REU 2016 Day 6 G. Musiker

Stable cluster variables and pyramid partition functions

REF: "Colored BPS pyramid partition functions, quivers and cluster transformations" (Eager–Franco)

§9.4 REU Prob. 2

§9.5 REU Prob. 6

1. Principal coefficients and F-polynomials
2. Pyramid partition functions for the conifold
3. Conjectural definition of stable cluster variables (Eager-Franco)
4. The case of $1 \Rightarrow 2$
5. Data for

$$\begin{tikzcd} 1 \arrow[r, Rightarrow] & 3 \arrow[d, Rightarrow] \\ 4 \arrow[u, Rightarrow] & 2 \arrow[l, Rightarrow] \end{tikzcd}$$

and its relatives.

① **DEF'N:** Let Q be a quiver with n vertices. Define a new quiver called the **framed quiver** $\widehat{Q}$, having $2n$ vertices $\{1, 2, \ldots, n, 1', 2', \ldots, n'\}$ and having every arrow from Q plus $\{i \to i' : i = 1, 2, \ldots, n\}$.

e.g. $Q = 1 \longrightarrow 2$

$$\widehat{Q} = \begin{tikzcd} \boxed{1'} & \boxed{2'} \\ 1 \arrow[u] \arrow[r] & 2 \arrow[u] \end{tikzcd}$$

frozen vertices - can't mutate at them!

<u>DEF'N:</u> $\hat{A} = \hat{A}(Q) =$ cluster algebra with principal coefficients defined by Q

IDEA: Mutate at vertices of $Q \subsetneq \hat{Q}$ as usual, but frozen vertices $\{y_1, \dots, y_n\}$.

$$\begin{tikzcd} \boxed{1'} & \boxed{2'} & & \boxed{1'} \arrow[d] & \boxed{2'} \\ 1 \arrow[u] \arrow[r] & 2 \arrow[u] & \overset{\mu_1}{\rightsquigarrow} & 1 & 2 \arrow[l] \arrow[u] \end{tikzcd}$$

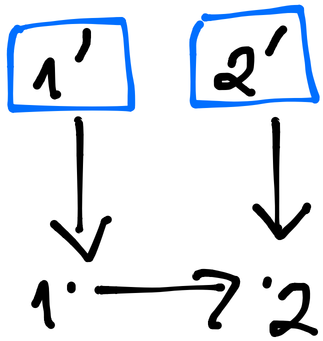
$\{x_1, x_2\}$

$$\left\{\frac{y_1 x_2 + 1}{x_1}, x_2, y_1, y_2\right\}$$

extended cluster

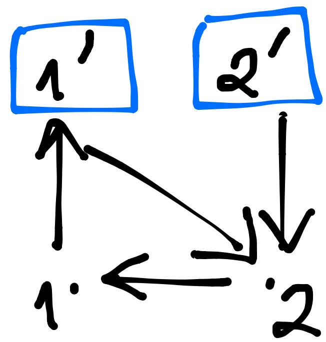
$\{x_1, x_2, y_1, y_2\}$

$\mu_2 \rightsquigarrow$



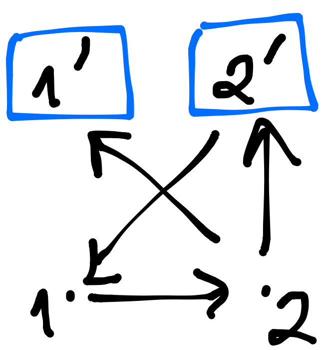
$$\left\{\frac{y_1x_2+1}{x_1},\ \frac{y_1y_2x_2+y_2+x_1}{x_1x_2}\right\}$$

$\mu_1 \rightsquigarrow$



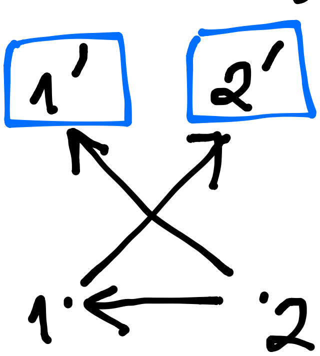
$$\left\{\frac{x_1+y_2}{x_2},\ \frac{y_1y_2x_2+y_2+x_1}{x_1x_2}\right\}$$

$\mu_2 \rightsquigarrow$



$$\left\{\frac{x_1+y_2}{x_2},\ x_1\right\}$$

$\mu_1 \rightsquigarrow$



$$\{x_2, x_1\}$$

DEF'N (Fomin-Zelevinsky)

F-polynomials are cluster variables with principal coefficients where we set all the $x_i = 1$.

e.g.

$$\begin{tikzcd} 1' & 2' & 1' & 2' \\ 1 \arrow[ur] & 2 \arrow[l] \arrow[ul] & 1 \arrow[u] \arrow[r] & 2 \arrow[u] \end{tikzcd}$$

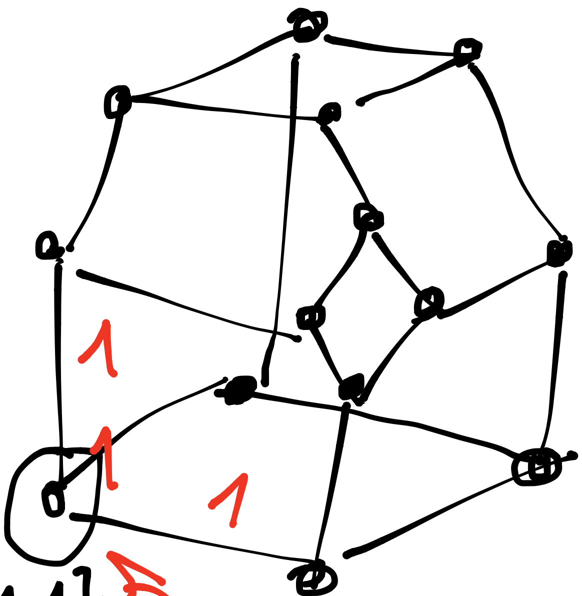
$$\begin{tikzcd}[column sep=small] & \{1,1\} \arrow[r, phantom, "\cong"] & \{1,1\} \arrow[dr, "\mu_1"] & \\ \{1+y_2, 1\} \arrow[ur, "\mu_1"] & & & \{y_1+1, 1\} \arrow[d, "\mu_2"] \\ \{1+y_2, y_1y_2+y_2+1\} \arrow[u, "\mu_2"] & & & \{y_1+1, y_1y_2+y_2+1\} \arrow[lll, "\mu_1"] \end{tikzcd}$$

