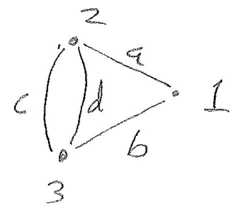
6/2/15 Matroid polytopes & Ehrhart polynomials

1. graphs
2. matroids
3. lattice path matroids
4. Ehrhart polynomials
5. Derksen–Fink invariants

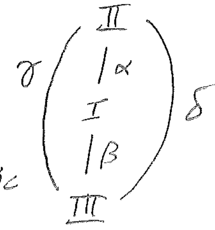
1. Graphs

Graph theory sometimes studies isomorphism invariants of graphs

e.g. $G = (V, E)$, $V = \{1,2,3\}$, $E = \{a,b,c,d\}$



$\cong$ isomorphic to



$G' = (V', E')$

$V' = \{I, II, III\}$

$E' = \{\alpha, \beta, \gamma, \delta\}$

G. Birkhoff (1912) introduced chromatic polynomial

$\chi(G,k) := \#(\text{proper vertex } k\text{-colorings of } G)$ (proper: no monochromatic edges)

$\left(= k(k-1)(k-2) = k^3 - 3k^2 + 2k\right)$ (G above)

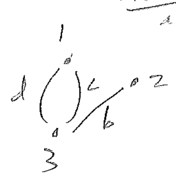
blue / red / yellow — $k = 100$ choices; blue — $98 = k-2$ choices; $k-1 = 99$ choices

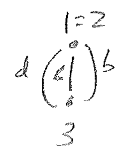
* If G has a loop, $\chi(G,k) = 0$

which satisfies a deletion-contraction recurrence:

$$\chi(G,k) = \chi(\underbrace{G\backslash a}_{\text{deletion at } a}, k) - \chi(\underbrace{G/a}_{\text{contraction at } a}, k)$$

if a is not a self-loop:





Remark: $\chi(G,k)$ is #P-hard to compute in general.

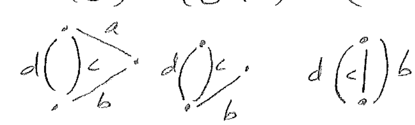
Another graph invariant (Kirchoff 1840's) (see Stanley Chap. 9, 11)

$\tau(G) := \#(\text{spanning trees of } G) = \#\{T \subset E : T \text{ is acyclic (i.e. contains no cycles) and connects all of } V\}$

$\left(= \#\left\{\right.\right.$  $\left.\left.\right\} = 5\right)$

for G as before.

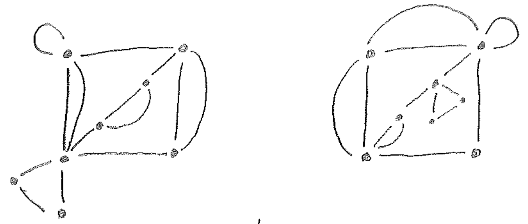
$\tau(G)$ also satisfies a deletion-contraction recurrence: $\tau(G) = \tau(G\backslash a) + \tau(G/a)$



$\mathcal{B}(M(G)) = \{bc, bd, ab, ac, ad\} = \underbrace{\{bc, bd\}}_{\mathcal{B}(M\backslash a)} \sqcup a\underbrace{\{b,c,d\}}_{\mathcal{B}(M/a)}$

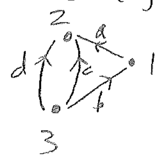
2. Matroids (1935 H. Whitney)

Whitney realized many graph invariants depend on G, but on the matroid $M(G)$.

e.g.  will have the same matroid.

$G_1 \not\cong G_2$

graph $G=(V,E)$ (orient edges) $\rightsquigarrow$ the $|V|\times|E|$ signed edge-node incidence matrix or a collection of column vectors $\{v_e\}_{e\in E} \subset \mathbb{R}^{|V|}$ $\rightsquigarrow$ matroid $M=M(\{v_e\}_{e\in E})$ on ground set E $=M(A)$



$$A=\begin{array}{cc} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} -1 & +1 & 0 & 0 \\ +1 & 0 & +1 & +1 \\ 0 & -1 & -1 & -1 \end{bmatrix} \end{array}$$

$\rightsquigarrow$ column indexing lin. indep. subsets.

specified by giving any of the collections of subsets of E:

- $\mathcal{I}(M)$ = independent sets of M

or

- $\mathcal{B}(M)$ = bases of M

or

$\mathcal{C}(M)$ = circuits of M

$a+b=c=d$

* column determined by a loop is the zero column vector.

