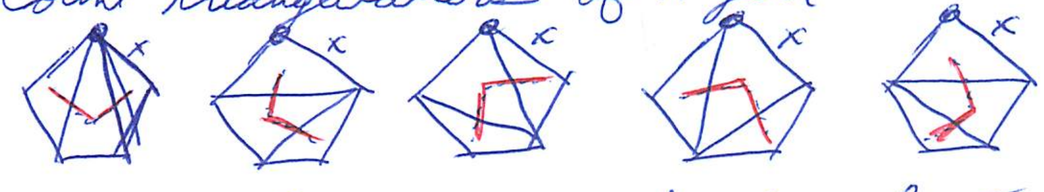
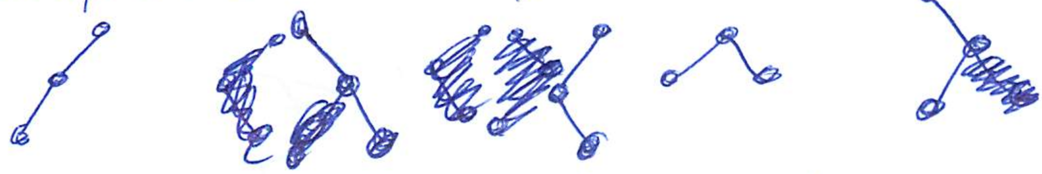
Catalan Numbers (Mentioned by Al)

Count triangulations of n-gon



We have broken symmetry by choosing a special vertex and an orientation, which gives a "starting edge"



Gives bijection w/ binary trees on n-2 nodes

These in bijection w/ complete binary trees w/ n-1 leaves

by adding all possible children to each node

How many such trees? Fix root, this partitions leaves into 2 parts, ~~one~~ each side being non-empty

n-1

So for (k, n-k-1), there are $C_k C_{n-k-1}$, where C_k = # trees

Sum over all possibilities. $\cancel{C_{n-1} = \sum_{k=1}^{n-2} C_k C_{n-k-1}}$

Reindexing, we get

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

C_n is the n-th Catalan number.

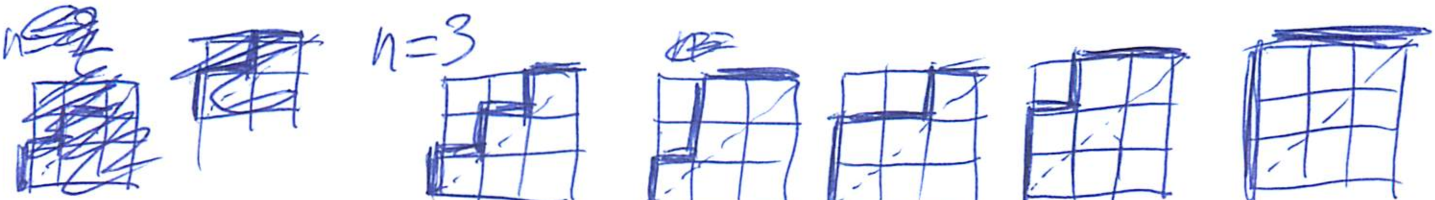
$C_1 = 1$, $C_2 = 2$, $C_3 = 5$ $C_4 = 14$ $C_5 = 42$

$C_0 = 1$

How to find closed form?

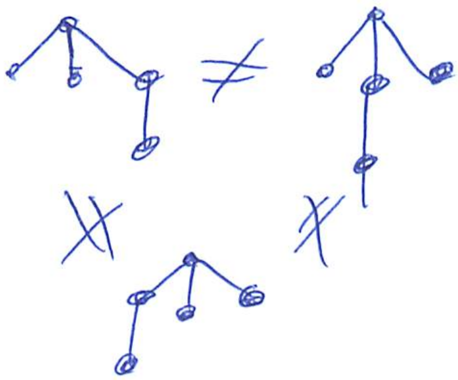
Dyck paths = walk on lattice by U and R steps that never crosses $y=x$ line that ends at (n,n)

$n=3$



In bijection w/ rooted planar trees (aka ordered trees) trees w/ orientation around each vertex and distinguished vertex, called the root.

