REU PROBLEM: NONCROSSING TREE PARTITIONS AND SHARD INTERSECTION ORDERS

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Contents

A **set partition** $\mathbf{B} = (B_1, \ldots, B_r)$ of $[n] := \{1, \ldots, n\}$ is a family of subsets $B_i \in [n]$ where $B_i \cap B_j = \varnothing$ and $\cup_{i=1}^r B_i = [n]$. A partition $\mathbf{B}$ is **noncrossing** if no two of its **blocks** have $i, k \in B_s$, $j, \ell \in B_t$ s.t. $i < j < k < \ell$. Let $\text{NC}(n)$ denote the set of noncrossing partitions of n.

It is a classical result that $\#\text{NC}(n) = C_n := \frac{1}{n+1}\binom{2n}{n}$. The number C_n is known as the nth **Catalan number**.

1. Lattices

A **poset** (or **partially ordered set**) is set P and a relation $\leqslant$ called a **partial order** that satisfies

- $x \leqslant x$,
- if $x \leqslant y$ and $y \leqslant x$, then $x = y$, and
- if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$

for any $x, y, z \in P$. We will write P instead of $(P, \leqslant)$, unless it is not clear which partial order on P is being used.

A subset C of P is called a **chain** if any two elements of C are comparable (under the partial order on P). The chain C is **maximal** if it is not contained in any larger chain of P.

A **lattice** L is a poset where any $x, y \in L$ have a **join** and a **meet**. A join (resp. meet) of x and y, denoted $x \vee y \in L$ (resp. $x \wedge y$), must satisfy the following

- $x, y \leqslant x \vee y$ (resp. $x \wedge y \leqslant x, y$) and
- if $z \in L$ where $x \leqslant z$ and $y \leqslant z$ (resp. $z \leqslant x$ and $z \leqslant y$), then $x \vee y \leqslant z$ (resp. $z \leqslant x \wedge y$).

We will only consider finite lattices and posets. All finite lattices have a unique maximal (resp. minimal) element, denoted $\hat{1}$ (resp $\hat{0}$). Ask students why.

2. Shellability

Assume that P is finite poset, that all maximal chains of P are of the same length r, and that $\hat{0}, \hat{1} \in P$ (P is a finite **graded** poset). Let $\text{Cov}(P) := \{(x, y) \in P^2 : \ x \to y \text{ in } P\}$ be the set of **covering relations** of P. A map $\lambda : \text{Cov}(P) \to Q$ where $(Q, \leqslant_Q)$ is some poset is called a(n) **(edge) labeling**. A maximal chain $C = c_1 < \cdots < c_{r+1}$ of P is **increasing** if $\lambda(c_1, c_2) \leqslant_Q \cdots \leqslant_Q \lambda(c_r, c_{r+1})$. Given two maximal chains $C = c_1 < \cdots < c_{r+1}$ and $C' = c'_1 < \cdots < c'_{r+1}$ in P, we say C is **lexicographically smaller** that C' if $(\lambda(c_1, c_2), \ldots, \lambda(c_r, c_{r+1}))$ lexicographically precedes $(\lambda(c'_1, c'_2), \ldots, \lambda(c'_r, c'_{r+1}))$.

Definition 2.1. A labeling $\lambda : \text{Cov}(P) \to Q$ is an **EL-labeling** (or edge lexicographical labeling) of P if for every **interval** $[x, y] := \{z \in P : \ x \leqslant z \leqslant y\}$ of P,

i) there is a unique increasing maximal chain C in $[x, y]$, and
ii) C is lexicographically smaller than any other maximal chain C' in $[x, y]$.

If P admits an EL-labeling, it is said to be **EL-shellable**.

Theorem 2.2 (Björner)**.** Let $(\mathbf{B}, \mathbf{B}') \in \text{Cov}(\text{NC}(n))$ and let $B_i, B_j \in \mathbf{B}$ be the blocks that are merged to produce $\mathbf{B}'$. Then the labeling $\lambda : \text{Cov}(\text{NC}(n)) \to [n]$ defined by $\lambda(\mathbf{B}, \mathbf{B}') := \max\{\min B_i, \min B_j\}$ is an EL-labeling. Thus the lattice $\text{NC}(n)$ is EL-shellable.

3. Noncrossing Tree Partitions

Let T be a tree embedded in the disk D^2 in such a way that a vertex of T lies on the boundary of D^2 if and only if that vertex is a leaf of T. The tree T has **boundary vertices** and **interior vertices**.

