A' and a plactic class of A'' is a union of plactic classes of A (identifying a class and the sum of its elements). It is in fact a direct consequence of the following

LEMMA 5.4.4. *Let I be an interval of A. Then*

$$w \equiv w' \Rightarrow w|_I \equiv w'|_I$$

Proof. It is enough to check the lemma in the case when w' differs from w by a single Knuth transformation, and this amounts to the observation that erasing x or z in 5.2.3 or 5.2.4, we are left with $xy = xy$ or $yz = yz$. ∎

Proof of Theorem 5.4.3. The words occurring in the shuffle are exactly those w such that $w|_{A'} \equiv t'$ and $w|_{A''} \equiv t''$. By Lemma 5.4.4 , this set of words is saturated with respect to the plactic congruence, hence is a union of plactic classes. ∎

We can now state the plactic version of the Littlewood-Richardson rule.

THEOREM 5.4.5. *The plactic Schur functions span a commutative subalgebra of $\mathbb{Z}[\mathrm{Pl}\,(A)]$ and we have*

$$S_\lambda(A)S_\mu(A) = \sum_\nu c^\nu_{\lambda\mu} S_\nu(A) \ ,$$

where the $c^\nu_{\lambda\mu}$ are the same as in (5.4.1). In particular $c^\nu_{\lambda\mu}$ is equal to the number of factorizations in $\mathrm{Pl}\,(A)$ of any tableau $t \in \mathrm{Tab}\,(\nu, A)$ as a product $t't''$ with $t' \in \mathrm{Tab}\,(\lambda, A)$ and $t'' \in \mathrm{Tab}\,(\mu, A)$.

Proof. We first work in the free associative algebra $\mathbb{Z}\langle A\rangle$ and consider a product $\mathbf{S}_{t'}(A)\mathbf{S}_{t''}(A)$ where t', t'' are arbitrary standard tableaux of respective shapes λ and μ, with $p = |\lambda|$, $q = |\mu|$. We identify as above a word w' of length p with a monomial in commutative biletters:

$$w' = \begin{pmatrix} 1\cdots p \\ w' \end{pmatrix} .$$

Then, by reordering biletters, we can write in view of Proposition 5.3.9

$$\mathbf{S}_{t'} = \sum_{Q(w')=t'} \begin{pmatrix} 1\cdots p \\ w' \end{pmatrix} = \overrightarrow{\sum_{P(u)=t'}} \begin{pmatrix} u \\ r' \end{pmatrix} ,$$

where the notation means that the second sum is over all words u and r' such that the biword $\begin{bmatrix} u \\ r' \end{bmatrix}$ is increasing for the lexicographic order with bottom priority, and that $P(u) = t'$. Similarly, using for w'' of length q the identification

$$w'' = \begin{pmatrix} (p+1)\cdots(p+q) \\ w'' \end{pmatrix}$$

we can express $\mathbf{S}_{t''}$ as

$$\mathbf{S}_{t''} = \sum_{P(v)=t''[p]}^{\longrightarrow} \begin{pmatrix} v \\ r'' \end{pmatrix},$$

where $t''[p]$ denotes the tableau obtained from t'' by adding p to all its entries. Now sorting lexicographically (with bottom priority) any of the biwords $\begin{bmatrix} u \\ r' \end{bmatrix}\begin{bmatrix} v \\ r'' \end{bmatrix}$, one gets a biword $\begin{bmatrix} w \\ r \end{bmatrix}$ such that w occurs in $u \sqcup\!\sqcup v$. Conversely, all increasing biwords $\begin{bmatrix} w \\ r \end{bmatrix}$ such that w occurs in $u \sqcup\!\sqcup v$ arise in this way from the sorting of a unique product $\begin{bmatrix} u \\ r' \end{bmatrix}\begin{bmatrix} v \\ r'' \end{bmatrix}$ of increasing biwords. Thus, by Theorem 5.4.5,

$$\mathbf{S}_{t'}\mathbf{S}_{t''} = \sum_t \sum_{P(w)=t}^{\longrightarrow} \begin{pmatrix} w \\ r \end{pmatrix},$$

where the outer sum is over all standard tableaux t which occur in the shuffle of t' and a of a word congruent to $t''[p]$. Hence

$$\mathbf{S}_{t'}\mathbf{S}_{t''} = \sum_t \mathbf{S}_t\,, \tag{5.4.2}$$

sum over the same tableaux t, and taking the plactic image we obtain

$$S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu \tag{5.4.3}$$

where $c^\nu_{\lambda\mu}$ is the number of standard tableaux of shape ν which occur in the shuffle of t' and of a word in the class of $t''[p]$. Taking the commutative image of (5.4.3), we see that the $c^\nu_{\lambda\mu}$ are the same as in (5.4.1), which implies that the plactic Schur functions span a subalgebra of $\mathbb{Z}[\mathrm{Pl}(A)]$ isomorphic to the commutative algebra spanned by the ordinary Schur functions. Finally the interpretation of $c^\nu_{\lambda\mu}$ in terms of factorizations in $\mathrm{Pl}(A)$ follows directly from the definition of plactic Schur functions. ∎

As an illustration of (5.4.2), one can check that for

$$t' = t'' = \begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array},$$

the product $\mathbf{S}_{t'}\mathbf{S}_{t''}$ is equal to $\sum_t \mathbf{S}_t$ where t ranges over the following tableaux:

$$
\begin{array}{llll} 3&6&& \\ 1&2&4&5 \end{array}
\qquad
\begin{array}{lll} 3&4&6 \\ 1&2&5 \end{array}
\qquad
\begin{array}{llll} 6&&& \\ 3&&& \\ 1&2&4&5 \end{array}
\qquad
\begin{array}{lll} 4&& \\ 3&6& \\ 1&2&5 \end{array}
$$

$$
\begin{array}{lll} 6&& \\ 3&4& \\ 1&2&5 \end{array}
\qquad
\begin{array}{ll} 4&6 \\ 3&5 \\ 1&2 \end{array}
\qquad
\begin{array}{lll} 6&& \\ 4&& \\ 3&& \\ 1&2&5 \end{array}
\qquad
\begin{array}{ll} 6& \\ 4& \\ 3&5 \\ 1&2 \end{array}
$$

COROLLARY 5.4.6. *Let $R(\lambda, k)$ (resp. $C(\lambda, k)$) be the set of partitions whose diagram is obtained by adding k boxes to the diagram of λ, no two of them being added in the same column (resp. in the same row). Then,*

$$
\begin{aligned}
S_\lambda S_{(k)} &= \sum_{\nu \in R(\lambda,k)} S_\nu \\
S_\lambda S_{(1^k)} &= \sum_{\nu \in C(\lambda,k)} S_\nu \,.
\end{aligned}
$$

Proof. Let $m = |\lambda|$. To calculate $\mathbf{S}_t \cdot \mathbf{S}_{12\cdots k}$, we have to look for the standard tableaux in the shuffle of the plactic class of t with the one element class

$$(m+1)(m+2)\cdots(m+k)\,.$$

Clearly, these tableaux can only be obtained by dispatching at the periphery of t the letters $(m+1), \ldots, (m+k)$ from left to right and in this order, and the resulting shapes are exactly those of $R(\lambda, k)$. The second formula is proved similarly. ∎

To recover the classical formulation of Littlewood and Richardson, we need the notion of a *Yamanouchi word*. We say that w is a Yamanouchi word on $A = \{1, 2, \ldots, n\}$ if any right factor v of w satisfies $|v|_1 \geq |v|_2 \geq \ldots \geq |v|_n$.

