REU 2016 Day 7 R. Patrias

Equality of skew Grothendieck polynomials

OUTLINE

1. Skew Schur
2. Skew coincidences
   - (i) Billera-Thomas-van Willigenburg
   - (ii) Reiner-Shaw-van W
   - (iii) McNamara-van W
3. Stable Grothendiecks + dual stable Grothendiecks
4. REU Problem 7 + EXERCISES

① Recall $f(x_1, x_2, \ldots)$ is symmetric if for any $\sigma \in S_n$

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}, x_{n+1}, x_{n+2}, \ldots) = f(x_1, x_2, \ldots)$$

and

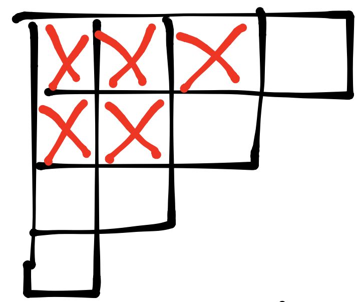
Schur functions

$$s_{\boxplus} = x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_5 x_8 x_{10} x_{15} + \ldots$$

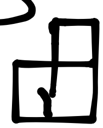
$$\begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array} \qquad \begin{array}{cc} 1 & 1 \\ 2 & 3 \end{array} \qquad \begin{array}{cc} 5 & 8 \\ 10 & 15 \end{array}$$

DEF'N: A **skew shape** is λ/μ for partitions $\mu \subseteq \lambda$

e.g. $(4,3,2,1)/(3,2)$



We define $s_{\lambda/\mu}$ in the natural way

$s_{(2,2)/(1)} = s$ 

skew Schur function

$= x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + \dots$

1	1	2
1 2	2 2	1 3

They are still symmetric.

QUESTION:

When is $s_{\lambda/\mu} = s_{\gamma/\nu}$?

Motivation:

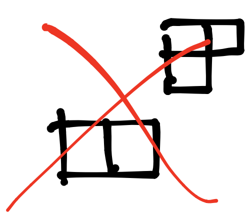
- $s_{\lambda/\mu} = \sum c^{\lambda}_{\mu,\nu} s_\nu$

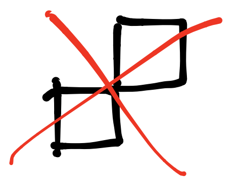
Littlewood-Richardson Coefficient

- distinguish $GL_N(\mathbb{C})$-modules, indexed by λ/μ having characters $s_{\lambda/\mu}$.

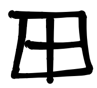
② (i)

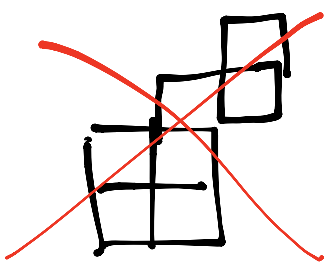
DEF'N: A skew diagram λ/μ is a **ribbon** if it is connected

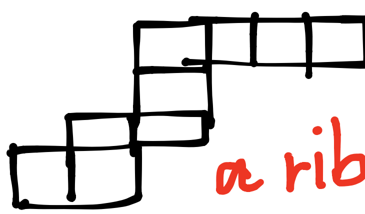




BAD; disconnected

and it contains no 2×2 subrectangle 





a ribbon

DEF'N: A composition of n is $(\beta_1, \dots, \beta_k) \in (\mathbb{Z}_+)^k$ with $n = \sum_{i=1}^{k} \beta_i$

EXAMPLE:

Compositions of 4

(4)

