REU 2016 Day 1 V. Reiner

Sandpile groups of cubes

1. **Matrix-Tree** Theorem(s)
2. Sandpile groups
3. Cube case
4. REU Problem 1

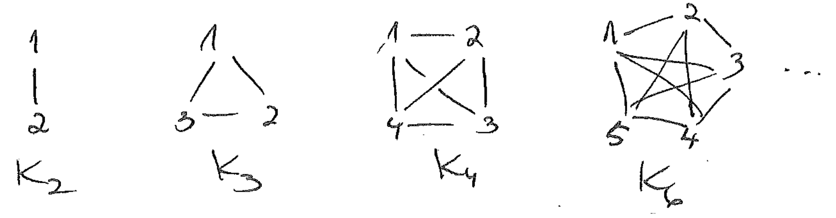
Matrix-Tree Thms

1. DEF'N: In a graph G, a spanning tree T is a subset of edges
(multiple edges **OK** , self-loops ~~not ok~~ )

containing no cycles, connecting all the vertices.

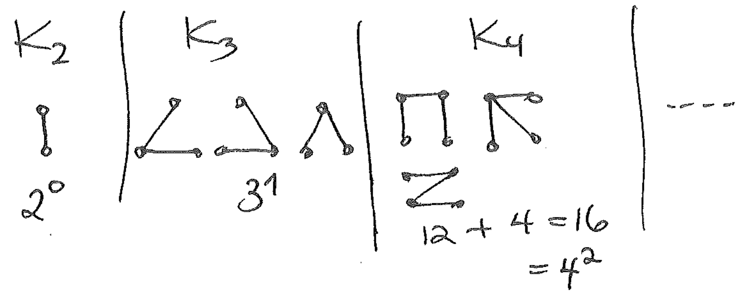
$\kappa(G) :=$ # spanning trees in G

EXAMPLE: Complete graphs K_n



have $\kappa(K_n) = n^{n-2}$ by a famous theorem of Cayley

e.g.



$\kappa(G)$ is easier to compute than one might think

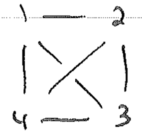
THM (Kirchhoff 1840's) $\kappa(G) = \det \overline{L(G)}^{i,i}$ for any vertex i of G where $L(G) :=$ Laplacian matrix of G having $L(G)_{i,j} := \begin{cases} \deg_G(i) & \text{if } i=j \\ -(\#\text{edges } i \text{—} j) & \text{if } i \neq j \end{cases}$ for vertices i,j

↗ see Stanley Thm 9.8

and $\overline{L(G)}^{i,i} :=$ reduced Laplacian $:= L(G) - \left\{ \begin{matrix} \text{row } i, \\ \text{column } i \end{matrix} \right\}$

(2) **EXAMPLE:**

$$L(K_4) = \begin{array}{c|ccc|c} & 1 & 2 & 3 & 4 \\ \hline 1 & 3 & -1 & -1 & -1 \\ 2 & -1 & 3 & -1 & -1 \\ 3 & -1 & -1 & 3 & -1 \\ \hline 4 & -1 & -1 & -1 & 3 \end{array} \qquad \overline{L(K_4)}^{4,4} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 3 & -1 & -1 \\ 2 & -1 & 3 & -1 \\ 3 & -1 & -1 & 3 \end{array}$$



$$\kappa(K_4) = \det \overline{L(K_4)}^{4,4} = 3\det\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} - (-1)\det\begin{bmatrix} -1 & -1 \\ -1 & 3 \end{bmatrix} + (-1)\det\begin{bmatrix} -1 & 3 \\ -1 & -1 \end{bmatrix}$$

$$= 3\cdot(9-1) + (-3-1) - (1+3)$$

$$= 24 - 4 - 4$$

$$= 16 \checkmark$$

COROLLARY: $\overset{\text{(easy)}}{}$ If $L(G)$ has eigenvalues $(\mu_1, \mu_2, \ldots, \mu_{n-1}, \underset{0}{\underset{\parallel}{\mu_n}})$ $\leftarrow \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \ker L(G)$ <u>always</u>!