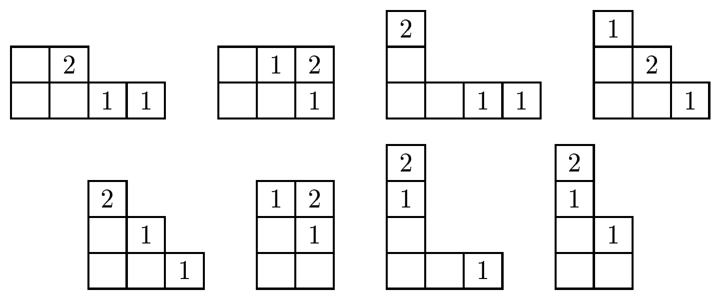
LEMMA 5.4.7. *The Yamanouchi words of a given evaluation $\mu = (\mu_1, \ldots, \mu_n)$ form a single plactic class whose representative tableau is the Yamanouchi tableau*

$$
\begin{array}{lllll} \cdots&&&& \\ 2&2&\cdots&2& \\ 1&1&\cdots&\cdots&1 \end{array}\,,
$$

that is, the unique tableau with shape and evaluation μ.

Proof. It is immediate to check that if w is a Yamanouchi word, and if w' is obtained from w by a single Knuth transformation, then w' is also a Yamanouchi word. Therefore, a plactic class which contains a Yamanouchi word contains only Yamanouchi words. Now, a tableau is a Yamanouchi word if and only if its bottom row contains only 1's, the next row contains only 2's, and so on. Hence there is a unique Yamanouchi tableau, namely, the unique tableau of shape μ and evaluation μ, and the lemma follows from Theorem 5.2.5. ∎

We can now see that the classical version of the Littlewood -Richardson rule is a direct consequence of (5.4.2). Indeed, to calculate $c^{\nu}_{\lambda\mu}$, we can choose for t' and t'' the standard tableaux of respective shapes λ and μ in which each row consists of consecutive integers. These tableaux are the standardized of the Yamanouchi tableaux of the same shapes, so that the words w'' in the plactic class of $t''[p]$ are precisely the shifted standardized of the Yamanouchi words y'' of evaluation μ. Hence, if one erases in the tableaux t the entries of t', which are irrelevant, and replaces the word w'' by the unique Yamanouchi word y'' of which it is the standardized, one obtains the classical Littewood-Richardson tableaux, i.e., the skew Yamanouchi tableaux of shape ν/λ and evaluation μ. Continuing the preceding example, one would obtain

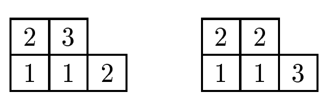


Another useful formulation of the rule is the following:

Corollary 5.4.8. *Let y_μ denote the unique Yamanouchi tableau of shape μ. Then $c^{\nu}_{\lambda\mu}$ is equal to the number of tableaux t of shape λ such that $t \cdot y_\mu$ is a Yamanouchi word of evaluation ν.*

Proof. By Theorem 5.4.5, $c^{\nu}_{\lambda\mu}$ is the number of factorizations $y_\nu = t \cdot t'$ in $\mathrm{Pl}\,(A)$, with $t \in \mathrm{Tab}\,(\lambda, A)$ and $t' \in \mathrm{Tab}\,(\mu, A)$. Equivalently, by Lemma 5.4.7, $c^{\nu}_{\lambda\mu}$ is the number of Yamanouchi words w of weight ν such that $w = t \cdot t'$ in A^*, for some $t \in \mathrm{Tab}\,(\lambda, A)$ and $t' \in \mathrm{Tab}\,(\mu, A)$. Then the right factor t' must be a Yamanouchi tableau, that is $t' = y_\mu$. ∎

For example, the coefficient $c^{(4,3,1)}_{(3,2),(2,1)}$ is equal to 2, corresponding to the following two tableaux t:



5.5. Coplactic operations

The set of words w having a given insertion tableau $t = Q(w)$ is called a *coplactic* class. In the preceding section we have seen that the sum $\mathbf{S}_t$ of the elements of a coplactic class is a pertinent lifting of a Schur function to the free algebra $\mathbb{Z}\langle A\rangle$. In this section, we show that coplactic classes can be endowed with a structure of colored graph.

We introduce linear operators e_i, f_i, σ_i, $i = 1, \ldots, n-1$, acting on $\mathbb{Z}\langle A\rangle$ in the following way. Consider first the case of the two-letters subalphabet $A_i = \{a_i, a_{i+1}\}$. Let $w = x_1 \cdots x_m$ be a word on A_i. Bracket every factor $a_{i+1}a_i$ of w. The letters which are not bracketed constitute a subword w_1 of w. Then bracket every factor $a_{i+1}a_i$ of w_1. There remains a subword w_2. Continue this procedure until it stops, giving a word w_k of type $w_k = a_i^r a_{i+1}^s = x_{j_1} \cdots x_{j_{r+s}}$. The image of w_k under e_i, f_i or σ_i is given by

$$e_i(a_i^r a_{i+1}^s) = \begin{cases} a_i^{r+1}a_{i+1}^{s-1} & (s \geq 1) \\ 0 & (s = 0) \end{cases}$$

$$f_i(a_i^r a_{i+1}^s) = \begin{cases} a_i^{r-1}a_{i+1}^{s+1} & (r \geq 1) \\ 0 & (r = 0) \end{cases}$$

$$\sigma_i(a_i^r a_{i+1}^s) = a_i^s a_{i+1}^s$$

Let $w'_k = x'_{j_1} \cdots x'_{j_{r+s}}$ denote the image of w_k. The image of the initial word w is then $w' = y_1 \cdots y_m$, where $y_i = x'_i$ if $i \in \{j_1, \ldots, j_{r+s}\}$ and $y_i = x_i$ otherwise.

