REU Day 6
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Positivity cones and cluster algebras

A log-concave sequence $(a_1, \ldots, a_n) \in (\mathbb{R}^{\geq 0})^n$ is one such that $a_i^2 > a_{i-1} a_{i+1} \; \forall_{1<i<n}$

Consider polynomials in $\mathbb{R}[a_1, \ldots, a_n]$
A polynomial will be called positive if it takes positive values on log concave sequences.

EXAMPLES:

1. 1
2. $a_i^2 - a_{i+1}a_{i-1}$
3. a_i
4. $a_i a_j - a_{i-1}a_{j+1}$ for $i \le j$

Let's restrict our attention to homogeneous polynomials

$$\xcancel{\underbrace{a_1}_{\text{degree } 1} + \underbrace{(a_2a_3 - a_1a_4)}_{\text{degree } 2}}$$

EXERCISE 17: Show that a polynomial is positive if and only if each of its homogeneous components is positive.

Note that if $(a_1, \ldots, a_n)$ is log-concave then so is $(ta_1, t^2a_2, t^3a_3, \ldots, t^na_n)$ for all $t > 0$.

Let's therefore restrict attention to polynomials whose sum of indices is fixed, say equal to k.

EXERCISE 17(b):

Show that if a polynomial is positive, then segregating it into sums of monomials where the sum of indices on the variables is fixed, each such component sum is also positive.

Let $P_{N,k}$ = {positive polynomial which are homogeneous of degree N and have monomials with sum of indices k}

This forms a <u>convex cone</u>, that is $\forall p,q \in P_{N,k}$ and $a,b \geq 0$

$$ap+bq \in P_{N,k}$$

Some cones are <u>polyhedral</u>, meaning $\exists p_1, p_2, \ldots, p_\alpha \in P$ such that any $p \in P$ can be written as $p = \sum_{i=1}^{\alpha} b_i p_i$ for some $b_i \in \mathbb{R}^{\geq 0}$

$P_{N,k}$ itself is likely to be hard to understand.

Instead, let's consider the <u>L-positive elements</u>, that is, those that can be expressed as <u>Laurent polynomials</u> in

$$\{a_i, a_i a_j - a_{i-1} a_{j+1}\}$$

with <u>positive</u> <u>coefficients</u>.

EXAMPLES:

(1) $$a_3^2 a_5 - a_1 a_4 a_6 = \frac{(a_2 a_5 - a_1 a_6) a_3^2 + a_1 a_6 (a_3^2 - a_2 a_4)}{a_2}$$

(2) $$a_1 a_3 a_5 + a_2 a_3 a_4 - a_1 a_4^2 - a_2^2 a_5 = \frac{a_2 a_4 (a_3^2 - a_2 a_4) + (a_2^2 - a_1 a_3)(a_4^2 - a_3 a_5)}{a_3}$$

Let $P^L_{N,k} := P_{N,k} \cap \left\{ \begin{matrix} \text{L-positive} \\ \text{polynomials} \end{matrix} \right\}$

REU Problem 6(a):

Describe $P^L_{N,k}$.

Is it polyhedral?

If so, what are its generators?

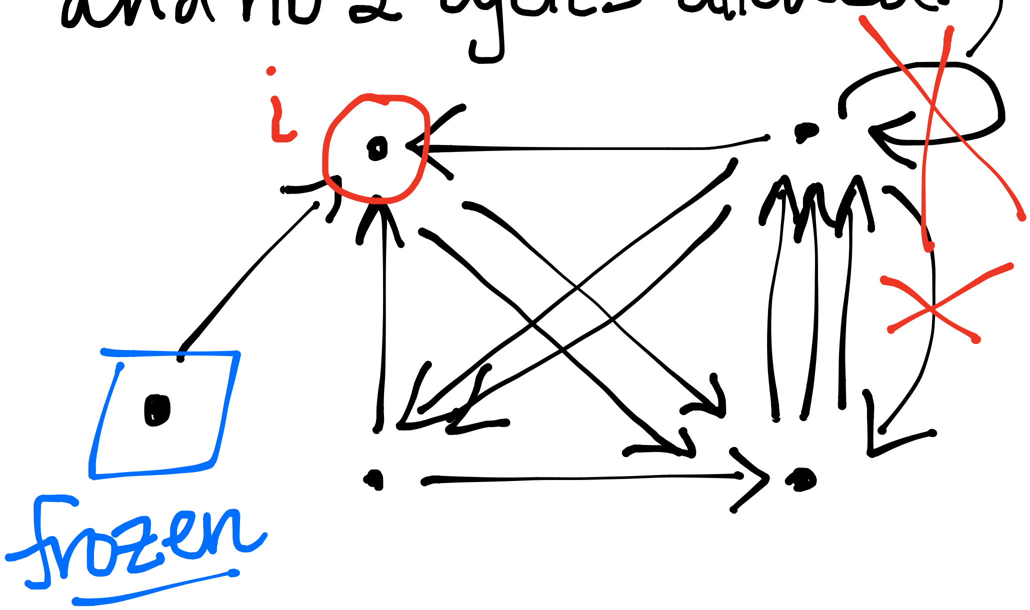
CONJECTURE: If $n \leq 4$ then $\forall N,k$ the cone $P_{N,k}$ contains only the polynomials in $\{a_i, a_i a_j - a_{i-1} a_{j+1}\}$.