↗ see Stanley Cor. 9.10

then $\kappa(G) = \dfrac{\mu_1\mu_2\cdots\mu_{n-1}}{n}$

EXAMPLE: To deduce Cayley's theorem $\kappa(K_n) = n^{n-2}$,

↗ Stanley Example 9.11

note that $L(K_n) = \begin{bmatrix} n-1 & -1 & -1 & -1 \\ -1 & n-1 & & \vdots \\ \vdots & & \ddots & -1 \\ -1 & \cdots & -1 & n-1 \end{bmatrix} = nI_{n\times n} - \underbrace{J_{n\times n}}_{\text{the all ones matrix}}$ $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \ddots & & \vdots \\ \vdots & & & \\ 1 & \cdots & & 1 \end{bmatrix}$

has the <u>0-eigenvector</u> $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, but then

also an $(n-1)$-dimensional <u>n-eigenspace</u> given by $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^{\perp} := \{\underline{x} \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$

since $(nI - J)\underline{x} = n\cdot\underline{x} - J\underline{x}$

$= n\cdot\underline{x}$ if $x_1 + \cdots + x_n = 0$.

So $L(K_n)$ has eigenvalues $\begin{pmatrix} \mu_1, & \ldots, & \mu_{n-1}, & \mu_n \\ \parallel & & & \parallel \\ n, & \ldots, & n, & 0 \end{pmatrix}$

and $\kappa(K_n) = \dfrac{\overbrace{n\cdot n\cdots n}^{n-1 \text{ factors}}}{n} = \dfrac{n^{n-1}}{n} = n^{n-2} \checkmark$

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2. Sandpile groups

Kirchhoff's Thm. tells us a fair bit out $\overline{L(G)}^{i,i}$, $L(G)$ as maps <u>over $\mathbb{Q}$</u>,

e.g. $\mathbb{Q}^{n-1} \underset{\text{isomorphism}}{\overset{\overline{L(G)}^{i,i}}{\hookrightarrow\!\!\!\twoheadrightarrow}} \mathbb{Q}^{n-1}$ if G is connected (so it has a spanning tree)

$$\underset{\mathbb{Q}}{\overset{\cong}{\ker L(G)}} \hookrightarrow \mathbb{Q}^n \xrightarrow[\text{co-rank }1]{L(G)} \mathbb{Q}^n \twoheadrightarrow \underset{\mathbb{Q}}{\overset{\cong}{\mathbb{Q}^n/\operatorname{im} L(G)}} = \operatorname{coker} L(G)$$

$$\operatorname{coker} L(G) := \mathbb{Q}^n/\operatorname{im} L(G)$$
$$\cong \mathbb{Q}^n/\mathbb{Q}^{n-1}$$
$$\cong \mathbb{Q}$$

But what about over <u>other</u> <u>fields</u>, or even <u>other</u> rings R

i.e. $$\ker L(G) \hookrightarrow R^n \xrightarrow{L(G)} R^n \twoheadrightarrow \operatorname{cok} L(G)$$

~~where~~ $\operatorname{coker} L(G) = R^n / \operatorname{im} L(G)$?

One gets a <u>universal</u> <u>answer</u> by doing a computation with $R = \mathbb{Z}$:

DEF'N: For any $A \in \mathbb{Z}^{n\times n}$, $\exists\ P, Q \in \underline{GL_n(\mathbb{Z})}$ $= \left\{ \begin{matrix} n\times n\ \mathbb{Z}\text{-matrices} \\ \text{with } \det \in \mathbb{Z}^\times = \{\pm 1\} \end{matrix} \right\}$

such that $$\underbrace{P}_{\substack{\text{invertible}\\\text{row}\\\text{operations}\\\text{over } \mathbb{Z}}}A\underbrace{Q}_{\substack{\text{invertible}\\\text{column}\\\text{operations}\\\text{over } \mathbb{Z}}} = \underbrace{\begin{bmatrix} s_1 & & 0 \\ & s_2 & \\ 0 & & \ddots\ s_n \end{bmatrix}}_{\text{diagonal}} = S$$

diagonal, and if s_i divides s_{i+1} in $\mathbb{Z}$ (with $s_i \geq 0$) then S is <u>unique</u>, and called the <u>Smith normal form</u> of A