The tree T has an important set of subgraphs, which we will call segments. A **segment** $s = (v_0, \ldots, v_t) = [v_0, v_t]$ with $t \geqslant 1$ is a sequence of interior vertices of T that turn sharply at v_i for each $1 \leqslant i \leqslant t-1$. A vertex of T is not a segment. Let $\mathrm{Seg}(T)$ denote the set of segments of T.

A **red admissible curve** $\gamma : [0,1] \to D^2$ for a segment $s = [v_0, v_t]$ is a simple curve where

- its endpoints are v_0 and v_t,
- γ may only intersect edges of T of the form (v_{i-1}, v_i) where $i \in [t]$, and
- γ must leave its endpoints "to the right."

Two segments are **noncrossing** if they admit red admissible curves that do not intersect.

A **noncrossing tree partition** $\mathbf{B} = (B_1, \ldots, B_k)$ is a set partition of the interior vertices of T where

- there is a (unique) set of segments $\mathrm{Seg}_r(B_i) \subset \mathrm{Seg}(T)$ connecting the vertices in B_i and any two segments in $\mathrm{Seg}_r(B_i)$ may agree only at their endpoints and
- any segments $s_1 \in \mathrm{Seg}_r(B_i)$ and $s_2 \in \mathrm{Seg}_r(B_j)$ are noncrossing.

Theorem 3.1 (G.–McConville)**.** The set $\mathrm{NCP}(T) := \{\text{noncrossing tree partitions of } T\}$ partially ordered by **refinement** (i.e. if $\mathbf{B} = (B_1, \ldots, B_k) \leqslant \mathbf{B}' = (B'_1, \ldots, B'_\ell)$, then each **block** B_i is contained in some B'_j) is a lattice.

Exercise 3.2. Find a tree T where

a) $\#\mathrm{NCP}(T)$ is not equal to any Catalan number
b) $\#\mathrm{NCP}(T)$ is equal to a Catalan number, but $\mathrm{NCP}(T) \ncong \mathrm{NC}(n)$ for any n.

Problem 3.3. Let T be a tree embedded in a disk with n interior vertices so that the rank of $\mathrm{NCP}(T)$ is $n-1$.

a) Show that $\mathrm{NCP}(T)$ is EL-shellable.
b) Find a formula for the number of maximal chains of $\mathrm{NCP}(T)$.

Remark 3.4. By Problem 3.3, the simplicial complex $\Delta\left(\overline{\mathrm{NCP}(T)}\right)$ will be homotopy-equivalent to a wedge of $(n-3)$-dimensional spheres. The number of such spheres will be $\#\{\text{maximal chains of } \mathrm{NCP}(T)\} - 1$.

4. Shard Intersection Order of Biclosed Sets

A tree T defines another lattice whose combinatorics we want to further understand.

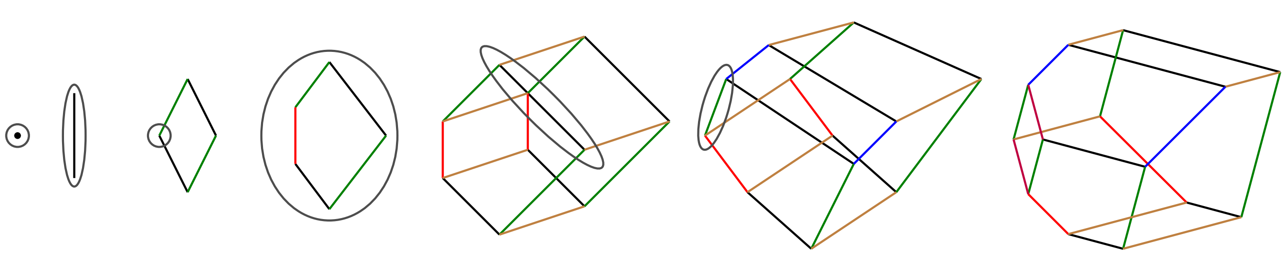
Two segments s_1 and s_2 are **composable** if $s_1 \circ s_2 \in \mathrm{Seg}(T)$. A set $B \subset \mathrm{Seg}(T)$ is **closed** if for any composable segments $s_1, s_2 \in B$, one has that $s_1 \circ s_2 \in B$. We say B is **biclosed** if B and $\mathrm{Seg}(T)\backslash B$ are closed. Let $\mathrm{Bic}(T)$ denote the set of biclosed sets of T partially ordered by inclusion.