(FALSE for $n=5$)

How might we produce more generators for $P^L_{N,k}$?

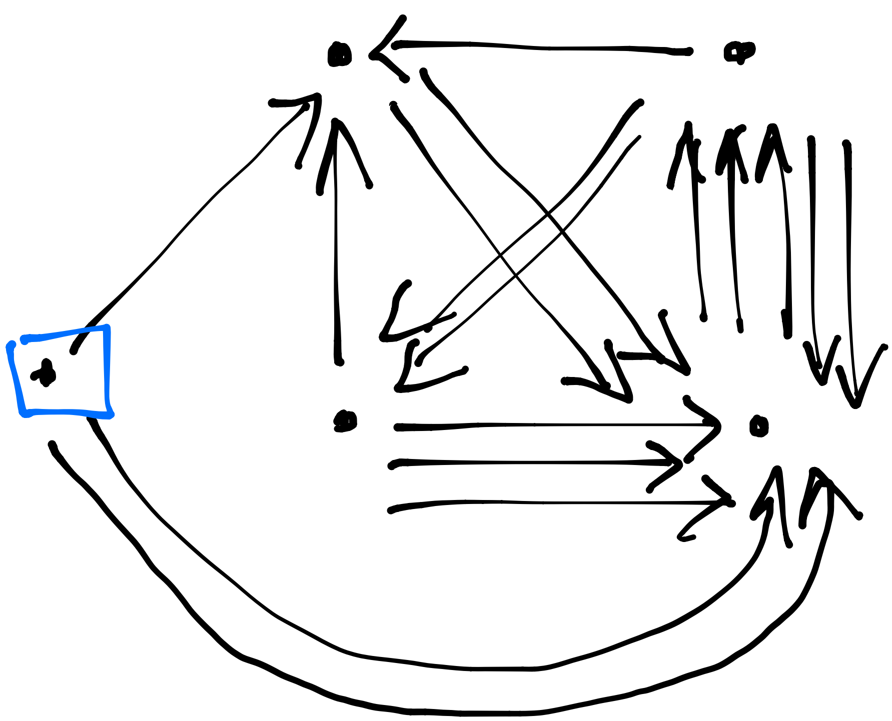
Cluster algebras

1. A quiver is a directed multigraph (but with no loops and no 2-cycles allowed.)

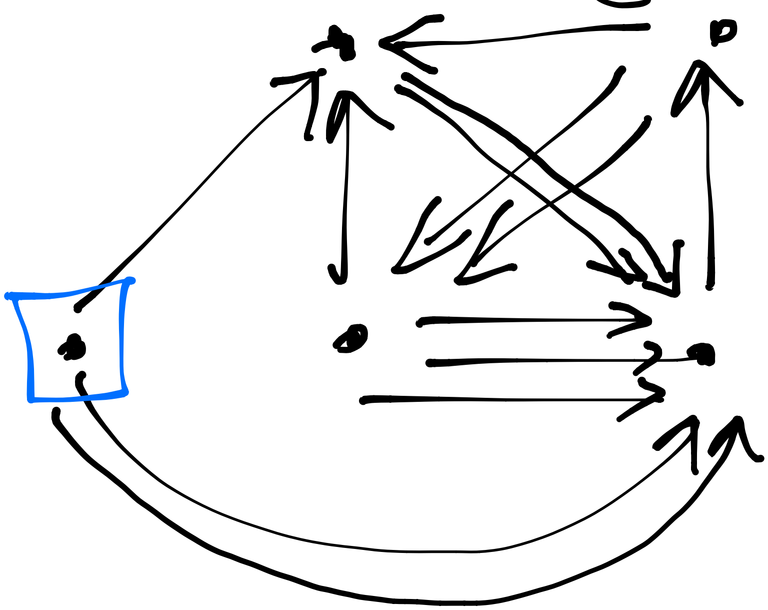


2. Quiver mutation (at i)
(not allowed to mutate at frozen nodes)

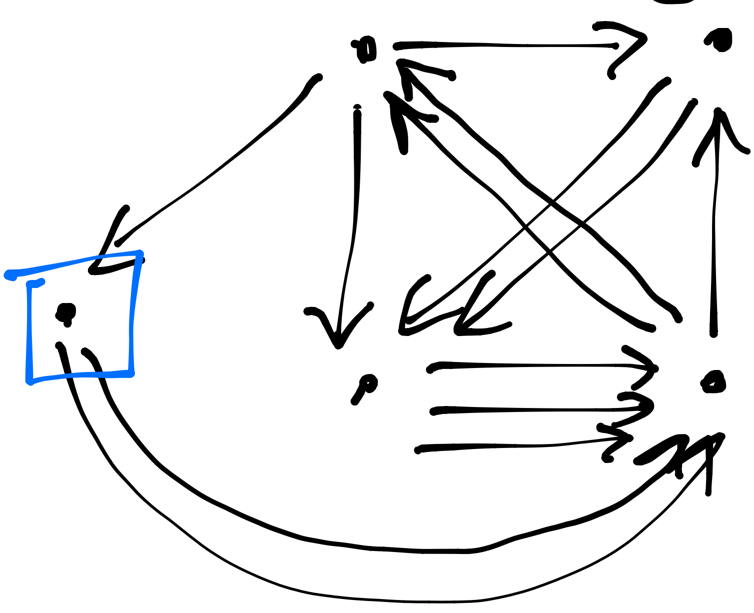
a. For any path $j \to i \to k$, add the arrow $j \to k$.



b. Remove 2-cycles.



c. Reverse all edges incident to i



EXERCISE 18(a): If we let $\mu_i :=$ mutation of the quiver at node i

then check that $\mu_i^2 = \mathrm{id}$.

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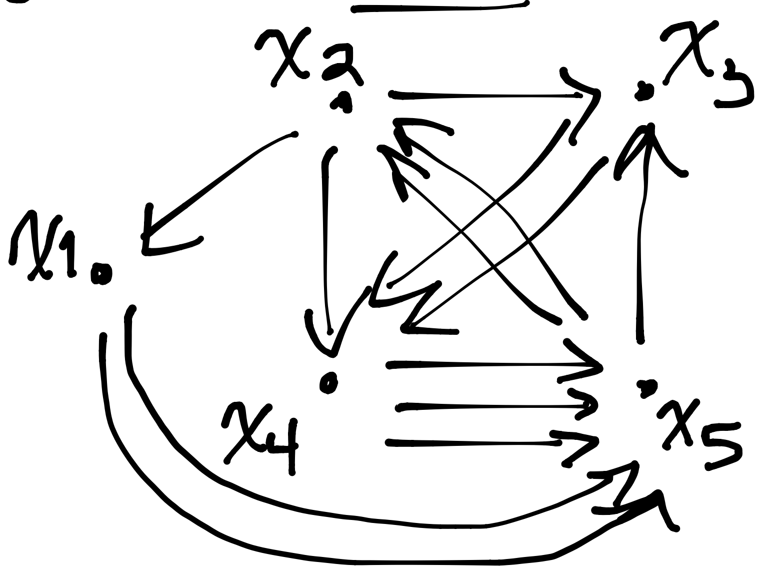
3. Cluster variables

Put variables on the nodes to form the <u>initial</u> <u>cluster</u>



When we mutate at x_3, we replace x_3 by x_3' where

$$x_3 x_3' = \underbrace{x_4^2}_{\substack{\text{product} \\ \text{of variabes} \\ x \text{ with } x_3 \to x}} + \underbrace{x_2 x_5}_{\substack{\text{product of} \\ \text{variables} \\ x \text{ with} \\ x_3 \leftarrow x}}.$$

This gives us a new cluster

$$\{x_1, x_2, x_3', x_4, x_5\}.$$

EXERCISE 18(b): Show mutation of variables is also an involution.

e.g. $x_3 \longmapsto x_3' \longmapsto x_3'' = x_3$

Exciting things about these new cluster variables that we produce:

1. The Laurent Phenomenon

All cluster variables are Laurent polynomials in the initial cluster, and the coefficients are nonnegative.

2. Finite type classification

A cluster algebra is of finite type if only finitely many cluster variables can appear.