Equivalently, one can compute
$$\operatorname{coker}(\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n) = \mathbb{Z}^n / \operatorname{im} A = \mathbb{Z}^n / \operatorname{im} \begin{bmatrix} s_1 & & 0 \\ & s_2 & \\ 0 & & \ddots\ s_n \end{bmatrix} \cong \bigoplus_{i=1}^{n} \mathbb{Z}/s_i\mathbb{Z}$$

(and then $\operatorname{coker}(R^n \xrightarrow{A} R^n) \cong \bigoplus_{i=1}^{n} R/s_iR$ for <u>any comm. ring</u> R)

<u>EXAMPLES</u>

① $$L(K_4) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \overset{P(-)}{\rightsquigarrow} \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{(-)Q}{\rightsquigarrow} \begin{bmatrix} \overbrace{\begin{matrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{matrix}}^{\overline{L(K_4)}^{4,4}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 0 \end{bmatrix} \overset{P(-)}{\rightsquigarrow} \left[\begin{array}{ccc|c} 0 & 8 & -4 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -4 & 4 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Big\downarrow (-)Q$$

$$\underset{\text{Smith normal form}}{S} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \leftsquigarrow \left[\begin{array}{ccc|c} 0 & 0 & 4 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \leftsquigarrow \left[\begin{array}{ccc|c} 0 & 8 & -4 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \operatorname{coker} L(K_4) \cong \mathbb{Z}/1\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}$$
$$= \mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z})^2$$
$$\left(\text{and } \operatorname{coker} \overline{L(K_4)}^{4,4} \cong (\mathbb{Z}/4\mathbb{Z})^2\right)$$

② $$A = \underset{\substack{\text{already diagonal,}\\\text{but not in Smith normal}\\\text{form}}}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \overset{(-)Q}{\rightsquigarrow} \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} \overset{P(-)}{\rightsquigarrow} \begin{bmatrix} 2 & 2 \\ -2 & 1 \end{bmatrix} \overset{(-)Q}{\rightsquigarrow} \begin{bmatrix} 6 & 2 \\ 0 & 1 \end{bmatrix} \overset{P(-)}{\rightsquigarrow} \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} = \underset{\text{Smith form}}{S}$$

already <u>diagonal</u>, but <u>not</u> in Smith normal form

$$\Rightarrow \begin{matrix} \operatorname{coker} A & \cong & \operatorname{coker} S \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} & \cong & \mathbb{Z}/6\mathbb{Z} \end{matrix}$$ "Chinese Remainder Thm"

DEF'N: For a connected graph G,

its sandpile group is $K(G) \cong \operatorname{coker} \overline{L(G)}^{i,i}$ for any vertex i

(critical / Jacobian / Picard)

or equivalently

$$\mathbb{Z} \oplus \underbrace{K(G)}_{\substack{\uparrow \\ (\text{the torsion part of coker } L(G))}} \cong \operatorname{coker} L(G)$$

In particular, since $PAQ = S \Rightarrow$

$$\underbrace{\det P}_{\pm 1} \det A \underbrace{\det Q}_{\pm 1} = \det S$$

COR: $$\left| K(G) \right| = \det \overline{L(G)}^{i,i} = \kappa(G)$$

EXAMPLE: $$K(K_4) = \operatorname{coker} \overline{L(K_4)}^{4,4} = (\mathbb{Z}/4\mathbb{Z})^2 \qquad (\text{and } \kappa(K_4) = 16 = 4^2)$$

$$(\not\cong \mathbb{Z}/16\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^4, \ldots)$$

REU EXERCISE 1:

(a) Prove $K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$ via a Smith normal form calculation

– OR –

(b) Approach it differently, as follows...