We introduced the lattice of noncrossing tree partitions $\mathrm{NCP}(T)$ in order to describe the **shard intersection order** of $\overrightarrow{FG}(T)$. Now we want to understand the shard intersection order of $\mathrm{Bic}(T)$.

Exercise 4.1. Let $B_1, B_2 \in \mathrm{Bic}(T)$.

a) Describe $B_1 \vee B_2$.
b) Use a) to show that $\mathrm{Bic}(T)$ is a lattice.

Theorem 4.2 (G.–McConville)**.** The lattice $\mathrm{Bic}(T)$ is a **congruence-uniform** lattice (i.e. it can be constructed from the one element lattice by a finite sequence of interval doublings (this definition is a result of Day)). Also, it is graded by cardinality of biclosed sets.



Proposition 4.3 (essentially Reading)**.** A lattice is congruence-uniform if and only if it admits a **CU-labeling**.

Definition 4.4. A labeling $\lambda : \mathrm{Cov}(L) \to Q$ is a **CN-labeling** if L and its dual L^* satisfy the following: For elements $x, y, z \in L$ with $(z,x),(z,y) \in \mathrm{Cov}(L)$ and maximal chains C_1 and C_2 in $[z, x \vee y]$ with $x \in C_1$ and $y \in C_2$,

(CN1) the elements $x' \in C_1$, $y' \in C_2$ such that $(x', x \vee y), (y', x \vee y) \in \mathrm{Cov}(L)$ satisfy
$$\lambda(x', x \vee y) = \lambda(z,y), \ \lambda(y', x \vee y) = \lambda(z,x);$$
(CN2) if $(u,v) \in \mathrm{Cov}(C_1)$ with $z < u$, $v < x \vee y$, then $\lambda(z,x), \lambda(z,y) <_Q \lambda(u,v)$;
(CN3) the labels on $\mathrm{Cov}(C_1)$ are pairwise distinct.

We say that λ is a **CU-labeling** if, in addition, it satisfies

(CU1) for any elements $j, j' \in L$ that cover unique elements $j_*, j'_* \in L$, respectively, one has that $\lambda(j_*, j) \neq \lambda(j'_*, j')$;
(CU2) for any elements $m, m' \in L$ that are covered by unique elements $m^*, m^{*\prime} \in L$, respectively, one has that $\lambda(m, m^*) \neq \lambda(m', m^{*\prime})$.

Theorem 4.5 (G.–McConville)**.** The labeling $\lambda : \mathrm{Cov}(\mathrm{Bic}(T)) \to \mathrm{Seg}(T)$ defined by $\lambda(B, B \sqcup \{s\}) = s$ is a CN-labeling (here $\mathrm{Seg}(T)$ has the partial order $s_1 \leqslant_{\mathrm{Seg}(T)} s_2$ if s_1 is a subsequence of s_2).

Remark 4.6. Someone should present the part of Oriented Flip Graphs & Noncrossing Tree Partitions about the shard intersection order of $\overrightarrow{FG}(T)$. They should explain the CU-labeling of $\overrightarrow{FG}(T)$ that we construct and how it is intrinsic to $\overrightarrow{FG}(T)$.

Remark 4.7. Someone should present Petersen's On the shard intersection order of a Coxeter group paper (using some basic definitions from Reading's Noncrossing partitions the shard intersection order).

Definition 4.8 (Reading)**.** Let L be a congruence-uniform lattice with CU-labeling $\lambda : \mathrm{Cov}(L) \to P$. Let $x \in L$ and let $y_1, \ldots, y_k$ be the elements of L satisfying $(y_i, x) \in \mathrm{Cov}(L)$. Define the **shard intersection order** of L $\Psi(L)$ to be the collection of sets of the form
$$\begin{aligned}\psi(x) &:= \{\text{labels appearing between } \textstyle\bigwedge_{i=1}^k y_i \text{ and } x\} \\ &= \{\lambda(w,z) : \textstyle\bigwedge_{i=1}^k y_i \leqslant w < z \leqslant x, \ (w,z) \in \mathrm{Cov}(L)\}\end{aligned}$$
partially ordered by inclusion.

Problem 4.9. Describe the shard intersection order of $\mathrm{Bic}(T)$.

a) Construct a CU-labeling $\lambda : \mathrm{Cov}(\mathrm{Bic}(T)) \to S$ where S is variation of the poset $\mathrm{Seg}(T)$.
a) Is $\Psi(\mathrm{Bic}(T))$ a lattice?
b) Is it EL-shellable?