REU EXERCISE 15

$1 \leftarrow 2 \leftarrow 3$

9 faces $\updownarrow$ cluster variables

14 vertices $\updownarrow$ clusters

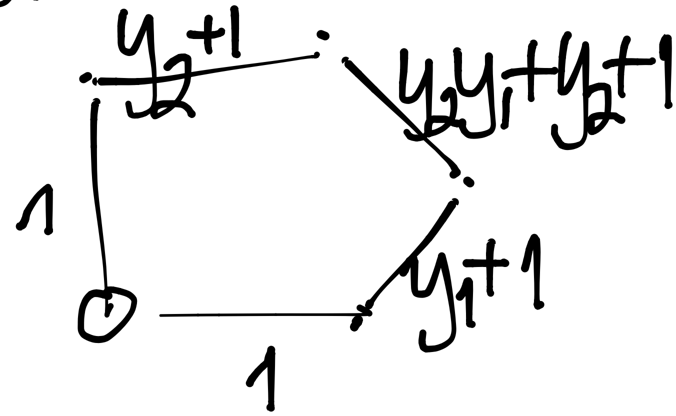


$\{1,1,1\}$

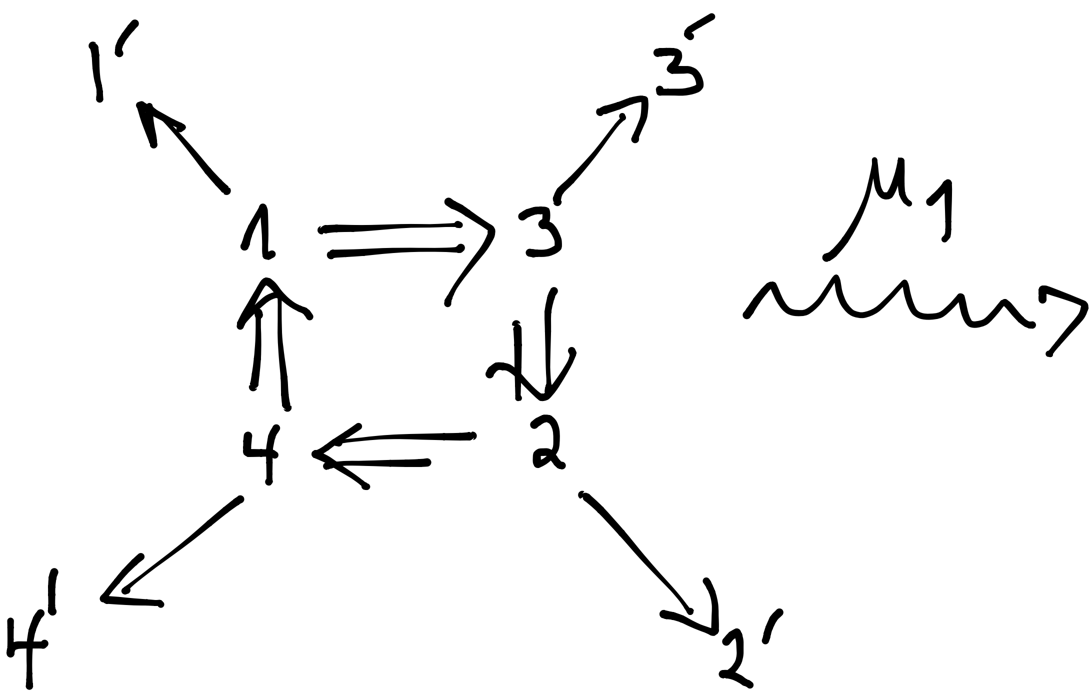
$1 \leftarrow 2 \leftarrow 3$

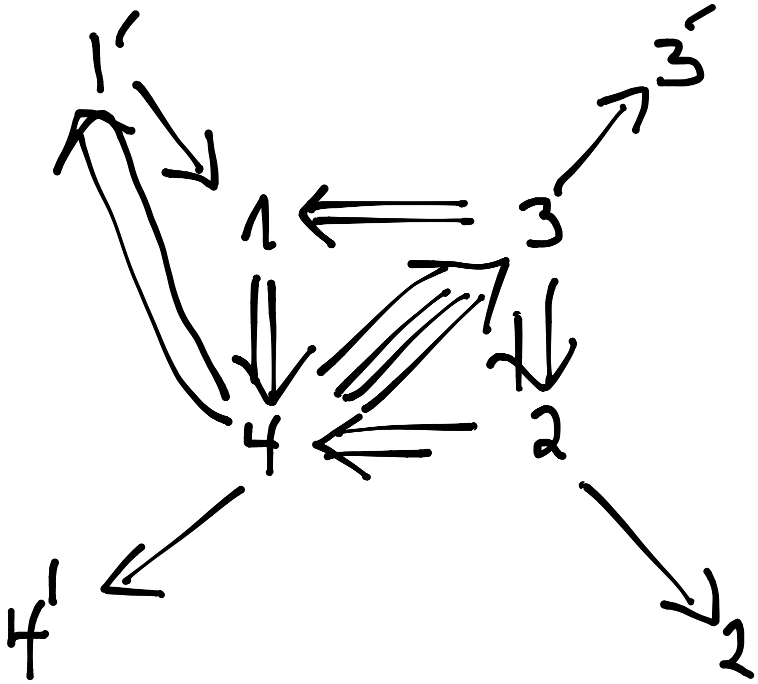
label the 3 faces incident to this vertex with F-polynomials 1, 1, 1; then fill in the face labels with the rest of the F-polynomials.