For example, if $w = (a_2a_1)a_1a_1a_2(a_2a_1)a_1a_1a_1a_2$, we have

$$w_1 = a_1a_1(a_2a_1)a_1a_1a_2 \quad \text{and} \quad w_2 = a_1a_1a_1a_1a_2\,.$$

Thus,

$$e_1(w) = a_2a_1\ \underline{a_1}\underline{a_1}\ a_2a_2a_1a_1\ \underline{a_1}\underline{a_1}\underline{a_1}$$

$$f_1(w) = a_2a_1\ \underline{a_1}\underline{a_1}\ a_2a_2a_1a_1\ \underline{a_1}\underline{a_2}\underline{a_2}$$

$$\sigma_1(w) = a_2a_1\ \underline{a_1}\underline{a_2}\ a_2a_2a_1a_1\ \underline{a_2}\underline{a_2}\underline{a_2}\ ,$$

where the underlined letters are those of the subword w'_2. Finally, the general action of the operators e_i, f_i, σ_i on w is defined by the previous rules applied to the subword $w|_{A'_i}$, the other letters remaining unchanged.

THEOREM 5.5.1. *Let h be anyone of the operators e_i, f_i, σ_i.*

(i) *Let $w \in A^*$ and suppose that $h(w) \neq 0$. Then $Q(h(w)) = Q(w)$.*

(ii) *Let w' be congruent to w. Then $h(w) \equiv h(w')$.*

Proof (i) Suppose first that $A = \{a_1, a_2\}$, and let us give the proof in the case $h = f_1$. Let $w \in A^*$ be such that $f_1 w \neq 0$. This means that $w = ua_1v$ where $u \equiv (a_2a_1)^k a_1^{r-1}$ $(r \geq 1)$, $v \equiv a_2^s(a_2a_1)^l$ and that we have $f_1(w) = ua_2v$. Clearly, $Q(ua_2) = Q(ua_1)$. Next, the insertion of v into $P(ua_2)$ will produce the same sequence of shapes as the insertion of v into $P(ua_1)$. Indeed, write $v = v_1 \cdots v_k$ and assume by induction that $P(ua_1v_1 \cdots v_{r-1})$ and $P(ua_2v_1 \cdots v_{r-1})$ have the same shape. If $v_r = a_2$, then clearly $P(ua_1v_1 \cdots v_r)$ and $P(ua_2v_1 \cdots v_r)$ will also have the same shape. If $v_r = a_1$, then since $v \equiv a_2^s(a_2a_1)^l$, we see that $r \geq 2$ and that the tableau $P(ua_1v_1 \cdots v_{r-1})$ has at least one a_2 in its bottom row. Thus the insertion of a_1 in both tableaux will produce again two tableaux of the same shape.

The proof is similar in the case $h = e_1$, and this also implies the case $h = \sigma_1$ since $\sigma_1 w$ is either of the form $f_1^p w$ or $e_1^q w$.

Consider now the general case $A = \{a_1, \ldots, a_n\}$, and suppose that $h = f_i, e_i$ or σ_i. By Corollary 5.3.7, we have to prove that $P(\mathrm{std}\,(h(w))^{-1}) = P(\mathrm{std}\,(w)^{-1})$. Recall that $\mathrm{std}\,(w)^{-1}$ is the word u'' obtained from the representation of w as the biword $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathrm{id} \\ w \end{bmatrix}$ (see Section 5.3). Set $w_1 = h(w)$ and $\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \mathrm{id} \\ w_1 \end{bmatrix}$. Then, we can write $v'' = \alpha a_i^r a_{i+1}^s \beta$ where a_i and a_{i+1} do not occur in α and β, $v_1'' = \alpha a_i^{r'} a_{i+1}^{s'} \beta$ $(r + s = r' + s')$, $u'' = \gamma\varepsilon\delta$ where $|\alpha| = |\gamma|$ and $|\beta| = |\delta|$, and finally $u_1'' = \gamma\varepsilon_1\delta$. By the above proof for a two letter alphabet, $\varepsilon_1 \equiv \varepsilon$. Therefore, $u_1'' \equiv u''$ as required.

(ii) Suppose that w' differs from w by a single Knuth transformation, and let us take for example $h = f_i$. Write $w = \alpha xzy\beta$ and $w' = \alpha zxy\beta$, where we assume that $x < y < z$. Let a (resp. a') be the letter a_i of w which is changed into a_{i+1} by f_i. We claim that if a is a letter of α (resp. β), then a' is the letter occupying the same position in w'. This is clear because the transformation $xzy \to zxy$ does not modify the relative positions of consecutive letters a_i and a_{i+1}. Therefore, $f_i(w) \equiv f_i(w')$ trivially if a is a letter of α or of β. Otherwise, a is one of the letters x, y, z of w and a' is the same letter in w'. Hence, according to $a = x, y$ or z, we have

$$f_i(w) = \begin{cases} \alpha a_{i+1} zy\beta \\ \alpha xza_{i+1}\beta \\ \alpha xa_{i+1}y\beta \end{cases} \equiv f_i(w') = \begin{cases} \alpha za_{i+1}y\beta \\ \alpha zxa_{i+1}\beta \\ \alpha a_{i+1}xy\beta \end{cases} .$$

Note that in the case $a = y$, we must have $z \geq a_{i+2}$, because if $z = a_{i+1}$, $y = a_i$, then zy would be put between brackets. In the case $w = \alpha xyx\beta$ and $w' = \alpha yxx\beta$, the reasoning given above remains unchanged, except when $x = a_i$, $y = a_{i+1}$, and a does not belong to α or β. In this case, we have

$$f_i(w) = f_i(\alpha a_i a_{i+1} a_i \beta) = \alpha a_{i+1} a_{i+1} a_i \beta\,,$$

and

$$f_i(w') = f_i(\alpha a_{i+1} a_i a_i \beta) = \alpha a_{i+1} a_i a_{i+1} \beta \equiv f_i(w)\,.$$

The case of a Knuth transformation $yxz \equiv yzx$ $(x < u \leq z)$ is treated similarly. ■

We shall now make use of the operators e_i, f_i to define a graph Γ on A^*. The vertices of this graph are all the words $w \in A^*$, and we put labelled arrows between words according to the following rule:

$$(w \xrightarrow{i} w') \iff (f_i w = w') \ .$$

Note that if $f_i w = w' \neq 0$, then $e_i w' = w$, hence at each vertex w there is at most one incident arrow of color i (and also, by definition, at most one outgoing arrow of color i). Hence the subgraph obtained by erasing all arrows of color $j \neq i$ is extremely simple: it is just a collection of disjoint *i-strings*

$$w_1 \xrightarrow{i} w_2 \longrightarrow \cdots \xrightarrow{i} w_k$$

of various lengths $k \geq 0$. However, when all the colors are considered simultaneously, a rich combinatorial structure emerges. Let us call "connected components of Γ" the connected components of the underlying non-oriented non-labelled graph.