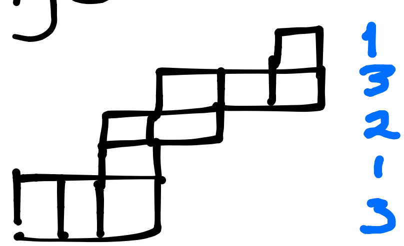
$(3,1)$ $(2,2)$ $(1,3)$

$(2,1,1)$ $(1,2,1)$ $(1,1,2)$

$(1,1,1,1)$

Compositions biject with ribbons

$(3,1,2,3,1) \longleftrightarrow$



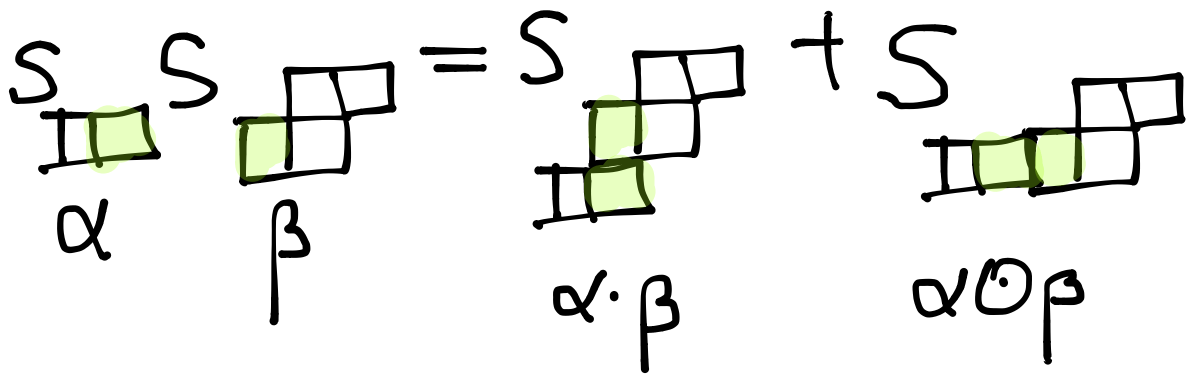
The ribbon Schur functions s_α are indexed by α compositions.

FACTS:

1) $\{s_\alpha\}$ generate the symmetric functions as an algebra.

2) $s_\alpha s_\beta = s_{\alpha\cdot\beta} + s_{\alpha\odot\beta}$

e.g.



DEF'N: $\alpha \geq \beta$ if α is obtained from β by adding adjacent parts.

$$(1,4,2,4) \leq (5,2,4) \leq (11)$$

DEF'N:

$$M(\beta) := \{\lambda(\alpha) : \alpha \geq \beta\}$$

regard the right side as a multiset

rearrange α into weakly decreasing order

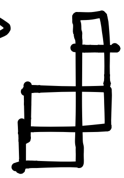
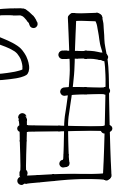
e.g.

$$M(2,1,1,1) =$$
$$\{(2,1,1,1),(3,1,1),(2,2,1),(3,2,1),(3,2),(4,1),(3,2),(5)\}$$

THM (B-T-vW):

$$S_\beta = S_\gamma \iff M(\beta) = M(\gamma).$$

e.g. $M(1,2,1,1) \neq M(2,1,1,1)$

$\Rightarrow S$  $\neq S$ 

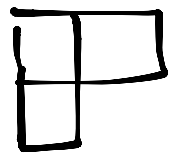
Composition of compositions

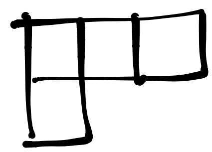
Combine $\cdot$ and $\odot$ to get $\circ$

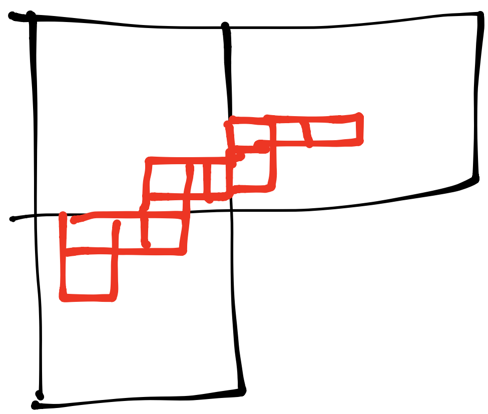
$$\alpha^{\odot n} = \alpha \odot \alpha \odot \cdots \odot \alpha$$

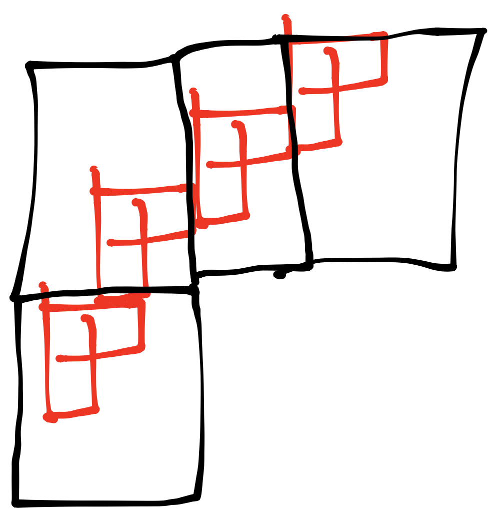
$$\alpha \circ \beta = \beta^{\odot \alpha_1} \cdot \beta^{\odot \alpha_2} \cdot \ldots \cdot \beta^{\odot \alpha_k}$$