e.g. the previous pentagon would have been labeled

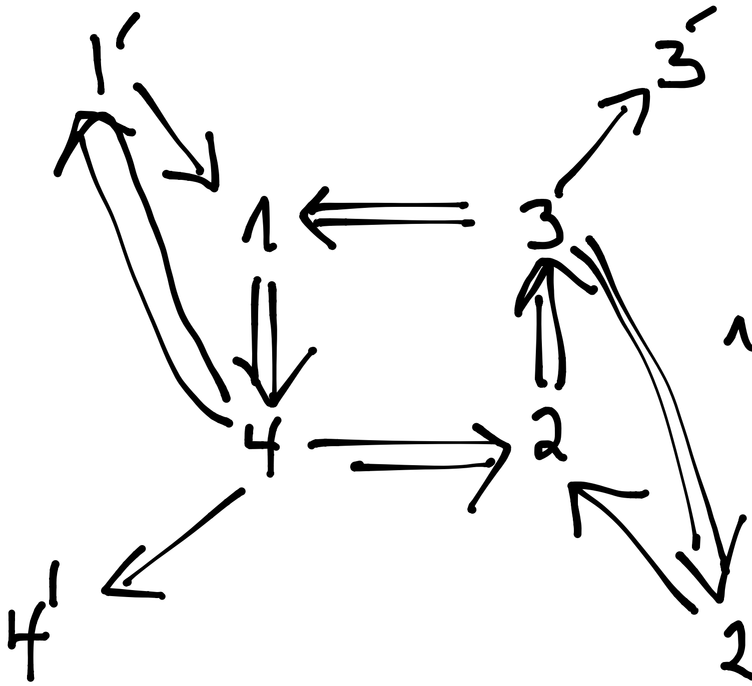


Now let's do this one

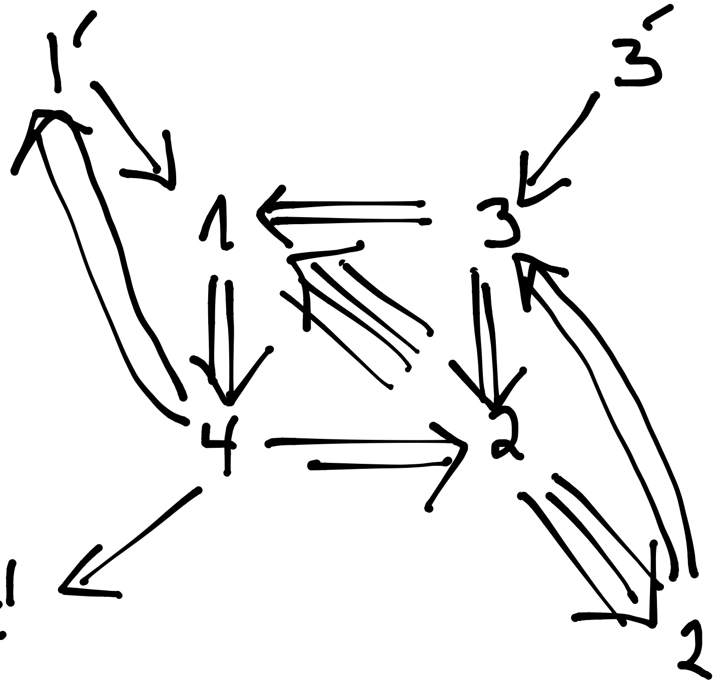




μ_2



μ_3



One can check that

$$x_n x_{n-4} = \begin{cases} x_{n-2}^2 + x_{n-3}^2 \prod_i y_i, & n \text{ odd}, \\ x_{n-1}^2 + x_{n-2}^2, & n \text{ even}. \end{cases}$$

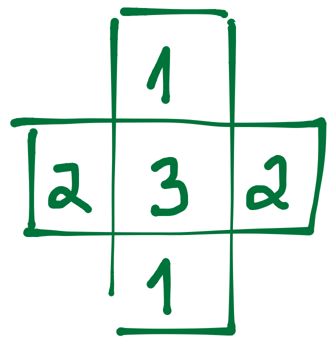
CLAIM:

$$F_{4n+1}F_{4n-3} = F_{4n-1}^{2}\, y_1^{2n-1}\, y_4^{2n-2} + F_{4n}^{2}$$

for other subscripts mod 4 there are analogous identities.

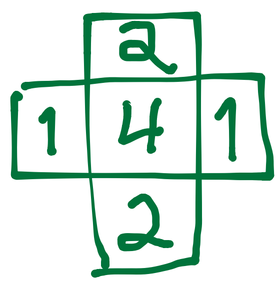
$F_5 = 1 + y_1$ $\boxed{1}$ 1,1,2,8,64,…