PROPOSITION 5.5.2. (i) *The connected components of Γ are the coplactic classes.*

(ii) *Two coplactic classes are isomorphic as subgraphs of Γ if and only if they are indexed by two standard tableaux of the same shape.*

Proof. (i) By Theorem 5.5.1 (i), any connected component of Γ is contained in a coplactic class. Conversely, let w be a a non-Yamanouchi word. Then there exists an index i such that $e_i w \neq 0$. If $w' = e_i w$ is not a Yamanouchi word, we can again find j such that $e_j w' = w'' \neq 0$. Iterating this procedure, we construct a chain of arrows connecting w to the unique Yamanouchi word in its coplactic class. Hence any two words of the same coplactic class are connected by a sequence of arrows going through the same Yamanouchi word.

(ii) It follows from Theorem 5.5.1 (ii) that two coplactic classes indexed by standard tableaux of the same shape are isomorphic as subgraphs. Conversely, if two coplactic classes C, C' correspond to two standard tableaux t, t' of respective shapes $\lambda \neq \lambda'$, then the Yamanouchi words of these classes have evaluation λ and λ'. It is easy to check from the definition of f_i that for a Yamanouchi word of evaluation $\lambda = (\lambda_1, \ldots, \lambda_k)$, one has

$$\max\{p \mid f_i^p y \neq 0\} = \lambda_i - \lambda_{i+1} \ .$$

Hence the unique vertices of C and C' with no incident arrows have outgoing strings of different lengths, and C and C' are not isomorphic. ∎

As an illustration Figure 5.1 shows the graph structure of the coplactic class of $t = 2211$ for $A = \{1, 2, 3, 4\}$. These graphs are examples of *crystal graphs* in the sense of Kashiwara.

5.6. Cyclage and canonical embeddings

In this section we investigate the behavior of the previous constructions under circular permutations on words. We denote by ζ the bijection on A^* defined by $\zeta(x_1 x_2 \cdots x_n) = x_2 \cdots x_n x_1$ $(x_i \in A)$.

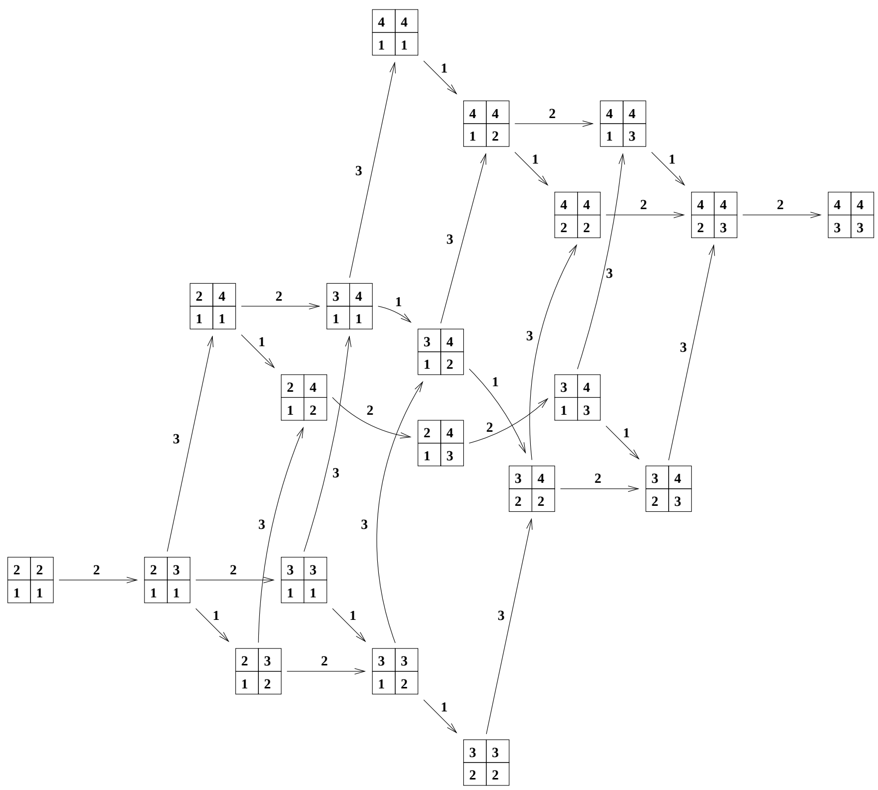


Figure 5.1. The graph structure of the coplactic class of $t = 2211$.

PROPOSITION 5.6.1. *The cyclic shift ζ commutes with the maps σ_i.*

Proof. We have to prove that $\zeta\sigma_i(w) = \sigma_i\zeta(w)$, $w \in A^*$. If the first letter x_1 of w is different from a_i and a_{i+1} there is nothing to prove. Otherwise we distinguish 4 cases. Let us say that a letter x_k of w is *free* if it does not occur inside a pair of mutually closing brackets at the end of the bracketing procedure described in Section 5.5. We then have the following cases: (*i*) $x_1 = a_i$ and no a_{i+1} is free; (*ii*) $x_1 = a_i$ and at least one a_{i+1} is free; (*iii*) $x_1 = a_{i+1}$ is free; (*iv*) $x_1 = a_{i+1}$ is not free. In each case, the verification is immediate. ■

LEMMA 5.6.2. *Let $t \in A^*$ be a tableau and σ be any product of σ_i. Then the following conditions are equivalent:*

(i) $\sigma(t) = t$

(ii) $\sigma(P(\zeta(t))) = P(\zeta(t))$.

Proof. Since ζ is bijective,

$$\sigma(t) = t \Leftrightarrow \zeta(\sigma(t)) = \zeta(t)\,.$$

By Proposition 5.6.1, $\zeta(\sigma(t)) = \sigma(\zeta(t))$, which has the same Q-symbol as $\zeta(t)$ by Theorem 5.5.1 (i). Thus

$$\sigma(t) = t \Leftrightarrow P(\sigma(\zeta(t))) = P(\zeta(t))$$

because of Theorem 5.3.1. Now, again by Theorem 5.5.1, $P(\sigma(w)) = \sigma(P(w))$ for any $w \in A^*$ and the statement follows. ∎

THEOREM 5.6.3. *The operators σ_i satisfy the Moore-Coxeter relations*

$$\sigma_i^2 = 1\,, \tag{5.6.1}$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad (|i-j| > 1)\,, \tag{5.6.2}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\,. \tag{5.6.3}$$

In other words, the map ρ sending the elementary transposition $(i, i+1)$ onto σ_i is a linear representation of the symmetric group $\mathfrak{S}_n$ in $\mathbb{Z}\langle A\rangle$.