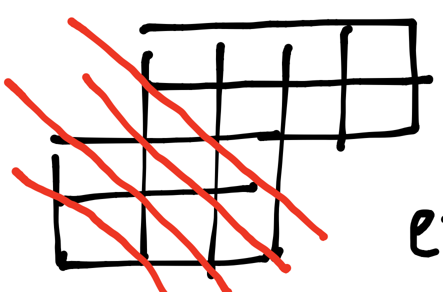
This occurs if and only if its quiver is <u>mutation-equivalent</u> (in its mutable/nonfrozen part) to an orientation of a <u>Dynkin</u> <u>diagram</u> of a <u>finite</u> <u>root system</u>.

e.g. Type A Dynkin diagram:

$$\bullet \rightarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet \leftarrow \bullet$$

<u>Our cluster algebra of interest</u>:

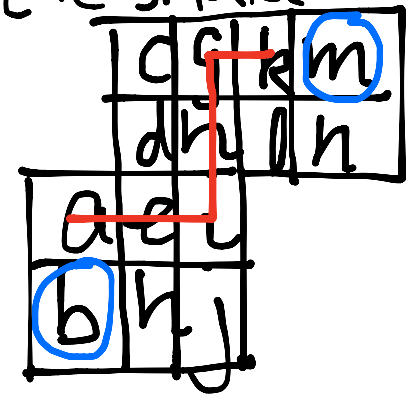
Start with a <u>width 2</u> <u>snake</u>:



All diagonals beside initial & final have exactly 2 boxes.

Upper/lower boundary are NE lattice paths.

Fill the snake with variables:



<u>Frozen</u> <u>variables</u>: b, m

and all 2x2 (adjacent square) determinants $\begin{vmatrix} a & e \\ b & h \end{vmatrix}$, $\begin{vmatrix} e & i \\ h & j \end{vmatrix}$, …

<u>Mutable</u> <u>variables</u>:

$\underbrace{a, e, i, h, g, k}$

start one square above b,
go right as far as possible,
then up as far as possible, then right, up, etc

To each square $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ associate

<u>the identity</u> $ad = bc + \begin{vmatrix} a & b \\ c & d \end{vmatrix}$.

To $\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline\end{array}$ associate

$$b\begin{vmatrix} a & c \\ d & f \end{vmatrix} = a\begin{vmatrix} b & c \\ e & f \end{vmatrix} + c\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

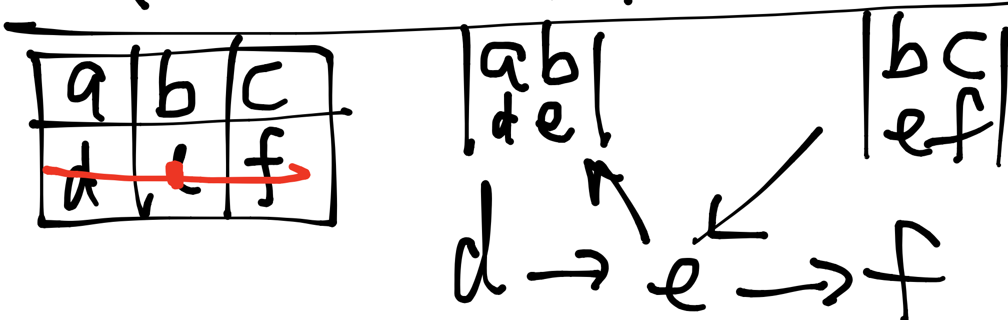
and $e\begin{vmatrix} a & c \\ d & f \end{vmatrix} = d\begin{vmatrix} b & c \\ e & f \end{vmatrix} + f\begin{vmatrix} a & b \\ d & e \end{vmatrix}$.

<u>Arrows of the quiver</u>

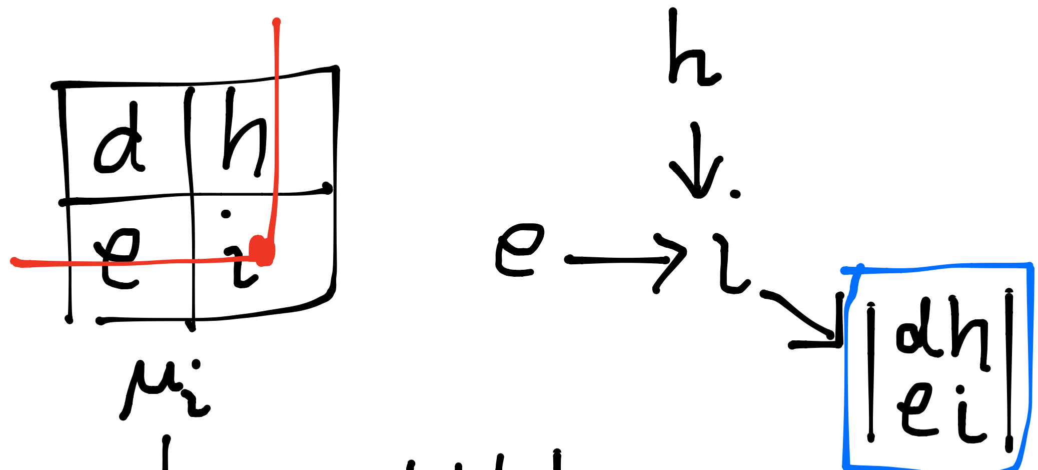
$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline\end{array} \rightsquigarrow \begin{tikzcd} a \arrow[r] & b \arrow[r] \arrow[dr] & c \\ \boxed{\begin{vmatrix} b & c \\ e & f \end{vmatrix}} \arrow[ur] & & \boxed{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \end{tikzcd}$$