$F_6 = 1 + y_2$ $\boxed{2}$



$$F_7 = y_2^2 y_3 (1+y_1)^2 + (1+y_2)^2$$

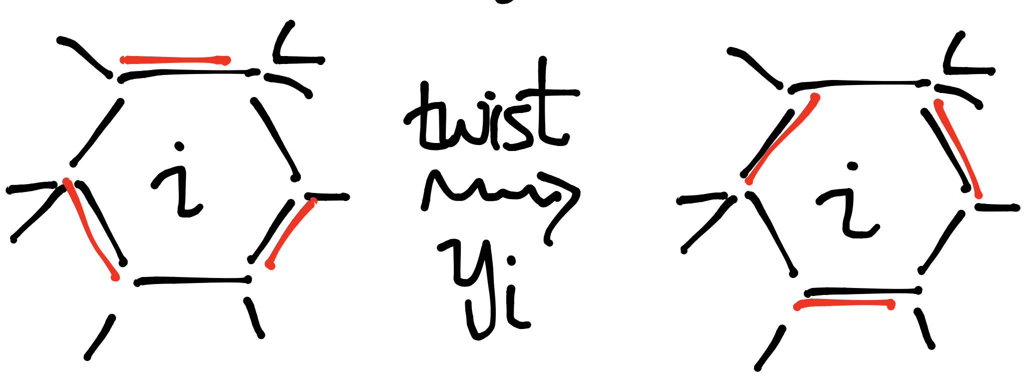
$$F_8 = y_1^2 y_4 (1+y_2)^2 + (1+y_1)^2$$



<u>DEF'N</u>: Given a perfect matching M of a graph G (with face labels) and a choice of minimal matching M_0, the <u>height</u>

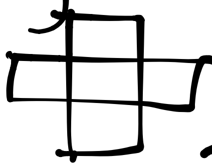
$$h(M) := y_{i_1} y_{i_2} \cdots y_{i_k}$$

where $\{i_1, \ldots, i_k\}$ is any sequence of labels of faces of G that can be twisted to get from $M_0 \longrightarrow M$

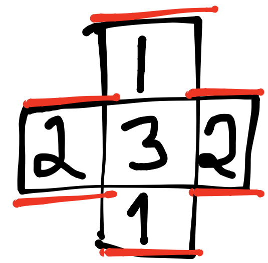


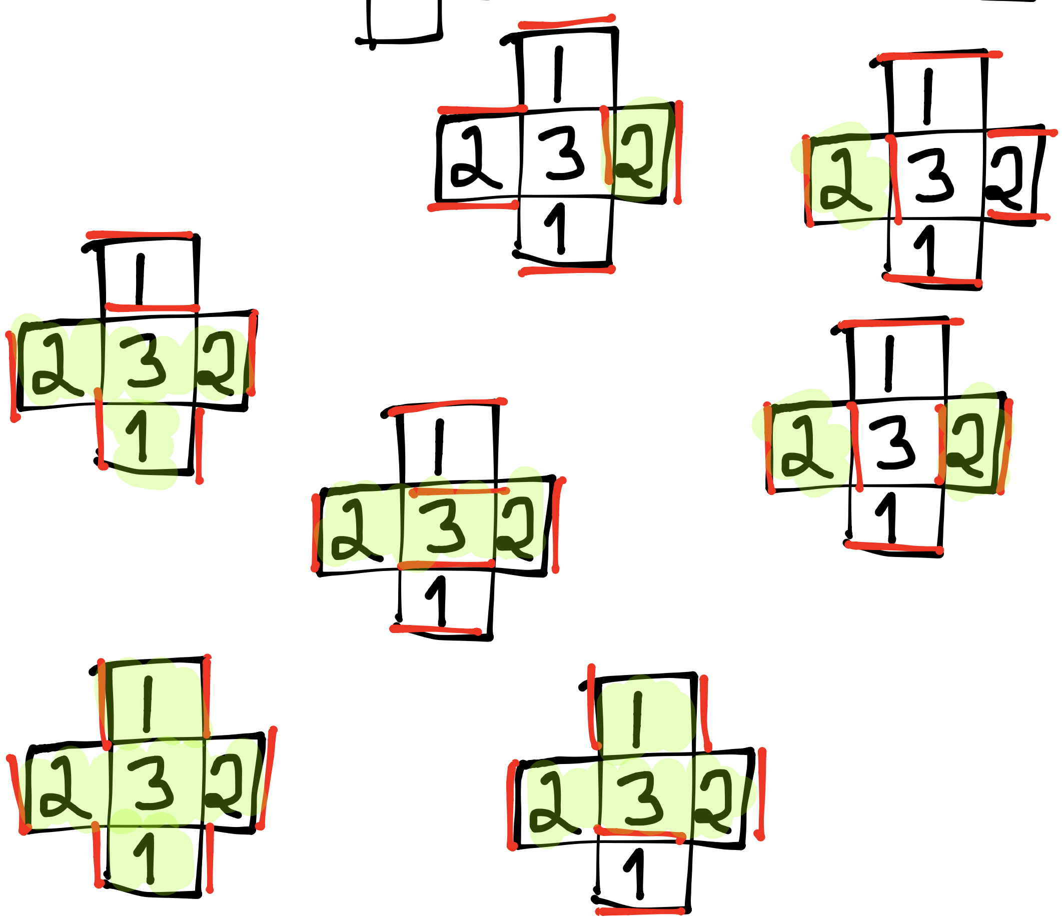
CLAIM:

$$F_7 = \sum_{\substack{\text{perfect} \\ \text{matchings } M \\ \text{of}}} h(M)$$ with