Proof. Relations (5.6.1) and (5.6.2) are obviously satisfied. To prove (5.6.3), we have to show that $(\sigma_i\sigma_{i+1})^3(w) = w$ for any $w \in A^*$. From Theorem 5.5.1, it is enough to check this when $w = t$ is a tableau. Let $t = uv$ where v is the bottom row of t. By Lemma 5.6.2, it is equivalent to show that $(\sigma_i\sigma_{i+1})^3 P(uv) = P(vu)$. Now, in the tableau $t' = P(vu)$ all the letters a_1, a_2 lie in the bottom row. Writing $t' = u'v'$ and $t'' = P(v'u')$, and iterating, we construct a sequence $t^{(k)}$ of tableaux such that all the letters $a_1, \ldots, a_{k+1}$ of $t^{(k)}$ are in its first row, and such that

$$(\sigma_i\sigma_{i+1})^3(t) = t \Longleftrightarrow (\sigma_i\sigma_{i+1})^3(t^{(k)}) = t^{(k)}\ .$$

But $t^{(n-1)}$ is a row, and $(\sigma_i\sigma_{i+1})^3(t^{(n-1)})$ has to be a row with the same evaluation, hence $(\sigma_i\sigma_{i+1})^3(t^{(n-1)}) = t^{(n-1)}$. ∎

COROLLARY 5.6.4. *The free Schur functions $\mathbf{S}_t$ are invariant under the above action of $\mathfrak{S}_n$. As a consequence, the commutative Schur functions $s_\lambda(\xi)$ are symmetric in the usual sense.*

We next investigate which transformations on tableaux arise when the map P is applied to circular permutations of words. Let Row (A) denote the subset of Tab (A) consisting of rows.

DEFINITION 5.6.5. Let t be a tableau which is not a row. We put

$$\mathcal{C}(t) = P(\zeta(t))\,.$$

The map $\mathcal{C} : \mathrm{Tab}\,(A) \setminus \mathrm{Row}\,(A) \to \mathrm{Tab}\,(A)$ is called *cyclage*.

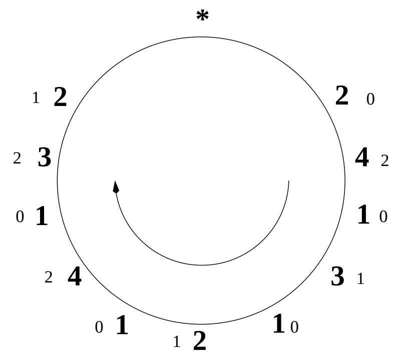


Figure 5.2. The calculation of the cocharge of $w = 23141213142$ (labels are written in small type)

To describe properties of the cyclage map, we need to use a plactic invariant on words called *cocharge*. Let w be a word. Let σ be any permutation such that $v = \sigma(w)$ has a dominant evaluation, that is

$$|v|_{a_1} \geq |v|_{a_2} \geq \cdots \geq |v|_{a_n} .$$

Write v on a circle, adding a "point at infinity" $*$ (see Figure 5.2). Then label each letter of v according to the following algorithm, reading the word clockwise.

1. start at $*$ and label the first unlabelled a_1 with 0.
2. after labelling an a_i with the number c, label the first unlabelled a_{i+1} with $c+1$ if it is obtained without crossing $*$, and with c otherwise. If there is no unlabelled a_{i+1}, go to the first step again, while there are still unlabelled letters.

The sum of all labels is called the *cocharge* of w, and is denoted by $\mathrm{coch}\,(w)$. The complementary statistic $\mathrm{ch}\,(w) = \max\{\mathrm{coch}\,(v) \mid \mathrm{ev}\,(v) = \mathrm{ev}\,(w)\} - \mathrm{coch}\,(w)$ is called the *charge* of w. For example, the cocharge of $w = 23141213142$ (whose evaluation is dominant) is equal to 9, as shown in Figure 5.2.

LEMMA 5.6.6. (i) *If* $\mathcal{C}(t) = t'$, *then for any* $\sigma \in \mathfrak{S}(A)$, $\mathcal{C}(\sigma(t)) = \sigma(t')$.
(ii) *If* $w \equiv w'$ *then* $\mathrm{coch}\,(w) = \mathrm{coch}\,(w')$.
(iii) *For* $t \in \mathrm{Tab}\,(A) \setminus \mathrm{Row}\,(A)$, *we have* $\mathrm{coch}\,(\mathcal{C}(t)) = \mathrm{coch}\,(t) - 1$.
(iv) *If* $\mathcal{C}(t) = \mathcal{C}(t')$ *and* $t \neq t'$, *then* t *and* t' *must have different shapes.*

Proof. (i) results clearly from Theorem 5.5.1 and Proposition 5.6.1.

As to (ii), we note that by definition $\mathrm{coch}\,(\sigma(w)) = \mathrm{coch}\,(w)$ for $\sigma \in \mathfrak{S}(A)$, hence using Theorem 5.5.1 (ii) we can assume that w and w' have a dominant evaluation. For such words, the above calculation of the charge proceeds by extracting from w a sequence of standard subwords $w^{(i)}$ such that

$$\operatorname{coch}(w) = \sum_i \operatorname{coch}(w_i)\,.$$

Now, it is clear that replacing a factor $a_i a_j$ by $a_j a_i$ when $|i-j| \neq 1$, does not change these subwords, and thus does not change the cocharge. Similarly, one checks that replacing a factor $a_{i+1}a_i a_i$ (*resp.* $a_{i+1}a_{i+1}a_i$) by $a_i a_{i+1} a_i$ (*resp.* $a_{i+1}a_i a_{i+1}$) does not modify these standard subwords. Hence, cocharge is invariant under plactic relations.

Let now $t = xw$, $x \in A$, be a tableau of dominant evaluation, which is not a row. Then $x \neq a_1$, and the order in which letters are labelled in the word xw is the same as in wx. Thus, all labels are preserved except the label of x which is decreased by 1, and

$$\operatorname{coch}(P(wx)) = \operatorname{coch}(wx) = \operatorname{coch}(xw) - 1$$

which proves (iii).

To prove (iv), assume that t and t' are two different tableaux of the same shape, and write $t = xw$, $t' = x'w'$ with $x, x' \in A$. Then w and w' also are two tableaux of the same shape, say λ. By Corollary 5.4.6, $S_\lambda\, S_{(1)}$ is a multiplicity-free sum of tableaux in $\mathbb{Z}[\mathrm{Pl}(A)]$, hence $wx \not\equiv w'x'$, that is, $\mathcal{C}(t) \neq \mathcal{C}(t')$. ∎

We shall now use the map $\mathcal{C}$ to define a graph structure on the set $\mathrm{Tab}\,(A)$. Namely, consider the oriented graph with set of vertices $\mathrm{Tab}\,(A)$ and edges defined by:

$$t \longrightarrow t' \qquad \Longleftrightarrow \qquad \mathcal{C}(t) = t'\,.$$

Since the cyclage map does not change the evaluation of tableaux this graph decomposes into the disjoint union of the subgraphs with sets of vertices $\mathrm{Tab}\,(\cdot,\mu)$ for all evaluations μ. The following theorem describes these subgraphs and shows how they can all be naturally embedded into the subgraph of standard tableaux.