$$bb' \overset{\mu_b}{=} a\begin{vmatrix} b & c \\ e & f \end{vmatrix} + c\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

(so $b' = \begin{vmatrix} a & c \\ d & f \end{vmatrix}$)

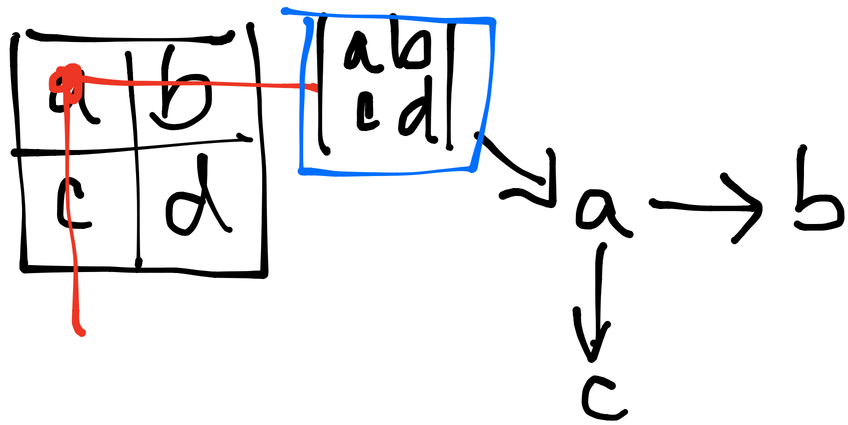


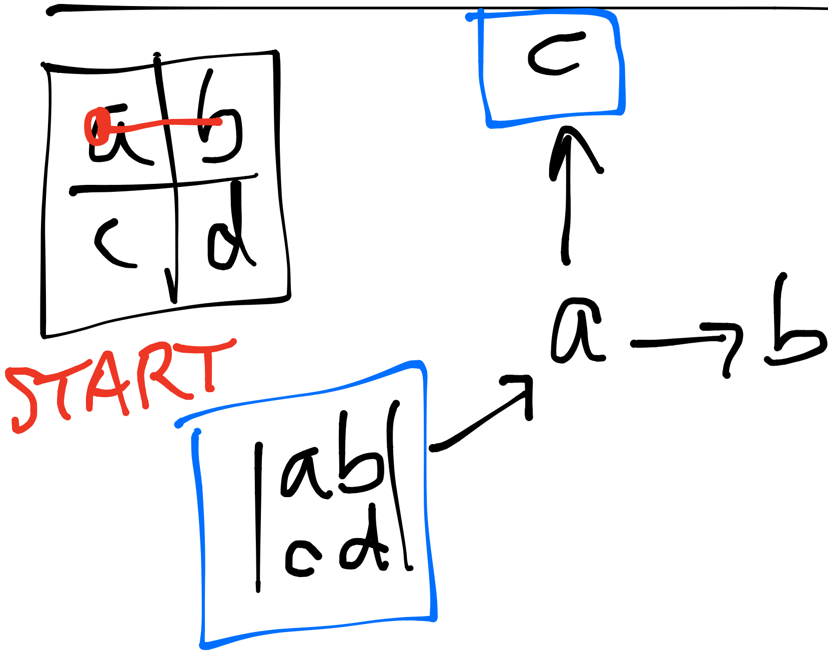
$$ee' = d\begin{vmatrix} b & c \\ e & f \end{vmatrix} + f\begin{vmatrix} a & b \\ d & e \end{vmatrix}, \text{ so } e' = \begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