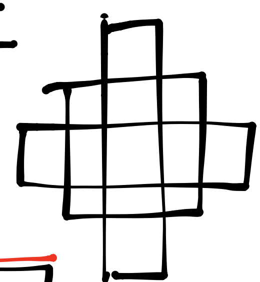


$M_0 =$

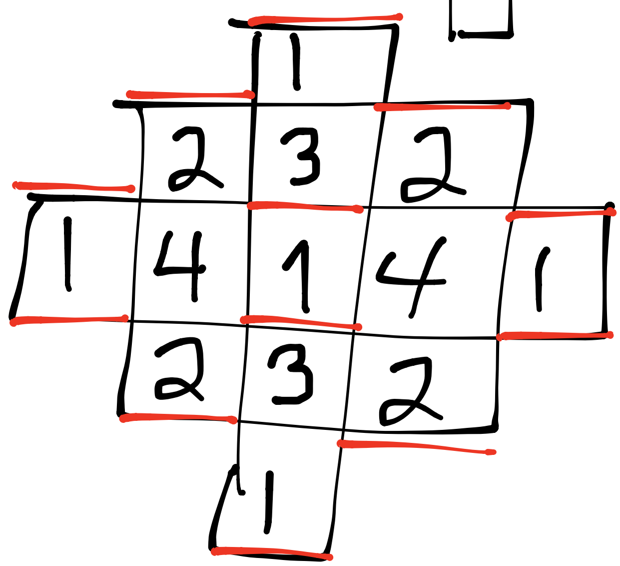




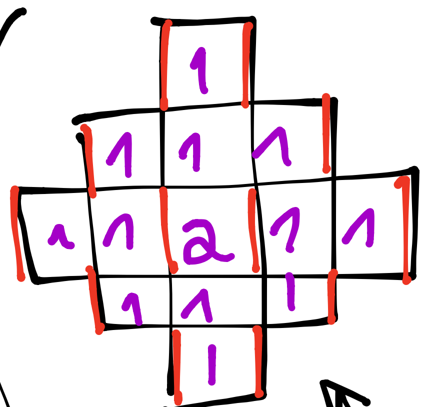
CLAIM: $$F_9 = \sum_{M \text{ of}} h(M)$$



with $M_0 =$



and $h\Big($



$\Big) = \prod_{\text{faces } F} y_F^{e(F)}$

$e(F)$ shown here

Generally, $e(F) = \#$ curves enclosing the face F when you superimpose the two matchings M_0, M

REU EXERCISE 16

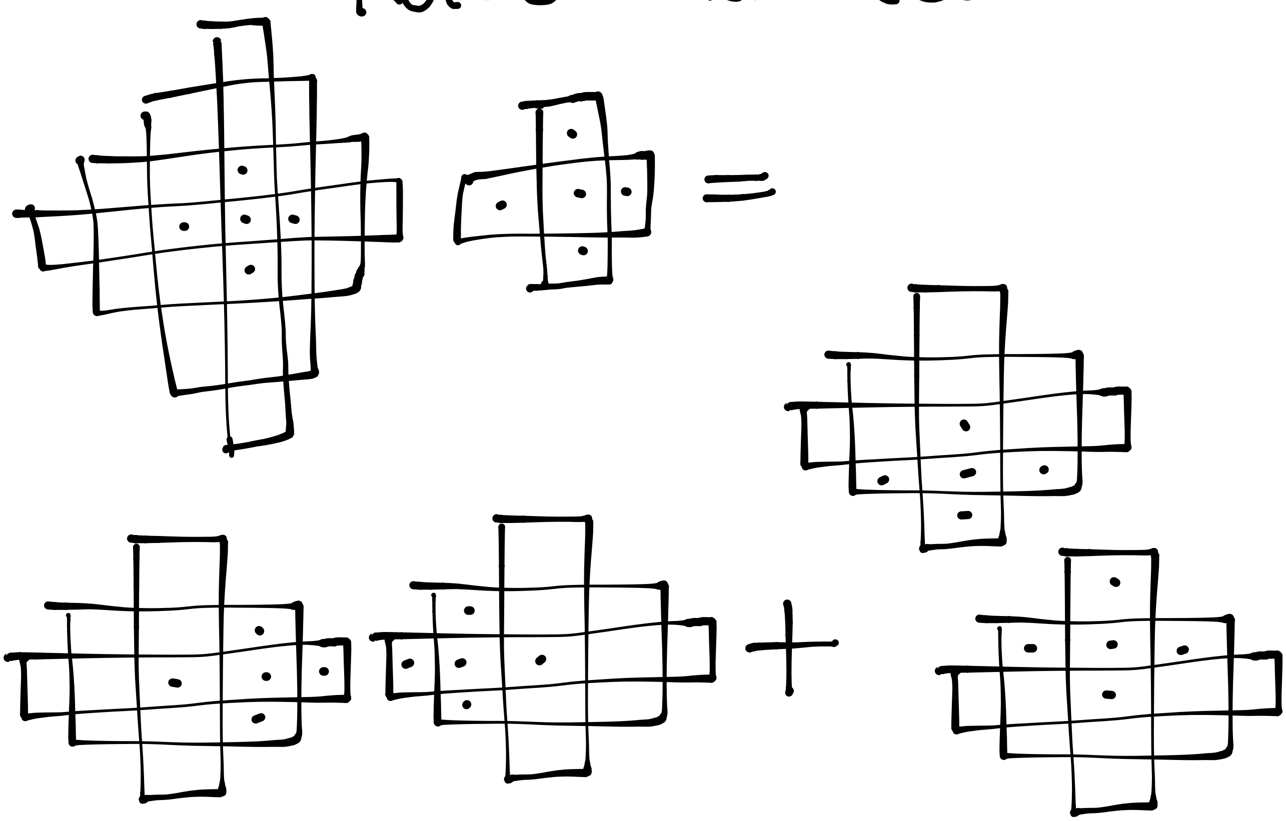
Prove $$F_n = \sum_{\substack{\text{perfect} \\ \text{matchings } M \\ \text{of "}AD_n\text{" with labels}}} h(M)$$