EXAMPLE $\alpha = (1,2)$ $\beta = (1,3)$

$\alpha = (1,2) =$ 

$\beta = (1,3) =$ 

$\alpha \circ \beta =$ 

$\beta \circ \alpha =$ 

Define $\alpha^* = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_2, \alpha_1)$

THM (B-T-vanW):

$S_\beta = S_\gamma \iff$

$\beta = \beta_1 \circ \beta_2 \circ \ldots \circ \beta_k$ ← irreducible under $\circ$

$\gamma = \gamma_1 \circ \gamma_2 \circ \ldots \circ \gamma_k$ ←

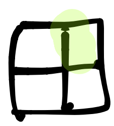
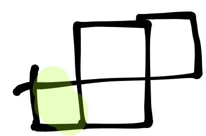
where $\gamma_i = \beta_i$ or $\gamma_i = \beta_i^*$ $\forall i$

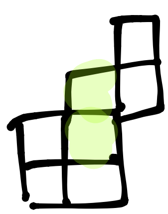
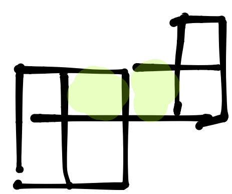
COR: An equivalence class of β will contain 2^r elements where $r = \#$ non-symmetric irreducible factors of β

(ii) Coincides among skew Schur functions

IDEA: Generalize $\circ$ of ribbons to composition of ribbon with a skew shape.

Define $\cdot$ and $\odot$ for skew shapes

$D_1 =$  and $D_2 =$ 

$D_1 \cdot D_2 =$  $D_1 \odot D_2 =$ 

These are associative,

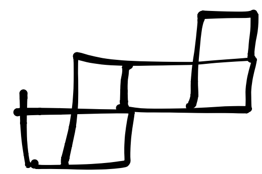
$$(D_1 *_1 D_2) *_2 D_3 = D_1 *_1 (D_2 *_2 D_3)$$

where $*_i$ are $\{\cdot, \odot\}$ anywhere

Now we will define $\underset{\text{ribbon}}{\alpha} \circ \underset{\text{skew}}{D}$

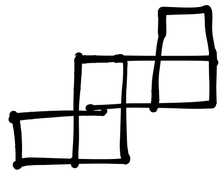
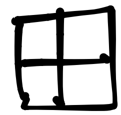
Note that ribbons are exactly the skew diagrams that can be written uniquely as

$$\square *_1 \square *_2 \square *_3 \cdots *_{k-1} \square$$

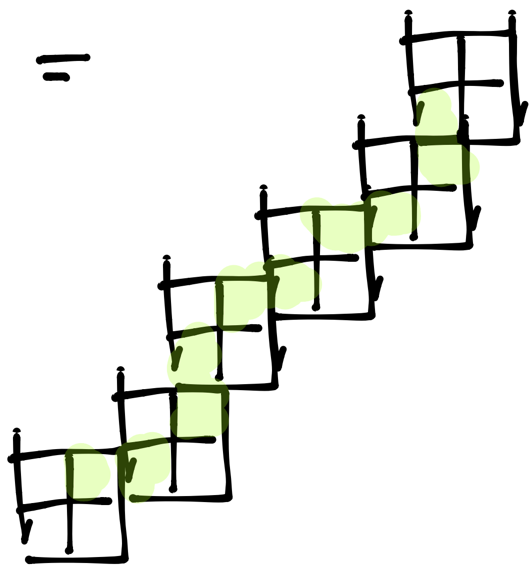
e.g. $\alpha = (2,3,1) =$ 

$$= \square \odot \square \cdot \square \odot \square \odot \square \cdot \square$$

Define $\alpha \circ D$ to be the result of replacing each $\square$ of α with D.

EXAMPLE: $\alpha =$  $D =$ 

$\alpha \circ D =$ 

$=$ 

Note this generalizes $\alpha \circ \beta$.
They also define $D \circ \beta$ in a way that generalizes $\alpha \circ \beta$.