(i) Show $$\mathbb{Z}^n \underset{\substack{\uparrow \\ \text{as abelian groups}}}{\cong} \mathbb{Z}[x]/(x^n - 1)$$

by showing this ring has $\mathbb{Z}$-basis $\{1, x, x^2, \ldots, x^{n-1}\}$

(ii) Then show

$$\underset{\substack{\| \\ (\mathbb{Z} \oplus K(K_n))}}{\operatorname{coker}\left(\mathbb{Z}^n \xrightarrow{L(K_n)} \mathbb{Z}^n\right)} \cong \mathbb{Z}[x]/\left(x^n - 1,\ n - (1 + x + x^2 + \cdots + x^{n-1})\right)$$

(iii) Then show $$\mathbb{Z}[x]/\left(x^n - 1, n - (1 + x + x^2 + \cdots + x^{n-1})\right)$$

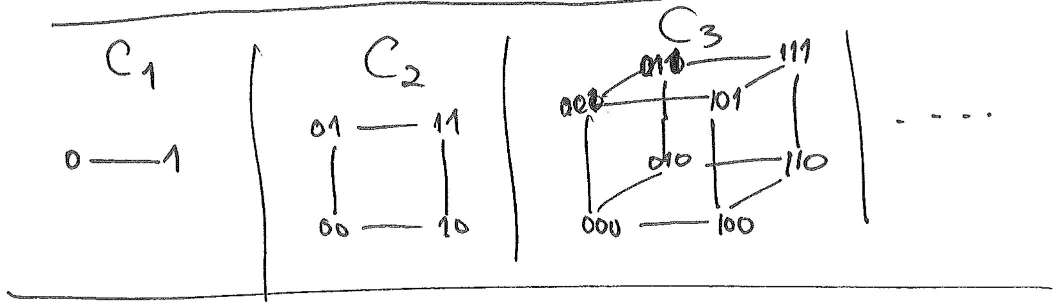
$$\cong \mathbb{Z}[x]/\left(n(x-1),\ n - (1 + x + x^2 + \cdots + x^{n-1})\right)$$

some monic polynomial in y of deg $n-1$

$$\cong \mathbb{Z}[y]/(n \cdot y,\ f(y)) \underset{\substack{\uparrow \\ \text{as abelian groups}}}{\cong} \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})^{n-2}$$

3. Cube case

DEF'N: The *n-cube graph* C_n has vertex set $\{0,1\}^n = (\mathbb{Z}/2\mathbb{Z})^n$ of size 2^n and edges $\underset{\substack{\| \\ (u_1,\ldots,u_n)}}{u}$ —— $\underset{\substack{\| \\ (v_1,\ldots,v_n)}}{v}$ when u,v differ in exactly 1 coordinate



(easy) THM: The vectors $\{\chi_u\}_{u\in(\mathbb{Z}/2\mathbb{Z})^n}$ lying in $\mathbb{Z}^{2^n}$

