REU Day 8

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Counting nondegenerate hypermatrices over $\mathbb{F}_q$

Ref: Gelfand, Kapranov, Zelevinsky "Discriminants, resultants and multidimensional determinants", Chapter 14.

THM 1. $|S_n| = n!$

e.g. $n=3$ $|S_3| = 3! = 6$

For $w \in S_n$, we've seen (Day 7)

$\ell(w) =$ # of inversions of w

e.g. $\ell(3412) = 4$

$\ell(1243) = 1$

THM 2. (MacMahon)

$$\sum_{w \in S_n} q^{\ell(w)} = [n]!_q := [1]_q [2]_q \cdots [n]_q$$

where

$$[k]_q = 1 + q + q^2 + \dots + q^{k-1}$$

e.g. $n=3$

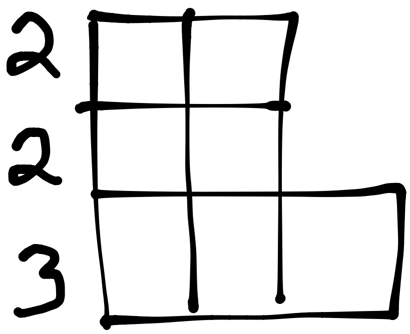
ω	$\ell(\omega)$
123	0
132	1
213	1
231	2
312	2
321	3

$$1+2q+2q^2+q^3 = 1\cdot(1+q)(1+q+q^2)$$
$$= [3]!_q$$

<u>THM</u> 3. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, the # of permutations in S_n that fit inside λ is

$$(\lambda_1-(n-1))(\lambda_2-(n-2))\cdots(\lambda_{n-1}-1)\lambda_n.$$

e.g. $\lambda = \langle 3,2,2 \rangle$



French convention for drawing Ferrers diagram of λ

$$\#\left\{ \begin{matrix} 213 \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix} , \begin{matrix} 123 \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix} \right\} = 2$$

$$(3-2)(2-1)(2-0) = 1 \cdot 1 \cdot 2 = 2 \checkmark$$

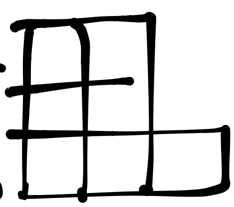
<u>THM</u> 4. Given $\lambda \subseteq \langle n^n \rangle$

$$\sum_{\substack{w \in S_n: \\ w \text{ fits inside } \lambda}} q^{\ell(w)} = [\lambda_1 - (n-1)]_q [\lambda_2 - (n-2)]_q \cdots [\lambda_n - 0]_q$$

e.g.

w	$\ell(w)$
123	0
213	1

$\lambda =$ 2, 2, 3



$$q^0 + q^1 = 1 + q = [2]_q$$
$$= [3-2]_q [2-1]_q [2-0]_q \checkmark$$

EXERCISE 23: Prove this.

Now let

$GL_n(\mathbb{F}_q) := \left\{ \begin{array}{l} n\times n \text{ invertible} \\ \text{matrices over } \mathbb{F}_q \end{array} \right\}$

THM 5.

$|GL_n(\mathbb{F}_q)| = (q^n-1)(q^n-q)(q^n-q^2)\cdots(q^n-q^{n-1})$

pick 1st column any vector in $\mathbb{F}_q^n$ not $\underline{0}$

pick 2nd column in $\mathbb{F}_q^n$ not in $\boxed{\text{span of 1st column}}$

etc...

a 1-dim'l subspace, so q^1 vectors to avoid.

Let's rewrite this as

$$|GL_n(\mathbb{F}_q)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$$

$$= q^{\binom{n}{2}} (q-1)^n [n]!_q$$

Can we explain this form?

<u>Observation</u>: $(q-1)^n$ had to appear since the orbit of any $g \in GL_n(\mathbb{F}_q)$ under left-multiplication by diagonal (invertible) matrices has size $(q-1)^n$.

In fact, there is a close connection S_n and $GL_n(\mathbb{F}_q)$ that will explain the whole story...