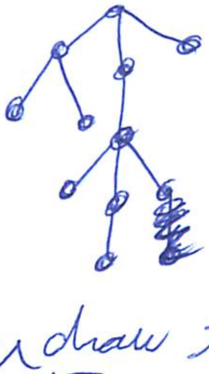
Ex



Bijection given by planar code of RPT

Start at root, walk around tree and put an U step when walk away from root and R when walk towards.

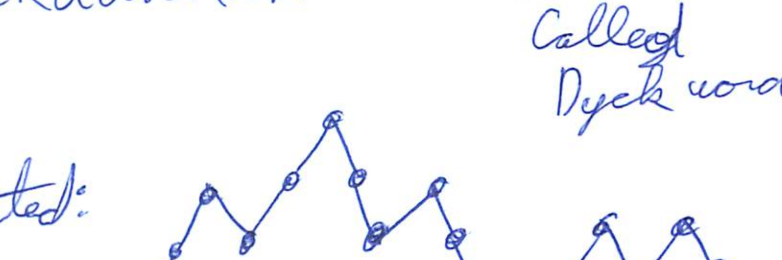
Ex



$\rightsquigarrow$ URUUUR UURRURRRUURURR

Called Dyck word

We sometimes draw this rotated:



In bijection w/ binary trees.

Note there are $\binom{2n}{n}$ paths in the $n\times n$ square. (Why?)

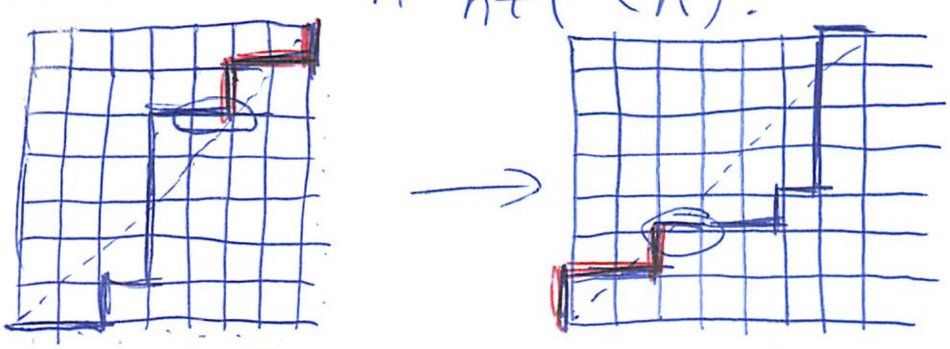
Next for a fixed path p, find first place it touches diagonal w/ R step. Move the entire path after step to ~~beginning~~ the beginning, then R step, then the remaining path.

This decreases # U above diagonal by 1.

It is also reversable.

Thus we split all paths into $n+1$ equally sized classes. $\Rightarrow$ $C_n = \frac{1}{n+1}\binom{2n}{n}$.

> **REU Exercise 1)**
> Describe a bijection between $2\times n$ standard tableaux w/out $(1,1,\dots,1)$ and permutations avoiding (321) i.e. have no length 3 decreasing subsequence



3 U step above diagonal $\longrightarrow$ 2 U steps above diagonal

There are 200+ Catalan objects.

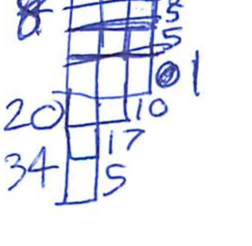
We will focus on 3: Dyck paths, RPT, and Rigged Configurations

Def/ A rigged configuration is a partition ν of n w/ integer on each row, called a rigging s.t.

$0 \le x \le P_\ell = 2n - 2\sum_j \min(\nu_j, \ell)$

(Up to reordering) P_ℓ called vacancy number.