THM (R-S-vW)

Assume we have ribbons α, α' and skew diagrams D, D', where $S_\alpha = S_{\alpha'}$ and $S_D = S_{D'}$. Then

i) $S_{\alpha \circ D} = S_{\alpha' \circ D}$

ii) $S_{D \circ \alpha} = S_{D' \circ \alpha}$

iii) $S_{D \circ \alpha} = S_{D \circ \alpha'}$

iv) $S_{\alpha \circ D} = S_{\alpha \circ D^*}$ ← 180° rotation

.

(iii) McN-vW generalize this further to $E \circ_\omega D$

skew → E; ribbon → $\circ_\omega$; skew → D

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③ Skew Grothendieck polynomials

DEF'N: A semistandard set-valued tableau is a filling of a partition shape with finite nonempty subsets of $\mathbb{Z}_+$ such that

- rows weakly increasing
  $A \le B$ iff $\max A \le \min B$
- columns strictly increasing.

$T =$

1,2	2	3,5,6
4,5	5,7	

(crossed out)

1,3	3
2,4	

$(-1)^{|\lambda|-|T|} = (-1)^{5-10} = -1$

$$x^T = x_1 x_2^2 x_3 x_4 x_5^3 x_6 x_7$$

DEF'N: The stable Grothendieck polynomial is

$$G_\lambda := \sum_{\substack{\text{semistandard} \\ \text{set-valued tableaux } T \\ \text{of shape } \lambda}} (-1)^{|\lambda|-|T|} x^T$$

EXAMPLE:

$$G_{(2,1)} = x_1^2 x_2 + 2x_1x_2x_3 - x_1x_2^2x_3 - 8x_1x_2x_3x_4$$

1	1
2	

1	2
3	

1	3
2	

1,2	2
3	

1,2	3
4	

$$- \dots + x_1x_2x_3^2x_4^2x_5 + \dots$$

1,2,3	3,4
4,5	

G_λ is...

- still symmetric
- an infinite sum
- unbounded in degree
- $s_\lambda \pm$ terms of higher degree
- generalizes to skew $G_{\lambda/\mu}$ straightforwardly

REMARK:

$s_\lambda \longleftrightarrow$ cohomology of Grassmannian

$G_\lambda \longleftrightarrow$ K-theory of Grassmannian

(s_λ:) in a certain sense, self-dual

(G_λ:) not self-dual in this sense, so they have a corresponding dual basis....

Dual stable Grothendieck polynomials

<u>DEF'N</u>: A **reverse plane partition** of shape λ is a filling of λ with $\mathbb{Z}_+$ so that entries weakly increase in rows and columns.

$T_1 =$

1	1	2
1	1	
1		

$T_2 =$

1	2	2
1	3	4
2	3	

$x^{T_1} = x_1^2 x_2^2$

$x^{T_2} = x_1 x_2^3 x_3 x_4$

where $x^{T} = \prod_i x_i^{\#\text{ of columns with an } i}$

DEF'N: The dual stable Grothendieck polynomial is

$$g_\lambda := \sum_{\substack{\text{reverse plane} \\ \text{partitions } T \text{ of} \\ \text{shape } \lambda}} x^T$$

Similarly define $g_{\lambda/\mu}$.

$$g_{(2,1)} = x_1^2x_2 + 2x_1x_2x_3 + x_1^2x_2 + x_1^2 + \dots$$

$$\begin{array}{cccc} \begin{matrix} 1 & 1 \\ 2 & \end{matrix} & \begin{matrix} 1 & 2 \\ 3 & \end{matrix} & \begin{matrix} 1 & 2 \\ 1 & \end{matrix} & \begin{matrix} 1 & 1 \\ 1 & \end{matrix} \\ & \begin{matrix} 1 & 3 \\ 2 & \end{matrix} & & \end{array}$$

Note g_λ is

- still symmetric
- an infinite sum
- terms of degree $\leq |\lambda|$
- s_λ + terms of lower degree.

REU PROBLEM 7

When is $G_{\lambda/\mu} = G_{\gamma/\nu}$?

$g_{\lambda/\mu} = g_{\gamma/\nu}$?

REU EXERCISE 17

(0) Show $S_D = S_{D^*}$ and

$S_\alpha S_\beta = S_{\alpha\cdot\beta} + S_{\alpha\odot\beta}$

(not to be presented)

(a) Prove multiplication formula for

$$G_\alpha \cdot G_\beta$$

(b) Prove a multiplication formula for

$$g_\alpha \cdot g_\beta$$

REU EXERCISE 18

Find an example α, β with $M(\alpha) = M(\beta)$ but (a) $G_\alpha \neq G_\beta$

(b) $g_\alpha \neq g_\beta$