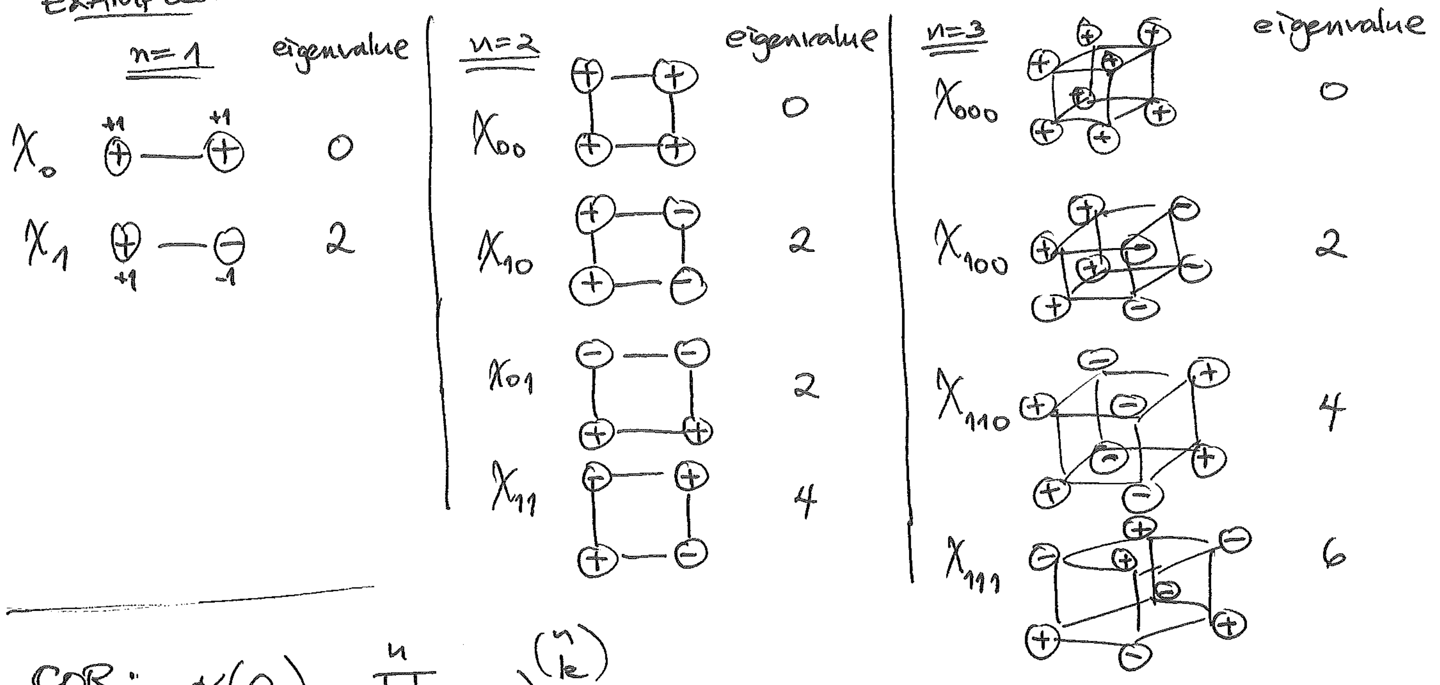
defined by $\chi_u(v) := (-1)^{u\cdot v} \in \{+1,-1\}$

give a complete set of orthogonal eigenvectors for $L(C_n)$,

with $L(C_n)\,\chi_u = 2k\cdot\chi_u$ where $k = \#$ones in u

see Stanley Results 2.1, 2.2, 2.3, 2.4

EXAMPLES:



COR: $\kappa(C_n) = \dfrac{\prod_{k=1}^{n} (2k)^{\binom{n}{k}}}{2^n}$

So what does $K(C_n)$ look like? ....

Try this in Sage Math Cell:

```
G = graphs.CubeGraph(3);
L = G.kirchhoff_matrix();
S = L.smith_form();
S[0]
```

n	1	2	3	4
eigenvalues of $L(C_n)$	0,2	0,2,2,4	0,2,2,2,4,4,4,6	0,2,2,2,2,4,4,4,4,4,4,6,6,6,6,8
$K(C_n)$	0	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/8\mathbb{Z})^2 \oplus \mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/8\mathbb{Z})^4 \oplus (\mathbb{Z}/32\mathbb{Z})^1 \oplus (\mathbb{Z}/3\mathbb{Z})^4$

The p-primary parts or p-Sylow subgroups $\bigoplus_{\substack{\text{primes}\\ p\neq 2}} \mathrm{Syl}_p K(C_n)$, away from $p=2$ are actually quite predictable...