HINT: prove inductively

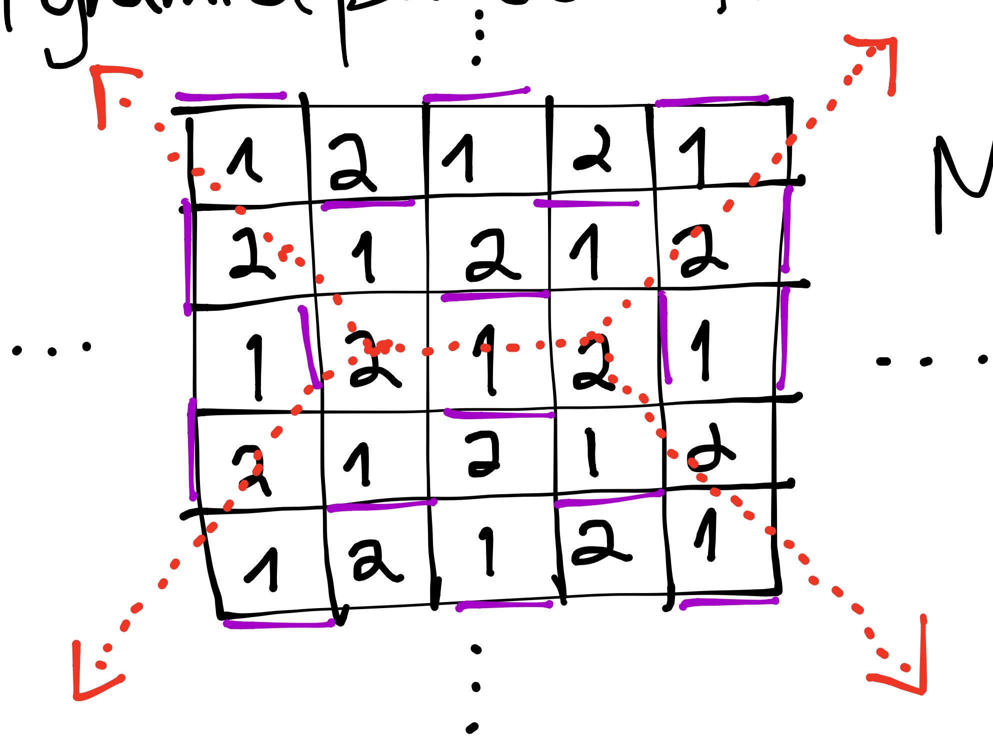
$$F_{4n+1}F_{4n-3} = F_{4n-1}^2\, y_1^{2n-1}\, y_4^{2n-2} + F_{4n}^2$$

implies $F_{4n+1} = \sum_M h(M)$.

...and "Kuo condensation"



Pyramid partition functions



M_0

$$PP_1 = \sum_{\substack{M \text{ reachable} \\ \text{by twisting} \\ \text{from } M_0}} h(M)$$

$$= 1 + y_1 + 2y_1y_2 + y_1y_2^2 + 4y_1y_2y_1 + \dots$$

REU PROBLEM 6 Idea

Find a weighting change for terms in F_n or PP_1, respectively, so that

$$\lim_{n\to\infty} F_n = PP_1 .$$

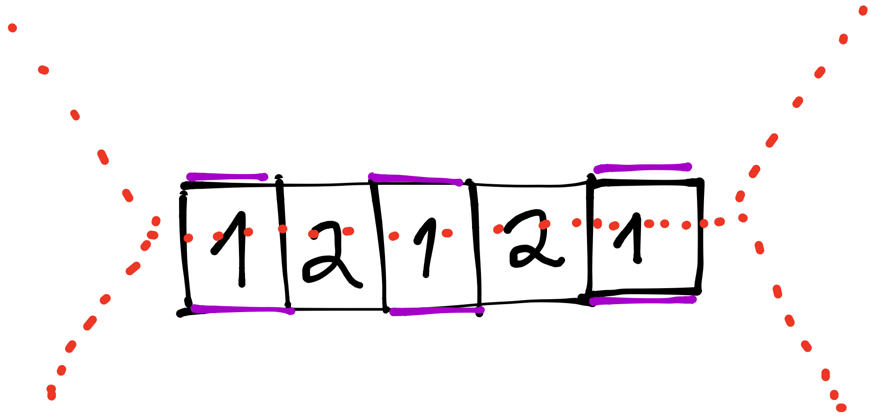
$\lim_{n\to\infty} PP_n$?

defined below

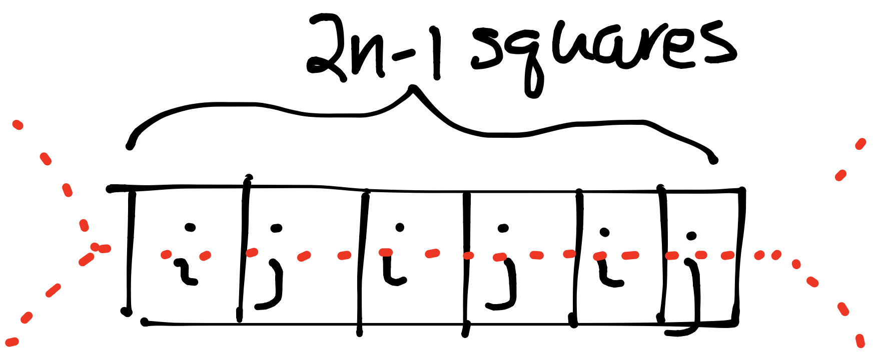
PP_2

2	1	2	1	2
1	2	1	2	1
2	1	2	1	2

PP_3

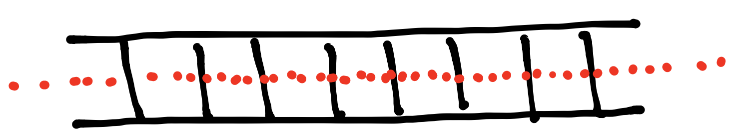


$PP_n =$



$\left.\begin{array}{l} i = 1 \text{ or } 2 \\ j = 2 \text{ or } 1 \end{array}\right\}$ depending upon parity of n

$\lim_{n\to\infty} PP_n$



THM:
(Benjamin Young-
paper presentation)