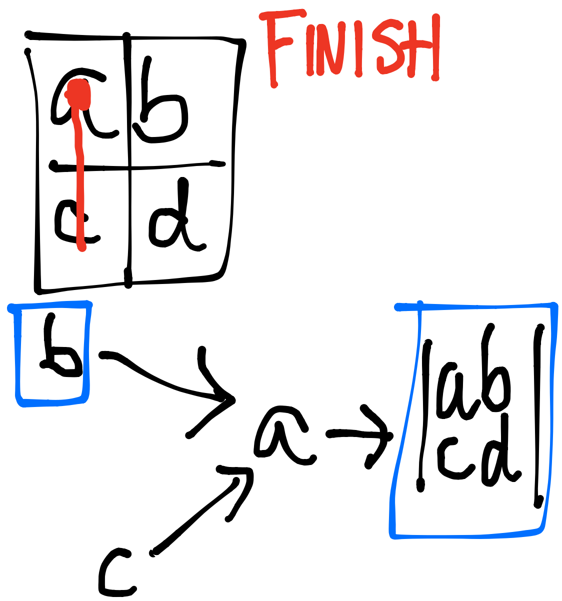


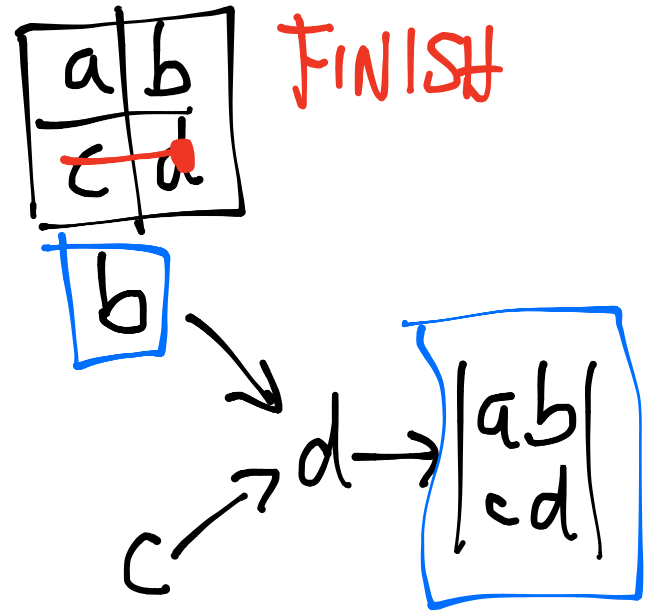
$$i\,i' \overset{\mu_i}{=} eh + \begin{vmatrix} d & h \\ e & i \end{vmatrix}$$

(so $d = i'$)