THM (H. Bai 2003) For $p \neq 2$ prime, $\mathrm{Syl}_p K(C_n) \cong \mathrm{Syl}_p\left(\bigoplus_{k=1}^{n} (\mathbb{Z}/k\mathbb{Z})^{\binom{n}{k}} \right)$

proof: REU EXERCISE 2:

(i) Show that for $A \in \mathbb{Z}^{n\times n}$ and $p \neq 2$ a prime, one has $\mathrm{Syl}_p(\mathrm{coker}(\mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n))$
$\cong \mathrm{Syl}_p(\mathrm{coker}(R^n \xrightarrow{A} R^n))$

where $R := \mathbb{Z}[\frac{1}{2}] \subset \mathbb{Q}$
$:= \{ \frac{a}{2^m} : a \in \mathbb{Z}, m = 0,1,2,\ldots \}$ (="$\mathbb{Z}$ localized away from the prime 2")

(ii) Show that the standard $\mathbb{Z}$-basis $\{f_u\}_{u\in(\mathbb{Z}/2\mathbb{Z})^n}$ for $\mathbb{Z}^{2^n}$ having $f_u(v) := \begin{cases} 1 & \text{if } u=v \\ 0 & \text{else} \end{cases}$

satisfies $f_u = \frac{1}{2^{n-1}} \sum_{v\in(\mathbb{Z}/2\mathbb{Z})^n} (-1)^{u\cdot v} \chi_u$ ← the $L(C_n)$-eigenvectors defined earlier

v not u

Typo - should be $\frac{1}{2^n}$

(iii) Explain why $\exists$ an R-basis for R^{2^n} in which $R^{2^n} \xrightarrow{L(C_n)} R^{2^n}$ acts diagonally with eigenvalues $2k = 0,2,4,\ldots,2n$ each with multiplicity $\binom{n}{k}$.

(iv) Deduce Bai's Thm. ■

(7) 4.

REU PROBLEM 1:

Describe $\mathrm{Syl}_2 K(C_n)$

We know several things about it (including data up to $n=11$)....

- **REU EXERCISE 3:**

  (i) Prove that $\mathbb{Z} \oplus K(C_n) \left(\cong \operatorname{coker}\left(\mathbb{Z}^{2^n} \xrightarrow{L(C_n)} \mathbb{Z}^{2^n}\right) \right.$

  $$\left. \cong \mathbb{Z}[x_1, \ldots, x_n] / (x_1^2-1, x_2^2-1, \ldots, x_n^2-1, n-(x_1+x_2+\ldots+x_n)) \right)$$

  (ii) Show that $\operatorname{coker}\left((\mathbb{Z}/2\mathbb{Z})^{2^n} \xrightarrow{L(C_n)} (\mathbb{Z}/2\mathbb{Z})^{2^n} \right) \cong (\mathbb{Z}/2\mathbb{Z})^{2^{n-1}}$

  (iii) Deduce that $\mathrm{Syl}_2 K(C_n) = \bigoplus_{i=1}^{2^{n-1}-1} \mathbb{Z}/2^{m_i}\mathbb{Z}$ for some $m_i \geq 1$,

  that is, $\mathrm{Syl}_2 K(C_n)$ has exactly $2^{n-1}-1$ invariant factors.

- H. Bai also showed that $\#\{i : m_i = 1\} =: a_n$
  $= \#$ summands $\mathbb{Z}/2\mathbb{Z}$ in $K(C_n)$

  has generating function $a_1 + a_2 x + a_3 x^2 + \ldots = \dfrac{1}{(1-2x)(1-2x^2)}$

- Ducey & Jalil (2013) conjectured, and Chandler, Sin, Xiang (2015) proved that not $L(C_n)$, but the adjacency matrix $A(C_n) \left(= n I_{2^n} - L(C_n)\right)$, has completely predictable Smith normal form (from the eigenvalues).

There are also other directions to generalize,
including abelian Cayley graphs, ring structures,
higher dimensional spanning trees, ...
quadratic forms $K(Q_n) \times K(Q_n) \to \mathbb{Q}/\mathbb{Z}$, ...