THEOREM 5.6.7. (i) *The subgraph* $\mathrm{Tab}\,(\cdot,\mu)$ *is a rooted-tree with root the unique row-tableau of evaluation* μ. *Two evaluations which differ by a permutation give rise to isomorphic trees.*

(ii) *Let* μ *and* ν *be two evaluations such that*

$$\begin{aligned} \mu_k &= \nu_k \quad \text{for } k \neq i, j,\\ \mu_i &> \mu_j,\\ \nu_i &= \mu_i - 1,\\ \nu_j &= \mu_j + 1. \end{aligned}$$

Then there exists a unique embedding $\mathcal{I}_{\mu\nu}$ *of* $\mathrm{Tab}\,(\cdot,\mu)$ *into* $\mathrm{Tab}\,(\cdot,\nu)$ *commuting with* $\mathcal{C}$ *and such that* $\mathcal{I}_{\mu\nu}(t)$ *has the same shape as* t *for all* t.

(iii) *Similarly, for any evaluation* μ *there exists a unique embedding* $\mathcal{I}_\mu$ *of* $\mathrm{Tab}\,(\cdot,\mu)$ *into* STab *preserving shapes and commuting with* $\mathcal{C}$.

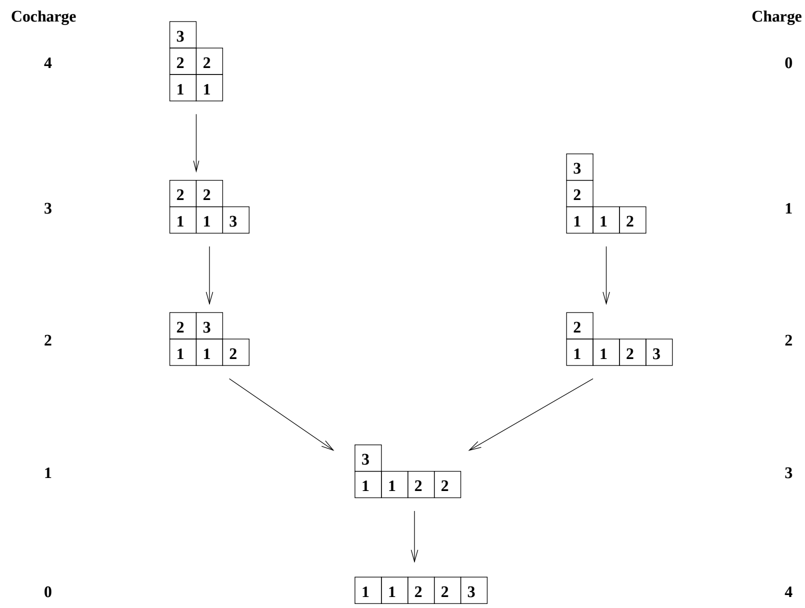


Figure 5.3. The tree structure of Tab $(\cdot, (2,2,1))$

Proof By Lemma 5.6.6 (iii), the map $\mathcal{C}$ decreases cocharge by 1. Hence, the cyclage graph has no cycle and is a union of trees. It is clear from the definition of cocharge that row-tableaux are the only words with cocharge 0. Therefore, the subgraph Tab $(\cdot, \mu)$ is a rooted-tree with root the unique row of evaluation μ. If $\nu = \sigma(\mu)$ for some $\sigma \in \mathfrak{S}(A)$, then, by Lemma 5.6.6 (i), Tab $(\cdot, \mu)$ and Tab $(\cdot, \nu)$ are isomorphic as trees, which proves (i).

Let $\sigma \in \mathfrak{S}(A)$ be any permutation such that $\sigma(a_i) = a_1$ and $\sigma(a_j) = a_2$. Let $\mu' = \sigma(\mu)$ and $\nu' = \sigma(\nu)$. Given $t = xw$ in Tab $(\cdot, \mu')$ its image under f_1 is non-zero and is the tableau in Tab $(\cdot, \nu')$ obtained by changing the rightmost a_1 into a_2. This operation clearly commutes with $\mathcal{C}$, since the letter x which is cycled does not interfere, in the computation of $P(wx)$, with the subtableau of w consisting of the occurrences of a_1 and a_2. Therefore, the image of Tab $(\cdot, \mu')$ under f_1 is a subtree of Tab $(\cdot, \nu')$. Moreover, if two tableaux of the same shape have the same image under cyclage, then they are identical according to Lemma 5.6.6 (iv). Hence there can be only one map from Tab $(\cdot, \mu')$ to Tab $(\cdot, \nu')$ preserving shape and commuting with $\mathcal{C}$. Finally, using σ^{-1}, one obtains from this embedding of Tab $(\cdot, \mu')$ in Tab $(\cdot, \nu')$ an embedding of Tab $(\cdot, \mu)$ in Tab $(\cdot, \nu)$ with the same properties, and (ii) is proved.

Composing the preceding embeddings, one obtains for each evaluation μ at least one embedding of Tab $(\cdot, \mu)$ into Tab $(\cdot, (1, \ldots, 1))$ preserving shapes and commuting with $\mathcal{C}$. The unicity of such an embedding is again ensured by Lemma 5.6.6 (iv). ∎

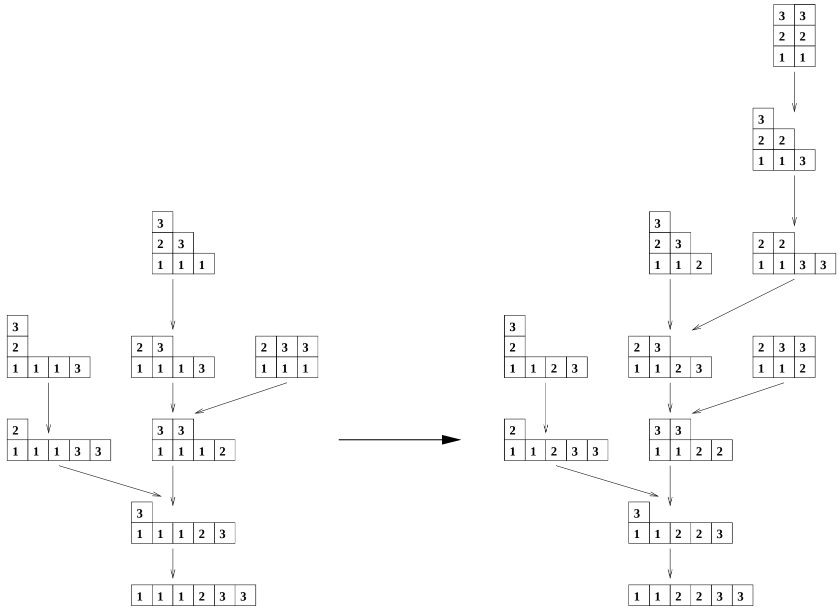


Figure 5.4. The embedding of Tab $(\cdot,(3,1,2))$ in Tab $(\cdot,(2,2,2))$

Figure 5.3 and Figure 5.4 illustrate Theorem 5.6.7 by displaying the tree structure of Tab $(\cdot,(2,2,1))$ and the canonical embedding of Tab $(\cdot,(3,1,2))$ in Tab $(\cdot,(2,2,2))$.