Ex/



Let $\nu = 1^{m_1} 2^{m_2} \cdots$

$P_\ell = 2n - 2\sum_i m_i \min(i, \ell)$

So only depends on row length

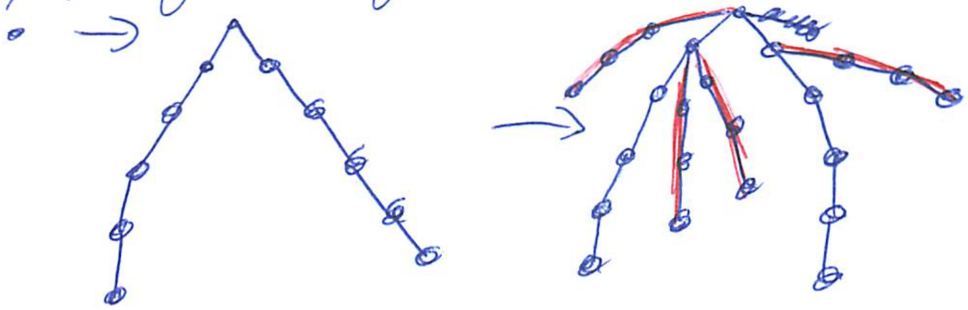
Can write ~~$C_n = \sum_{\nu \vdash n} \prod \binom{m_i + p_i}{m_i}$~~ $C_n = \sum_{\nu \vdash n} \prod_{i=1}^{\infty} \binom{m_i + P_i}{m_i}$

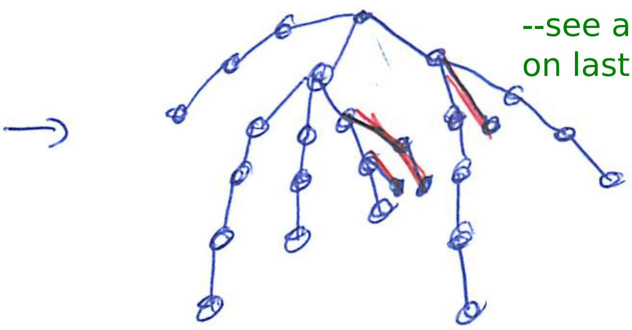
Fermionic formula version (at $q=1$)

In bijection w/ RPT by reading top to bottom, adding path of length ν_i at the X possible distinct location.

↑ rigging of ν_i

Ex/ RC from before





--see also expanded version of this example on last page of these notes.

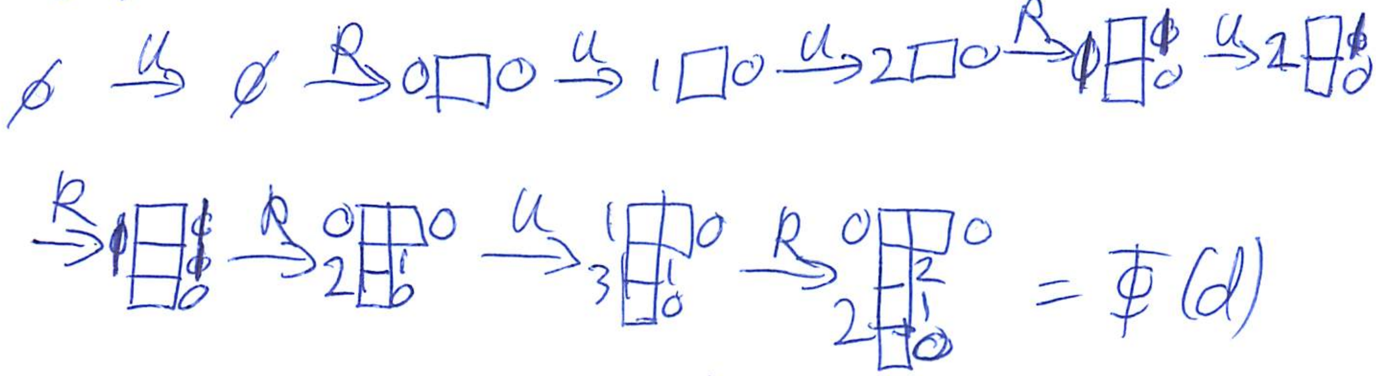
Let ~~denote~~ $\Psi : RC \to RPT$

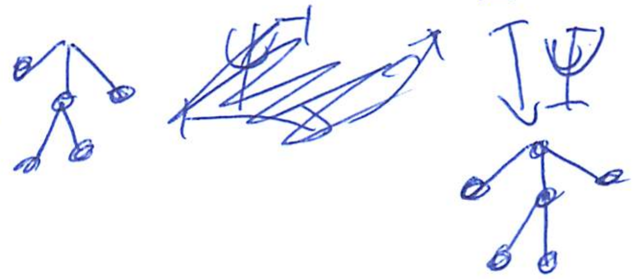
There is also bijection Φ: Dyck words $\to$ RC.