$PP_1 =$

$$\prod_{k\geq 1}\left(\frac{1}{1-y_1^k y_2^k}\right)^{2k}\left(1+y_1^k y_2^{k-1}\right)^{k}\left(1+y_1^k y_2^{k+1}\right)^{k}$$

$PP_n =$

$$\prod_{k\geq 1}\left(\frac{1}{1-y_1^k y_2^k}\right)^{2k}\left(1+y_1^{k-1} y_2^{k}\right)^{k+n-2}\left(1+y_1^{k+1} y_2^{k}\right)^{[k-n+2]}$$

$\max(k-n+2,0)$

Also see: Aganagic-Schaeffer (Appendix A)

<u>DEF'N</u>: Given a framed quiver $\hat{Q}$ and a mutation sequence

$\underline{\mu} = [\mu_1, \mu_2, \ldots, \mu_k]$

i) Compute F-polys $F_{n+1}, F_{n+2}, \ldots, F_{n+k}$

ii) Apply monomial-to-monomial transformation $F_{n+d} \to \tilde{F}_{n+d}$ defined as follows

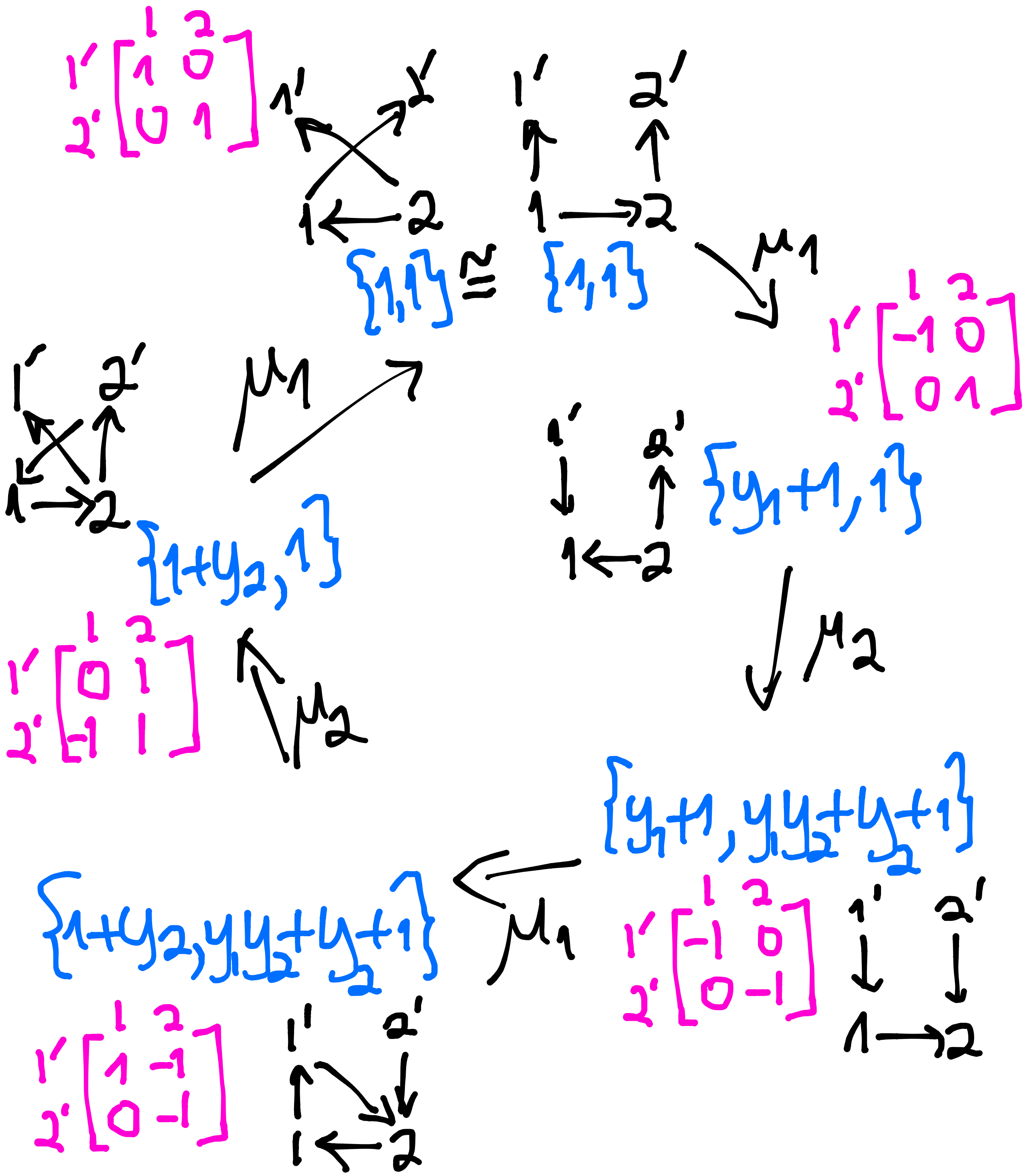
$C_d = C\text{-matrix} = [c_{ij}]$,

$c_{ij} = \#\text{arrows } i \longrightarrow j' \text{ in } \mu_d \cdots \mu_1 \hat{Q}$

> 0 if $i \to j'$

< 0 if $i \leftarrow j'$

EXAMPLE: (of C-matrices)



Then each term

$$M = y_1^{d_1} y_2^{d_2} \cdots y_n^{d_n} \text{ in } F_{n+k} \text{ is}$$

is replaced by $M' = y_1^{e_1} y_2^{e_2} \cdots y_n^{e_n}$

where
$$\begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = C_k^{-1} \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \in \mathbb{Z}^n$$

because C_k lies in $GL_n(\mathbb{Z})$ (not obvious)