e.g. $\{\emptyset, a, b, c, d, ab, ac, ad, bc, bd\} = \mathcal{I}(M)$

e.g. $\{ab, ac, ad, bc, bd\} = \mathcal{B}(M)$

bases for the column span $A \rightsquigarrow \mathcal{B}(M)$

e.g. $\{cd, abc, abd\} = \mathcal{C}(M)$

$A \rightsquigarrow \mathcal{C}(M)$ inclusion-minimal lin. dependent sets

Basis axioms: $\mathcal{B}\subset 2^E$ (B1) All $B\in\mathcal{B}$ have same cardinality $r =: \mathrm{rank}(M)$

(B2) $\forall B, B'$ and $\forall b\in B$ $\exists b'\in B'$ s.t. $(B\setminus\{b\})\cup\{b'\}\in\mathcal{B}$

Matroids have deletion-contraction: Given M via its bases $\mathcal{B}(M)$ and $e\in E$

$$\mathcal{B}(M\setminus e) := \{B\in\mathcal{B}(M) : e\notin B\} \text{ on } E\setminus e$$

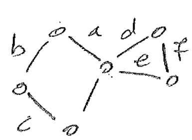
$$\mathcal{B}(M/e) := \{B\setminus e : e\in B\in\mathcal{B}(M)\} \text{ on } E\setminus e.$$

<u>Exercise 5:</u> (a) For a connected graph $G=(V,E) \rightsquigarrow$ vectors $\{v_e\}$ $\rightsquigarrow$ matroid $M=M(G)$

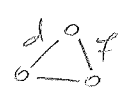
show $\mathcal{I}(M) = \{$<u>acyclic</u> subsets $I\subset E$ edges$\}$

$\mathcal{B}(M) = \{$spanning trees $T\subset E\}$

$\mathcal{C}(M) = \{$inclusion minimal cycles $C\subset E\}$

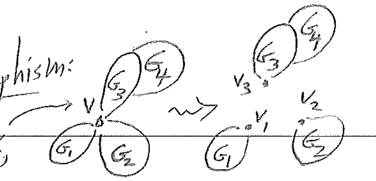


not inclusion minimal

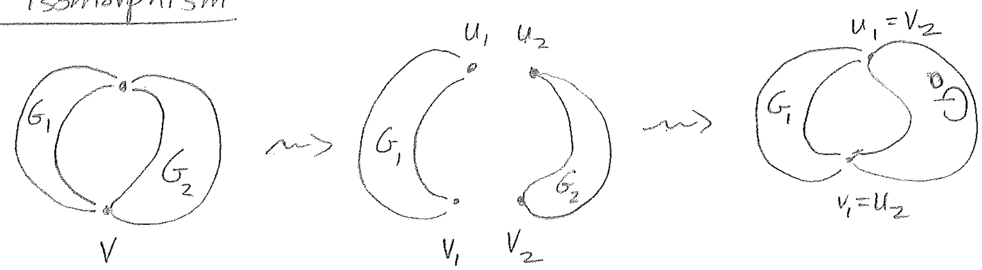
$\not\ni$ 

(b) Show these 2 operations <u>do not</u> affect $M(G)$

1-isomorphism: a <u>cut-vertex</u>, whose removal creates more connected components.

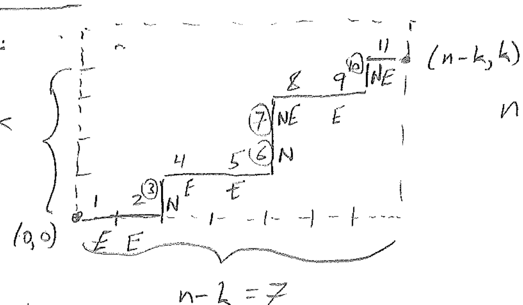


2-isomorphism:



(c) Show $M(G_1) = M(G_2)$ (Remark: (Whitney) These 2 operations basically characterize when $M(G_1) = M(G_2)$.)

3) Lattice path matroids: Fix k and n, which fixes a bounding $k \times (n-k)$ rectangle:



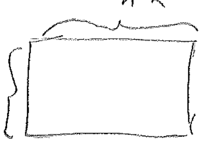
$\lambda = (\lambda_1, \dots, \lambda_k)$

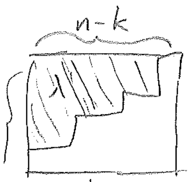
$n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$

We get bijections between:

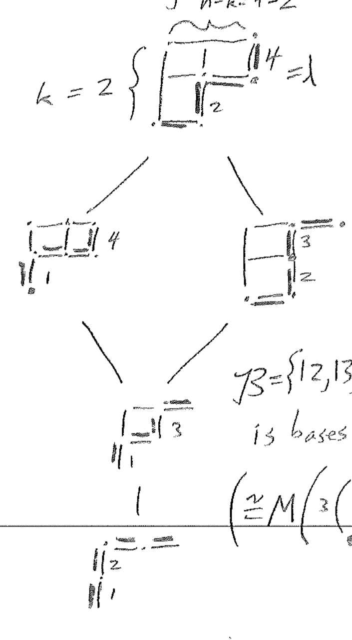
(a) lattice paths taking unit length north (N), east (E) steps $(0,0)$ to $(n-k,k)$

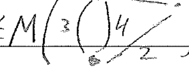
(b) subsets S of $\{1,2,\dots,n\}$ with $|B| = k$ via their N steps — e.g. $B = \{3,6,7,10\} = B(\lambda) \subset \{1,2,\dots,11\}$

(c) partitions $\lambda \subset$  (Stanley Chap. 6)

Exercise 6: Fix k, n and $\lambda \subset$ . Show that $\{B(\mu) : \mu \subseteq \lambda\}$ form the bases on $E = \{1,2,\dots,n\}$ called the lattice path matroid for λ, k, n.

e.g. $k=2$, $n=4$, $n-k = 4-2$



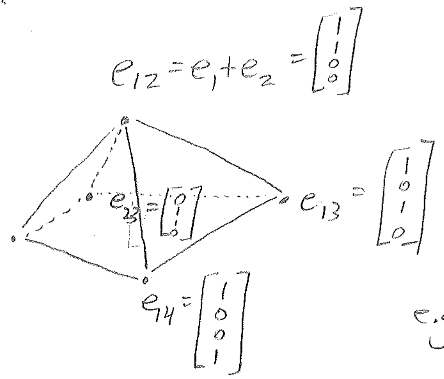
$\mathcal{B} = \{12, 13, 14, 23, 24\}$ is bases of a matroid ($\cong M($  $)$)

4) (2000's) A recent observation: if we fix the rank r of the matroid M and $|E| = n$, then many known matroid invariants behave like a valuation $\nu(A \cup B) = \nu(A) + \nu(B) - \nu(A \cap B)$ on the matroid base polytope!

$$P_M := \text{convex hull of } \{ e_B : B \in \mathcal{B}(M)\} \subseteq \mathbb{R}^{|E|}, \quad e_B = \sum_{i \in B} e_i$$

e.g. M above $\mathcal{B}=\{12,13,23,24,14\}$, $E=\{1,2,3,4\}$ has

$P_M =$



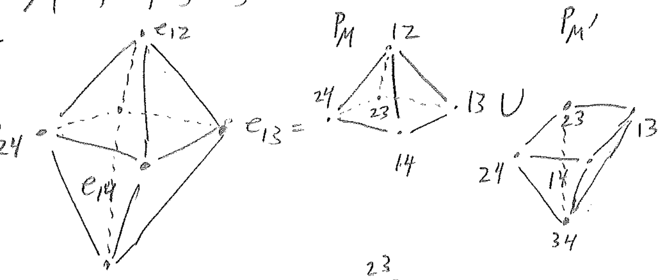
$\subset \mathbb{R}^4$ but inside affine hyperplane $\cong \mathbb{R}^3$

$\sum_{i=1}^{4} x_i = 2$

when all the pieces/intersection are themselves P_M's.

e.g. $\hat{M}=\{12,13,23,14,24,34\}$ has

$P_{\hat{M}} =$



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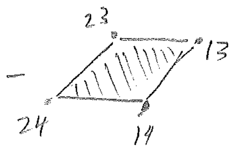
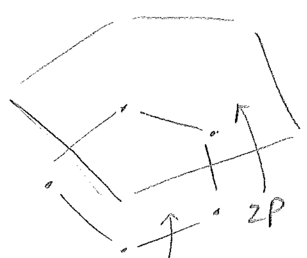


$P_{M''}$

$\#\mathcal{B}(\hat{M})=6$, $\#\mathcal{B}(M)=5$, $\#\mathcal{B}(M')=5$, $\#\mathcal{B}(M'')=4$

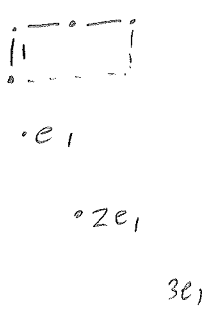
$6=5+5-4.$ ✓

Everyone's favorite valuation on polytopes P is volume $Vol(P)$, and for polytopes $P\subset\mathbb{R}^d$ with vectors in $\mathbb{Z}^d$ the 2nd favorite is $Ehr(P,m):=\#(\underbrace{mP}_{\text{dilation by } m}\cap\mathbb{Z}^d)$

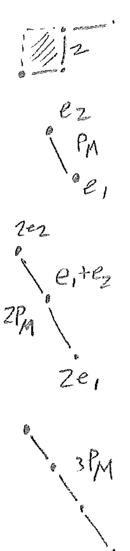


Thm (Ehrhart 1962) $Ehr(P,m)$ is a polynomial in m.