a b
c d
ab
cd
a
b
c


a b
c d
START
ab
cd
c
a
b


FINISH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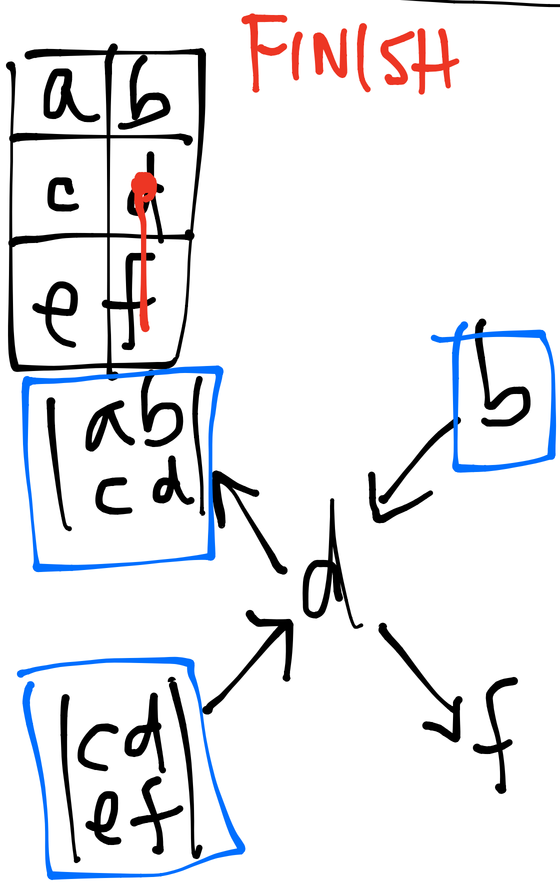
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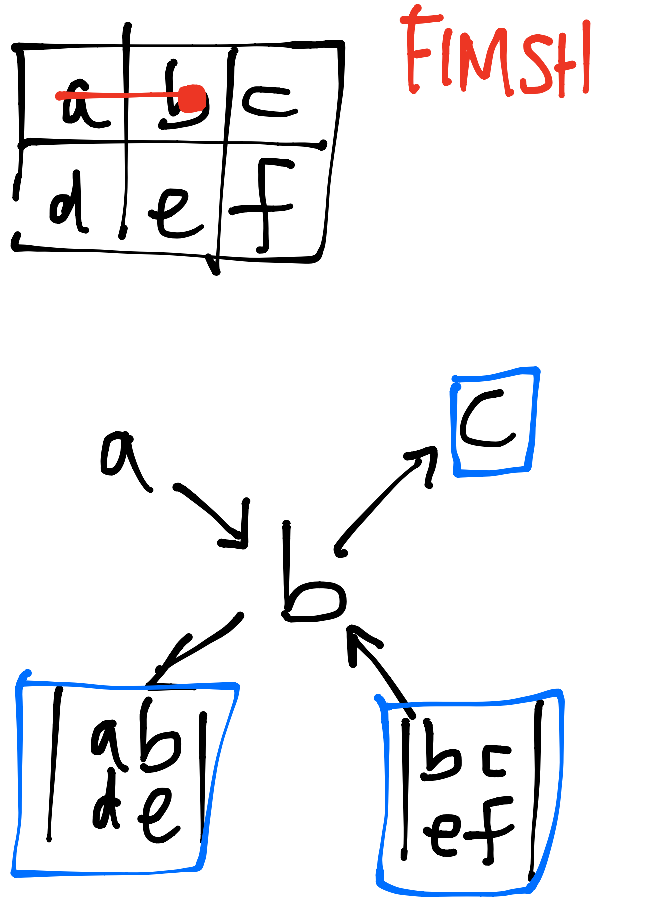
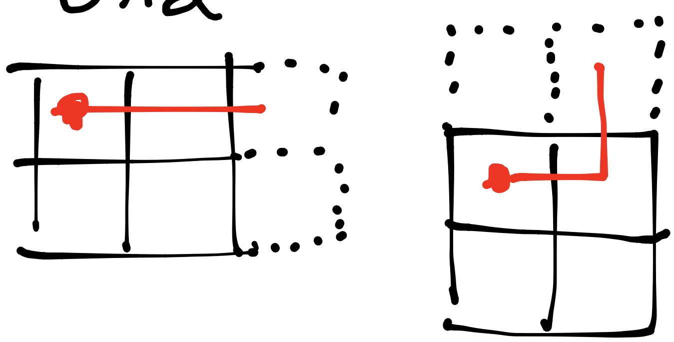
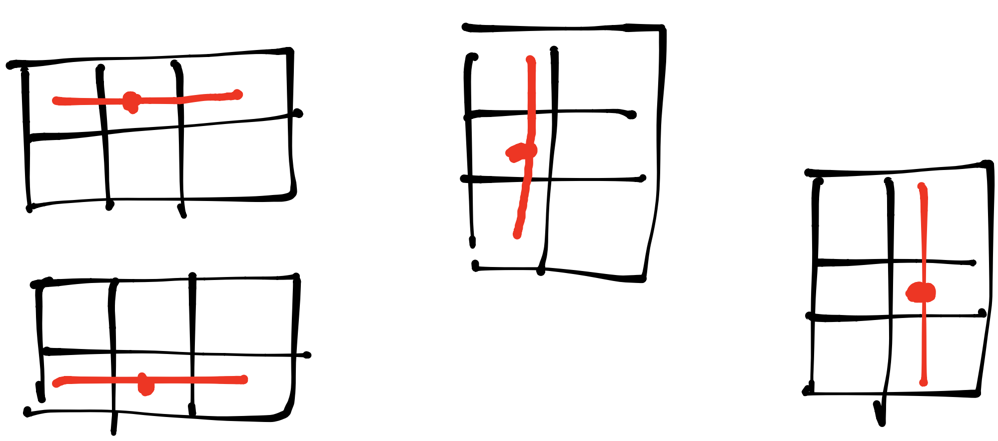


As one follows the red path through the mutable variables as nodes, the identities used for the mutations involve these 2x2 or $\begin{smallmatrix}2\times 3\\ 3\times 2\end{smallmatrix}$ rectangles:

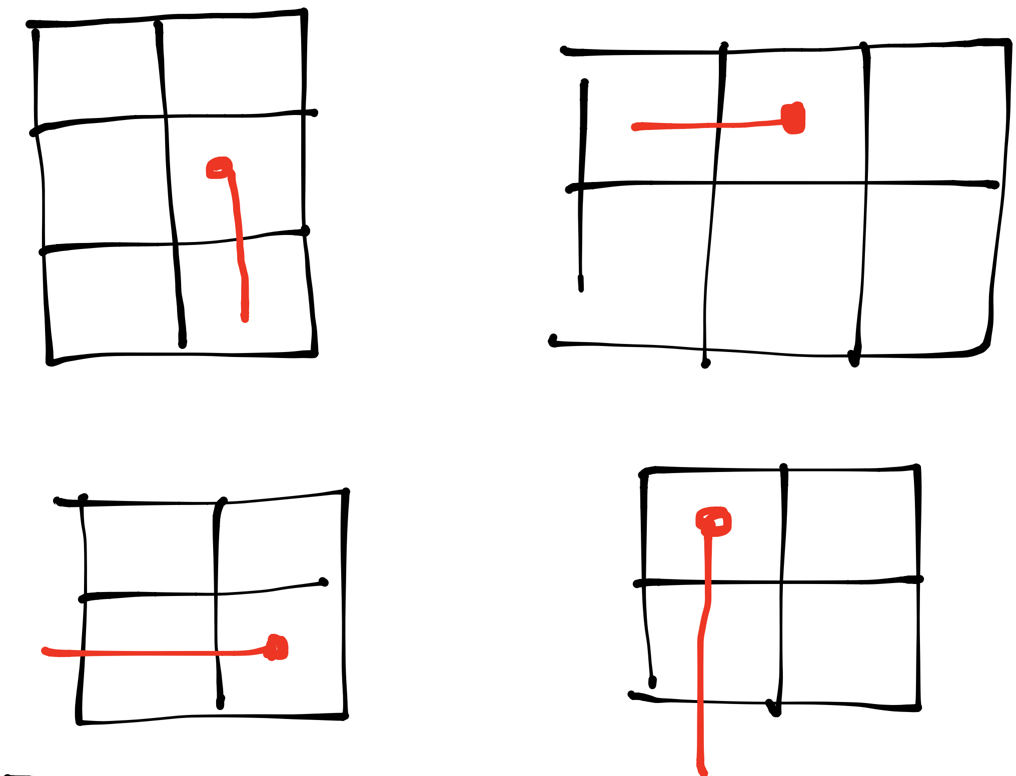
STARTS:



MIDDLE OF A LINE:

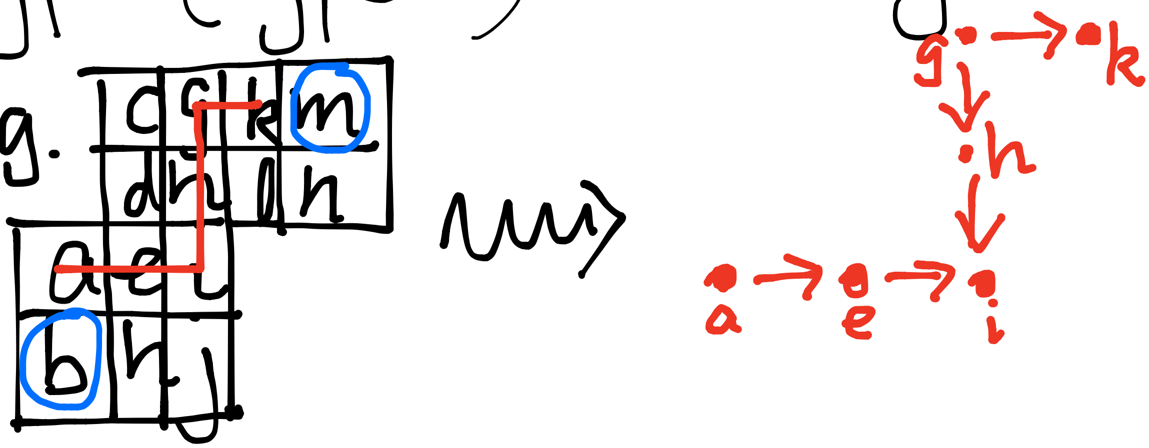


FINISHES:



It will end up being a finite type (Type A) cluster algebra

e.g.



A miracle occurs: all cluster variables will be <u>polynomials</u> in $a, b, c, d, e, \ldots\ldots$

Furthermore, if we make a substitution

		a_5	a_6
a_2	a_3	a_4	a_5
a_1	a_2	a_3	a_4

,

then frozen variables are the simple positive polynomials

$$\{ a_i, \; a_i a_j - a_{i-1} a_{j+1} \}.$$

REU Problem 6(b):

Describe all the cluster variables explicitly (before the above specialization) as polynomials in

$a, b, c, d, e, \ldots$

(as opposed to Laurent polynomials in the initial cluster variables, which is known; see Schiffler.)

EXERCISE 19:

	a_4	a_5
~~a_2~~	~~a_3~~	~~a_4~~
a_1	a_2	a_3

Mutate at a_3, a_4.

Show that you get a previously exhibited L-positive element.

CONJECTURE:

$P^L_{N,k}$ is the cone generated by the union of all cluster variables coming from all snake cluster algebras.