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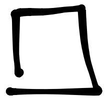
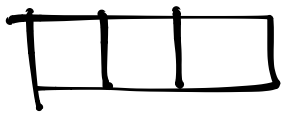
eg. $Q = \underset{0}{\cdot} \Longrightarrow \underset{1}{\cdot}$

$$\underline{\mu} = 0101010\cdots$$

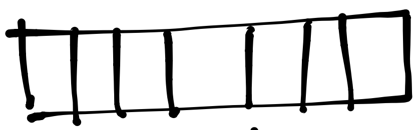


$$\tilde{F}_1 = y_0 + 1$$



$$\tilde{F}_2 = y_0^2 y_1^4 + 2y_0 y_1^2 + y_1 + 1$$

note minor fluctuation in subscripts 0,1 based on parity of n



$$\tilde{F}_3 = y_0^9 y_1^6 + 3y_0^6 y_1^4 + 2y_0^5 y_1^3 + 3y_0^3 y_1^2 + 2y_0^2 y_1 + y_0 + 1$$

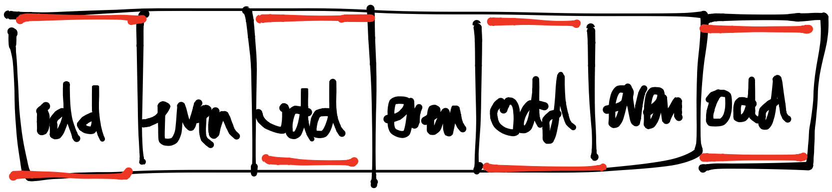
$$\lim_{n\to\infty} \tilde{F}_n = \dots + 4y_0^4 y_1^3 + 3y_0^3 y_1^2 + 2y_0^2 y_1 + y_0 + 1$$

Call this a stable cluster variable.

REU PROBLEM 6a

Show that $\tilde{F}_k$ are partition functions of perfect matchings of $2k \times 2$ grid graph, assuming that a matching has weight

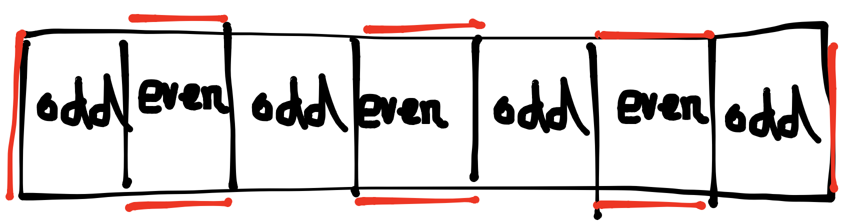
$$\frac{y_0^{k(k-1)} y_1^{k^2}}{\left(y_0^{k} y_1^{k+1}\right)^{\#\text{squares even indexed horizontal}} \left(y_0^{k-1} y_1^{k}\right)^{\#\text{squares odd indexed horizontal}}}$$



contributes

k odd squares

$$\frac{y_0^{k(k-1)}\, y_1^{k^2}}{\left(y_0^{k-1}\, y_1^{k}\right)^{k}} = 1.$$

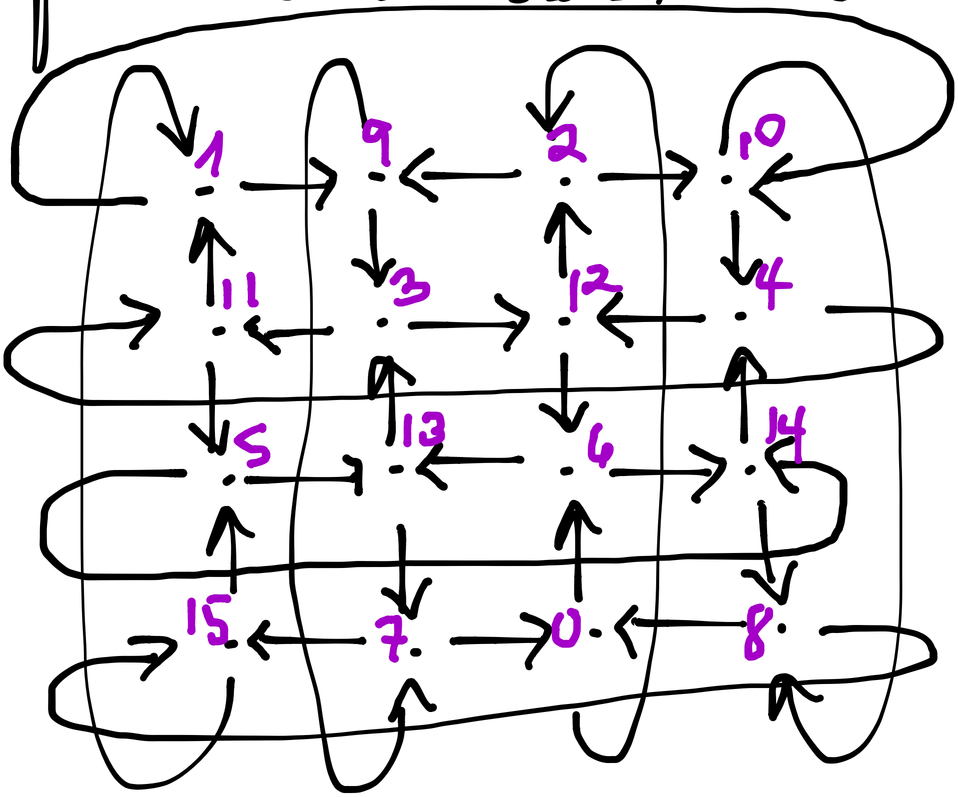


contributes

k-1 even squares

$$\frac{y_0^{k(k-1)}\, y_1^{k^2}}{\left(y_0^{k}\, y_1^{k+1}\right)^{k-1}} = y_1.$$

What if we instead try this quiver embeddable on atoms:



REU PROBLEM 6b

Combinatorially interpret the stable cluster variables for it.

See preliminary data and SAGE worksheet on REU page