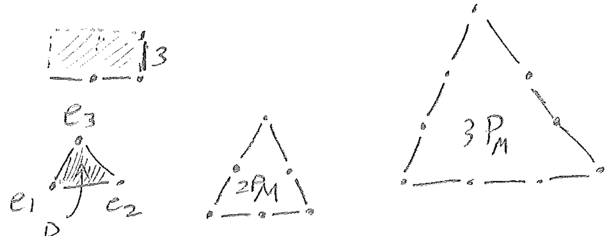
e.g. for lattice path matroids with $k=1$, $n-k=2$, $n=3$



$Ehr(P_M,m)=1$
$=\binom{m}{0}$



$Ehr(P_M,m)=\binom{m+1}{1}$



$Ehr(P_M,m)=\binom{m+2}{2}$
$=\frac{(m+1)(m+2)}{2}$
$=\frac{m^2+3m+2}{2}$

H. Bidkhori (2012) computed $Ehr(P_M, m)$ for lattice path matroids M.

⑥ Derksen-Fink invariant of M — Derksen (2009 conj.), Derksen-Fink (2010 proof) gave a universal (!) valuative invariant for matroids M: Fix $r = \mathrm{rank}(M)$ and $E = \{1, 2, \ldots, n\}$

$$\mathcal{G}_M := \sum_{\substack{\text{permutations} \\ (e_1, \ldots, e_n) \text{ of } \\ E}} \left[\text{rank jump subset of } e_1, e_1e_2, e_1e_2e_3, \ldots, e_1e_2\cdots e_n\right]$$

r positions — $r(\{e_1e_2e_3e_4e_5\}) - r(\{e_1, e_2, e_3, e_4\})$

e.g. $[0\,1\,1\,0\,0\,1\,0\,1]$

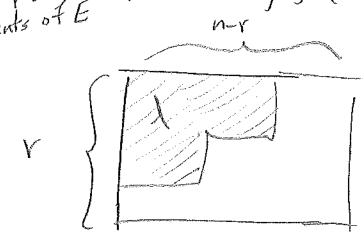
e.g. $r=1, n=3$	$\mathcal{G}_M$ coeff on $[100]$	$[010]$	$[001]$	
$\mathcal{B} = \{1,2,3\}$	6 (123, 132, 213, 312, 231, 321)	0	0	$\mathcal{G}_{M_1} = 6[100]$
$\mathcal{B} = \{1,2\}$	4 (123, 132, 213, 231)	2 (312, 321)	0	$\mathcal{G}_{M_2} = 4[100] + 2[010]$
	2 (123, 132)	2 (213, 312)	2 (321, 231)	$\mathcal{G}_{M_3} = 2[100] + 2[010] + 2[001]$

* $[1^r 0^{n-r}]$ coeff $\overset{?}{=} r!\,\#\mathcal{B}(M)$

Exercise 7: Fix rank r and $|E| = n$. Show every valuative invariant $F: M \mapsto F(M)$ on matroid base polytopes is completely determined once we know $\{F(M)\}$ M a lattice path matroid for r, n

on the set of matroids M of rank r and n elements of E

$\binom{n}{r}$



REU Problem 2: (a) Compute $\mathcal{G}_M$ explicitly for all lattice path matroids M.

(b) Use this together with Bidkhori's computation of $Ehr(P_M, m)$ for lattice path matroids to determine the specialization $\mathcal{G}_M \rightsquigarrow Ehr(P_M, m)$ for all matroids M, i.e. $[011010 1] \rightsquigarrow ?$

(c) Apply this to attack/decide some recent conjectures of De Loera et al. on $\mathrm{Ehr}(P_M, m)$:

(i) <u>Conj:</u> $\mathrm{Ehr}(P_M, m)$ has nonnegative coefficients in m

(ii) <u>Conj:</u> $\displaystyle\sum_{m\geq 0} \mathrm{Ehr}(P, m) t^m = \frac{h_0 + h_1 t + \cdots + h_N t^N}{(1-t)^d}$ has $h_0 \leq h_1 \leq \cdots \leq h_p \geq \cdots \geq h_N$ <u>unimodal</u>.