The main motivation for studying cyclage and the related plactic invariants given by charge and cocharge is to develop a combinatorial approach to the *Kostka-Foulkes polynomials* $K_{\lambda\mu}(q)$ which arise in many contexts, ranging from the character theory of the finite linear groups $\mathrm{GL}_n(\mathbf{F}_q)$ to the geometry of flag varieties or the solution of certain models in statistical mechanics. Actually, one has the following important result:

THEOREM 5.6.8. *The Kostka polynomial is equal to the generating function of the charge on the set* Tab (λ,μ) *of tableaux of shape* λ *and weight* μ*:*

$$\sum_{t\in \mathrm{Tab}\,(\lambda,\mu)} q^{\mathrm{ch}\,(t)} = K_{\lambda\mu}(q)\,.$$

The proof of this theorem is out the scope of this chapter.

Problems

Section 5.1

5.1.1 (The Erdös-Szekeres theorem). Prove that any permutation of n^2+1 elements contains a monotonic subsequence of length $n+1$. Show that there exist permutations of n^2 elements with no monotonic subsequence with length greater than n.

Section 5.2

5.2.1 Let $\overline{w}$ denote the mirror image of a word w. Let w be a standard word, and $t = P(w)$. Show that $P(\overline{w}) = t^T$, the transposed tableau of t.

5.2.2 Let w be a standard word. Show that the sequence w^n stabilizes in $\mathrm{Pl}\,(A)$, in the following sense: for n sufficiently large, $w^{n+1} \equiv c \cdot w^n$, where c is the column such that $\mathrm{ev}\,(c) = \mathrm{ev}\,(w)$.

5.2.3 Let w be a standard word. Let $V(w)$ be the set of words v such that $wv \equiv vr$, where r is a row. Show that the set of words of minimal length in $V(w)$ is a plactic class.

5.2.4 The *column reading* $C(t)$ of a tableau t is the word obtained by reading the planar representation of t column-wise, from left to right and from top to bottom. Show that for any tableau, $C(t) \equiv t$.

5.2.5 (Plactic monoid and quantum matrices). Let $\mathcal{A}$ be the associative unital $\mathbb{Q}[q, q^{-1}]$-algebra generated by elements $x_{11}, x_{12}, x_{21}, x_{22}$ subject to the relations:

$$\begin{aligned}
x_{12}x_{11} &= qx_{11}x_{12} \\
x_{21}x_{11} &= qx_{11}x_{21} \\
x_{22}x_{21} &= qx_{21}x_{22} \\
x_{22}x_{12} &= qx_{12}x_{22} \\
x_{12}x_{21} &= x_{21}x_{12} \\
x_{22}x_{11} &= x_{11}x_{22} + (q - q^{-1})x_{12}x_{21}
\end{aligned}$$

1) Show that $D = x_{11}x_{22} - q^{-1}x_{12}x_{21}$ commutes with the x_{ij}, hence is central in $\mathcal{A}$.

2) Introduce the $\mathbb{Z}[q]$-lattice $\mathcal{L}$ in $\mathcal{A}$ spanned by the elements $D^k x_{11}^l x_{22}^m$ $(k, l, m \in \mathbb{N})$.

(i) Show that every diagonal monomial $x_{i_1 i_1} \cdots x_{i_k i_k}$ $(i, j \in \{1,2\})$ belongs to $\mathcal{L}$. (Hint: prove that $x_{22}x_{11} = (1-q^2)D + q^2 x_{11}x_{22}$.)

(ii) Let $w = i_1 \cdots i_k$, $w' = j_1 \cdots j_k \in \{1,2\}^*$. Prove that

$$w \equiv w' \iff x_{i_1 i_1} \cdots x_{i_k i_k} \equiv x_{j_1 j_1} \cdots x_{j_k j_k} \bmod q\mathcal{L}$$

Section 5.3

5.3.1 Show that the number a_n of involutions in $\mathfrak{S}_n$ is equal to the number of standard tableaux of weight n. Show that

$$\sum_{n\geq 0} a_n \frac{z^n}{n!} = e^{z+\frac{z^2}{2}} .$$

Section 5.4

5.4.1 Show that if $\lambda = (k^l)$ and $\mu = (r^s)$ are partitions of rectangular shapes, all the coefficients $c^{\nu}_{\lambda\mu}$ are 0 or 1, and give a simple graphical description of the partitions ν such that $c^{\nu}_{\lambda\mu} = 1$.

5.4.2 For an integer k, let $h_k = s_{(k)}$ be the Schur function indexed by the one-part partition (k), and for a partition $\mu = (\mu_1, \dots, \mu_r)$, set $h_\mu = h_{\mu_1} h_{\mu_2} \cdots h_{\mu_r}$. The *Kostka numbers* $K_{\lambda\mu}$ are defined as the coefficients of the expansion $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$. Show that $K_{\lambda\mu}$ is equal to the number of tableaux of shape λ and evaluation μ.

5.4.3 Let $X = \{x_1, x_2, \dots, x_n\}$ be a set of commuting indeterminates, and let $E(t) = \prod_i (1 + tx_i) = \sum_k e_k t^k$, $H(t) = \prod_i (1 - tx_i)^{-1} = \sum_k h_k t^k$ be the generating functions of the elementary and complete symmetric functions of X. Let $p_k = \sum_i x_i^k$ be the power sums symmetric functions.

1) Show that $\sum_{k\geq 1} p_k t^{k-1} = H'(t)E(-t)$.

2) Deduce from 1) that $p_m = \sum_{k=0}^{m-1} (-1)^k s_{(m-k,1^k)}$.