<u>Bruhat</u> <u>decomposition</u>:

$\mathcal{B} := \{\text{lower triangular matrices}\}$

For $w \in S_n$, its <u>Bruhat</u> <u>cell</u> is

$$\mathcal{B} w \mathcal{B}$$

interpret w as a permutation matrix

and satisfies

$$|\mathcal{B} w \mathcal{B}| = q^{\binom{n}{2} + \ell(w)} (q-1)^n$$

and $$GL_n(\mathbb{F}_q) = \bigsqcup_{w \in S_n} \mathcal{B} w \mathcal{B}$$

THM 6:

Let $\mathrm{Mat}(\lambda) := \left\{ \begin{array}{l} g \in GL_n(\mathbb{F}_q): \\ \text{nonzero entries} \\ \text{of } g \text{ lie in } \lambda \end{array} \right\}$

Then $|\mathrm{Mat}(\lambda)|$

$$= q^{\binom{n}{2}} (q-1)^n [\lambda_1 - (n-1)]_q [\lambda_2 - (n-2)]_q \cdots [\lambda_n - 0]_q$$

(In particular, when $\lambda \not\supset \langle n, n-1, \ldots, 2, 1 \rangle$, you'll get 0, so there are <u>no</u> such g in $GL_n(\mathbb{F}_q)$)

<u>THM</u> 7 (L.-Liu-Morales-Panova-Sam-Zhang).

Given <u>any</u> board $S \subset \langle n^n \rangle$ the number of $g \in GL_n(\mathbb{F}_q)$ whose nonzero entries lie in S is a q-analogue of the # of $w \in S_n$ lying on S, in an appropriate <u>technical</u> <u>sense</u>.

(see the above 6 author paper for the precise statement)

Let's now try to extend the story to hypermatrices...

DEF'N: An r-dimensional hypermatrix of format $(k_1+1)\times(k_2+1)\times\dots\times(k_r+1)$ $k_i \geq 1$ over a field $\mathbb{F}$ is an array $A = \left(a_{i_1 i_2 \cdots i_r}\right)_{0 \leq i_j \leq k_j}$ of elements of $\mathbb{F}$.

When $2k_j \le \sum_{i=1}^{r} k_i$ for all j, there is associated a hyperdeterminant $\mathrm{Det}(A)$ with the following properties:

- It is a polynomial in the entries of A with $\mathbb{Z}$ coefficients
- It is <u>homogeneous</u> in the entries of each <u>slice</u> (:= fix one index, let the other $r-1$ of them vary)
- irreducible

...

- interchanging parallel slices will change Det(A) by a predictable sign.
- if two parallel slices are proportional then $\mathrm{Det}(A) = 0$.
- it is unchanged by adding a multiple of one slice to a parallel slice.

Alternatively:

$GL_{k_1+1} \times \ldots \times GL_{k_r+1}$ acts on and Det(A) is invariant up to a product of powers of determinants

Det(A) is the resultant of the system of equations

$$\frac{\partial f}{\partial x_i^{(j)}} = 0 \quad \text{for } j = 1, 2, \ldots, r \quad i = 0, 1, \ldots, k_j$$

where

$$f\left(\{x_i^{(j)}\}_{\substack{j=1,\ldots,r \\ i=0,\ldots,k_j}}\right) = \sum_{i_1,\ldots,i_r} a_{i_1\ldots i_r} x_{i_1}^{(1)} x_{i_2}^{(2)} \cdots x_{i_r}^{(r)}$$

In particular, over an alg. closed field, $\mathrm{Det}(A)=0$ if and only if this system has a nontrivial solution (i.e. no variable set $x_0^{(j)}, x_1^{(j)}, \dots, x_{k_j}^{(j)}$ is all zero.)

(GKZ prove this over $\mathbb{C}$; we assume alg. closed suffices)

EXAMPLE: $r=2$, $k_1=k_2=1$

$$f = f(\underbrace{x_0^{(1)}}_{x_0}, \underbrace{x_1^{(1)}}_{x_1}, \underbrace{x_0^{(2)}}_{y_0}, \underbrace{x_1^{(2)}}_{y_1})$$

$$= a_{00}x_0y_0 + a_{10}x_1y_0 + a_{01}x_0y_1 + a_{11}x_1y_1$$

The above system becomes

$$\begin{cases} a_{00}y_0 + a_{01}y_1 = 0 \\ a_{10}y_0 + a_{11}y_1 = 0 \\ a_{00}x_0 + a_{10}x_1 = 0 \\ a_{01}x_0 + a_{11}x_1 = 0 \end{cases}$$

and having a nontrivial solution, i.e. $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$

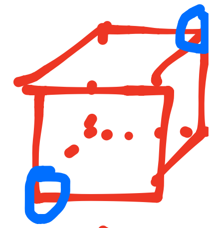
is equivalent to $A = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$

having $\mathrm{Det}(A) = a_{00}a_{11} - a_{10}a_{01} = 0$.