~~Write~~ Generalize $P_\ell = |d_k| - 2\sum_i \min(\nu_i, \ell)$ $\quad d_k$ = partial Dyck word.

Φ given by adding box to largest singular row (where $p_\ell = X$) if R-step, else do nothing. and make result singular

Ex  = URUURURRUR = d

 $= \Phi(d)$

$P^{-1}(d) =$ 

Planar code

Thm (~~Reynolds~~ Reynolds '15)

The diagram

$$\begin{tikzcd} RC \arrow[r, "\Psi"] & RPT \arrow[d, "P"] \\ & D \arrow[ul, "\Phi"] \end{tikzcd}$$

Commutes

REU Exercise 13

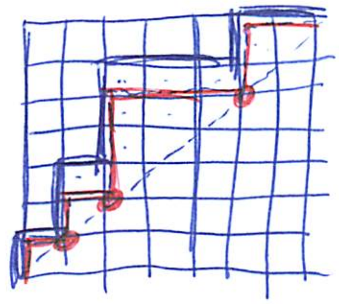
Show Ψ and Φ are bijections directly

There are 2 statistics on Dyck paths

Area = # complete boxes below diagonal

Bounce = sum of positions of bounce points

Take



Area = 8

Bounce = 1+2+5 = 8

Dinv = 5

Obtain ~~generating~~ q,t-~~Dyck~~ Catalan numbers

$$C_n(q,t) = \sum_{d \in D_n} q^{a(d)} t^{b(d)}$$

Thm / HHL

$$C_n(q,t) = C_n(t,q)$$

Other statistic

Dinv = diagonal inversions

= # rows the same length or differ by 1 w/ longer row below

Haglund ~~B~~ BS map:

(area, bounce) → (dinv, area)

Open Problem (Not REU problem)

Show this combinatorially.

REU Problem 5

Determine area and bounce and dinv on RC under Φ.

I.e., find statistics α, β, δ on RC s.t. ~~$a = \Phi$~~ $a = \alpha \circ \Phi$,

$b = \beta \circ \Phi$,

$d = \delta \circ \Phi$,

REU Exercise 14

(i) Find bijection binary trees and RPT or Dyck paths

(ii) Find area on RPT under Φ.

Cor/ The involution which swaps $x \to p_\ell - x$ intertwines w/ path reversal of Dyck path under Φ.

$[n]_q = \frac{1-q^n}{1-q}$

$[n]_q! = [n]_q \cdots [1]_q$

$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$

Cor/ RC's w/ ~~first~~ shape of length ℓ is in bijection w/ Dyck words w/ ℓ peaks = Narayana numbers

Cor/ ~~Len of 1st row~~ Height of $d = \nu_1$ under Φ

Thm/ (KKR) The major index of d is the cocharge of (ν, J) under Φ.

$\widetilde{C}_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$

Cor/ $C_n(q) = \sum_{\nu \vdash n} q^{cc(\nu)} \prod_{\ell=1}^{\infty} \begin{bmatrix} m_\ell + p_\ell \\ m_\ell \end{bmatrix}_q = \sum_{(\nu, J)} q^{cc(\nu, J)}$, where $cc(\nu)$ is the cocharge of ν.

$maj(d) = \sum$ position of peaks of d

$cc(\nu) = \frac{1}{2} \sum_{i,j=1}^{\infty} 2 m_i m_j \cdot \min(i,j)$ $\qquad cc(\nu, J) = cc(\nu) + \sum_{\text{riggings}} x$

In fact, bijection extends to all seq of U and R steps. We give U a weight of 1 and R a weight of -1.

The weight of a word is sum of the weights of the letters

The weight of a RC $= p_\infty = p_{\nu_1}$. then Φ preserves weights.

REU Programming Practice

Program the bijections Φ and Ψ (preferable in Sage).