3) The *character table* of the symmetric group $\mathfrak{S}_n$ is a square matrix χ^λ_μ indexed by pairs of partitions of n, in which χ^λ_μ is equal to the coefficient of s_λ in the product of power sums $p_\mu = p_{\mu_1} p_{\mu_2} \cdots p_{\mu_r}$. Using 2) and the Littlewood-Richardson rule, compute the character tables of the groups $\mathfrak{S}_n$ for $n \leq 6$.

Section 5.5

5.5.1 Let $w = x_1 \cdots x_m \in A^*$. One says that the integer $i < m$ is a *descent* of w if $x_i > x_{i+1}$. The *major index* $\mathrm{maj}\,(w)$ of w is the sum of its descents. We denote by $\mathrm{Des}\,(w)$ the descent set of w.
A *recoil* of a standard tableau t is an entry i of t such that $i + 1$ occurs in a higher row. Let $\mathrm{Rec}\,(t)$ be the set of recoils of t. The *index* of a tableau is $\mathrm{ind}\,(t) \sum_{i\in \mathrm{Rec}\,(t)} i$.
It is customary to encode a subset $E = \{e_1, \dots, e_{r-1}\} \subseteq \{1, 2, \dots, m-1\}$ by a *composition* of m, *i.e.* a vector $I = (i_1, \dots, i_r)$ of positive integers with sum $|I| = m$. The encoding $I = C(E)$ of E is specified by $e_k = i_1 + i_2 + \cdots + i_k$. The composition $I = C(\mathrm{Des}\,(w))$ is called the *descent composition* of w. Conversely, the set E defined in this way from a

composition I is called the descent set of I and denoted by $\mathrm{Des}\,(I)$. As above, on sets $\mathrm{maj}\,(I) = \sum_k e_k$.

1) Show that for any word, $\mathrm{Des}\,(w) = \mathrm{Rec}\,Q(w)$.
2) For a composition I, define the *noncommutative ribbon Schur function* $R_I \in \mathbb{Z}\langle A\rangle$ by

$$R_I = \sum_{\mathrm{Des}\,(w)=\mathrm{Des}\,(I)} w\,.$$

a) Show that $R_I = \sum_{\mathrm{Rec}\,(t)=\mathrm{Des}\,(I)} \mathbf{S}_t$.
b) Show that $w \mapsto Q(w)$ defines a bijection between the set of Yamanouchi words of evaluation λ and $\mathrm{STab}\,(\lambda)$.
c) Let r_I be the commutative image of R_I, and $r_I = \sum_\lambda c_\lambda^I s_\lambda$ its expansion in the Schur basis. Show that r_I is equal to the number of Yamanouchi words of evaluation λ with descent composition I.
3) Prove the identity between formal series

$$\overrightarrow{\prod_{k\geq 0}}\prod_{i\geq 1}(1-q^k a_i)^{-1} = \sum_{m\geq 0}\frac{1}{(q)_m}\sum_{|w|=m} q^{\mathrm{maj}\,(w)}w\,,$$

where $(q)_m = (1-q)(1-q^2)\cdots(1-q^m)$.
4) By taking the commutative image of the above identity, and applying Cauchy's identity to the alphabets $Q = \{1, q, q^2, \dots\}$ and X, show that $\sum_{|I|=m} c_\lambda^I q^{\mathrm{maj}\,(I)} = (q)_m s_\lambda(Q)$ and obtain the generating function of the major index on the set of standard tableaux of a given shape:

$$\sum_{t\in\mathrm{STab}\,(\lambda)} q^{\mathrm{maj}\,(t)} = (q)_m s_\lambda(Q)\,.$$

This is equal to the Kostka polynomial $K_{\lambda,1^m}(q)$.

Section 5.6

5.6.1 (Catabolism). Let $k : \mathrm{Tab} \to \mathrm{Tab}$ be the map $t = t'v \mapsto vt'$ where v is the bottom row of t. Let $\varphi(t)$ be the sequence of shapes of $t, k(t), k^2(t), \dots$.
1) Show that the restriction of φ to STab is one-to-one.
2) Show that φ is invariant under the action of $\mathfrak{S}(A)$ (*i.e.*, $\varphi(\sigma(t)) = \varphi(t)$).
3) Show that φ is invariant under the canonical embeddings $\mathrm{Tab}\,(\lambda) \hookrightarrow \mathrm{Tab}\,(1^n) = \mathrm{STab}$.

Notes

The name *plactic monoid* was coined by Schützenberger with reference to the *tectonique des plaques*. The basic theory of the plactic monoid was systematically developed in Lascoux and Schützenberger 1981.

Schensted's algorithm appeared in Schensted 1961. It was realized later that Robinson, in an attempt to prove the Littlewood-Richardson rule, had already formulated in Robinson 1938 the correspondence (5.3.1), which is essentially equivalent to Schensted's result (Theorem 5.3.1).

Theorem 5.2.5 is due to Knuth 1970. Greene's invariants were introduced in Greene 1974. Theorem 5.3.3 appears in Schützenberger 1963. It was already stated, without proof, in Robinson 1938.

The left-hand side of 5.3.10 can be interpreted as the sum of the characters of all irreducible polynomial representations of $\mathrm{GL}_n(\mathbb{C})$. Using this interpretation, Theorem 5.3.10 is a classical identity of Schur (see Littlewood 1950).

For an account of the theory of symmetric functions see Littlewood 1950 or Macdonald 1995. The proof of the Littlewood-Richardson rule given in Section 5.4 first appeared in Schützenberger 1977. Corollary 5.4.6 is known by geometers as the *Pieri rule*.

Lascoux and Schützenberger 1988 is the basic reference for the material of Section 5.5, with emphasis on the operators σ_i. Our exposition here, which stresses the role played by the operators e_i and f_i, is strongly influenced by Kashiwara's theory of crystal bases (see Kashiwara 1991, Kashiwara 1994, Lascoux, Leclerc, and Thibon 1995, Leclerc and Thibon 1996). The connection between Robinson-Schensted correspondence and quantum groups was first observed in Date, Jimbo, and Miwa 1990.

Concerning the statistics charge and cocharge, the cyclage, and their applications to Kostka-Foulkes polynomials, see Schützenberger 1978, Lascoux and Schützenberger 1980, Lascoux 1991. Another combinatorial description of the Kostka-Foulkes polynomials in terms of the geometry of crystal graphs was given in Lascoux et al. 1995.

The Littlewood-Richardson rule and the plactic monoid have been generalized to other root systems by Littelmann (see Littelmann 1994, Littelmann 1996). A monoid associated in a similar way to Gessel's quasi-symmetric functions has been introduced in Krob and Thibon 1997.

Problem 5.1.1 is a classical result that appears for instance in Knuth 1973. Problem 5.2.5 is from Leclerc and Thibon 1996. More on character tables (Problem 5.4.3) can be found in Macdonald 1995. Problem 5.5.1 is from Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon 1995.