EXAMPLE: $r=3$

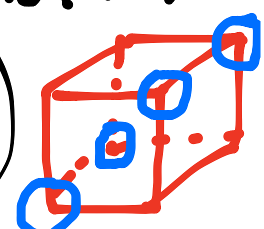
$k_1 = k_2 = k_3 = 1$

$$A = (a_{ijk})_{0 \le i,j,k \le 1}$$

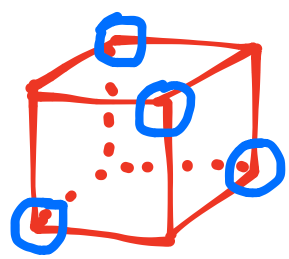


$$\mathrm{Det}(A) = \left(a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{011}^2 a_{100}^2 \right)$$

$$-2\left(a_{000} a_{001} a_{110} a_{111} + a_{000} a_{010} a_{101} a_{111} + \underbrace{\ldots}_{4 \text{ more terms}} \right)$$



$$+4\left(a_{000} a_{011} a_{101} a_{110} + a_{001} a_{010} a_{100} a_{111} \right)$$



REU Problem 8a:

How many hypermatrices of format $(k_1+1)\times \cdots \times (k_r+1)$ over $\mathbb{F}_q$ have nonzero hyperdeterminant?

THM (Musiker-Yu, unpublished)

When $r=3$, $k_1=k_2=k_3=1$ the answer is

$$q^3(q-1)^2\,[4]_q$$

"THM" (L. Sam)

For $r=3$, $k_1=k_2=1$
$k_3=2$

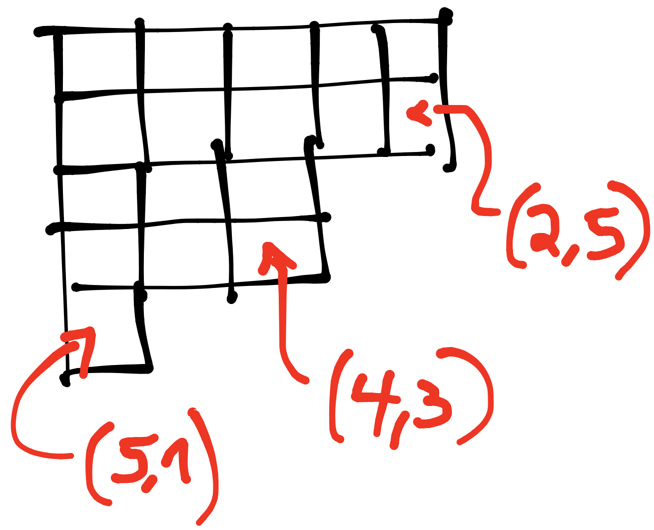
the answer is

$$q^4(q-1)^4[3]_q\left([2]_q\right)^2.$$

<u>An idea</u>:

A generalized plane partition in dimension r is a finite subset P of $\mathbb{N}^r$ such that $(x_1,\dots,x_r)\in P$ and $y_1\le x_1,\dots,y_r\le x_r$
$\Rightarrow (y_1,\dots,y_r)\in P$.

e.g. $r=2$



PROBLEM 8b:

Let λ be a g.p.p. $\subseteq \{0,\dots,k_1\} \times \dots \times \{0,\dots,k_r\}$. How many hypermatrices A of this format with $\det(A) \neq 0$ are supported on λ?

A possible elementary tool:

EXERCISE 24:

Show that if λ has 2 equal parts, so

$$\lambda = \langle \lambda_1 \geq \dots \geq \lambda_i = \lambda_{i+1} > \lambda_{i+2} \geq \dots \geq \lambda_n \rangle$$

and if one defines

$$\lambda^- := \langle \lambda_1, \dots, \lambda_i, \lambda_{i+1} - 1, \lambda_{i+2}, \dots, \lambda_n \rangle$$

$$\lambda^{--} := \langle \lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1} - 1, \lambda_{i+2}, \dots, \lambda_n \rangle$$

then

$$|\mathrm{Mat}(\lambda)| - (q+1)|\mathrm{Mat}(\lambda^-)| + q|\mathrm{Mat}(\lambda^{--})| = 0.$$

REU Problem 8c:

Is there an analogue of permutations here (e.g. generalizing LCMPSZ?) that interprets these $q=1$ limits?:

$$q^3(q-1)^2 \underbrace{[4]_q}_{q \to 1 \text{ gives } 4}$$

$$q^4(q-1)^4 \underbrace{[3]_q ([2]_q)^2}_{q \to 1 \text{ gives } 12}$$

REU Problem 8d:

(Special case of 8b)
When is $Det(A) = 0$?

G-K-Z have some guidance, perhaps, in certain boundary cases.