(1)

REU 2016 Day 8 V. Reiner

Faces of Gelfand-Tsetlin polytopes

1. GT-patterns & polytopes
2. Polytope review $+ \frac{2}{3}$(REU Problem 8)
3. Flagnumbers & cd-index $+ \frac{1}{3}$(REU Problem 8)

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1. GT-patterns

Recall $s_\lambda(x_1,\dots,x_n) :=$ Schur function $= \sum x^T$

$\underbrace{}$ Call this set $SST(\lambda,n)$ $\to$ {semistandard tableaux T of shape λ, entries in $\{1,2,\dots,n\}$}

e.g. $s_{(3,1,1)}(x_1,x_2,x_3) = x_1^3x_2x_3 + x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1x_2^3x_3 + x_1x_2^2x_3^2 + x_1x_2x_3^3$

$$\left\{ \begin{matrix}111\\2\\3\end{matrix}, \begin{matrix}112\\2\\3\end{matrix}, \begin{matrix}113\\2\\3\end{matrix}, \begin{matrix}122\\2\\3\end{matrix}, \begin{matrix}123\\2\\3\end{matrix}, \begin{matrix}133\\2\\3\end{matrix} \right\} = SST(\lambda, 3)$$

$\lambda = (1 \le 1 \le 3)$

PROP (Gelfand-Tsetlin 1950) (easy) If we pad λ with initial zeroes to make it have length n $(0 \le 0 \le \dots \le \lambda_n)$, there is a bijection

$$SST(\lambda,n) \xrightarrow{\sim} GT_{\mathbb{Z}}(\lambda) = \text{(integer) GT-patterns with top row } \lambda$$

$$T \longmapsto (\lambda^{(i)})_{i=1}^{n}, \qquad \lambda^{(i)} = shape(T|_{\{1,2,\dots,i\}})$$

$$\begin{array}{rccccccc}
\lambda=\lambda^{(n)} = & x_{1n} & \le & x_{2n} & \le & \dots\ x_{n-1,n} & \le & x_{nn} \\
& & \ge\ \le & & \ge\ \le & \dots & \ge\ \le & \\
\lambda^{(n-1)} = & & x_{1,n-1} & & x_{2,n-1} & \dots & x_{n-1,n-1} & \\
& & \vdots & & & & \vdots & \\
\lambda^{(3)} = & & & x_{13} & x_{23} & x_{33} & & \\
\lambda^{(2)} = & & & & x_{12}\ \ x_{22} & & & \\
\lambda^{(1)} = & & & & x_{11} & & &
\end{array}$$

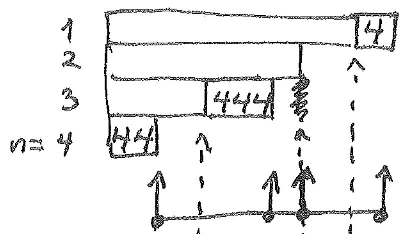
e.g. $T = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&4&4&5\\ \hline 3&5\\ \cline{1-2}\end{array}$ $\longmapsto$

$$\begin{array}{ccccccccccl}
0 & & 0 & & 2 & & 4 & & 4 & = & \lambda^{(5)} = \lambda \\
& 0 & & 1 & & 3 & & 4 & & = & \lambda^{(4)} \\
& & 1 & & 1 & & 4 & & & = & \lambda^{(3)} \\
& & & 1 & & 3 & & & & = & \lambda^{(2)} \\
& & & & 2 & & & & & = & \lambda^{(1)}
\end{array}$$

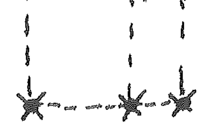
$n=5$

$\lambda = (0,0,2,4,4)$

<u>proof idea</u>: Think about how far in rows $1,2,\ldots,n$ the values extend



versus how far in rows $1,2,\ldots,n-1$ the values $\leq n-1$ extend:



The latter ~~distances~~ distances <u>interlace</u> the former:

(and it's reversible) ∎

DEF'N: The GT-polytope $GT(\lambda) := GT_{\mathbb{R}}(\lambda)$ is the solution set in $\mathbb{R}^{\binom{n}{2}}$ with coordinates $(x_{i,j})_{1 \leq i \leq j \leq n-1}$ to the same inequalities

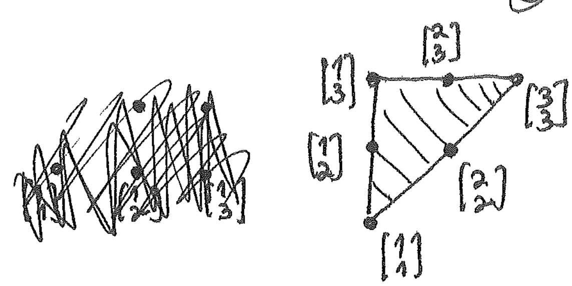
$$
\begin{array}{ccccccc}
\lambda_1 & & \lambda_2 & \cdots & & & \lambda_n \\
 & \leq x_{1,n-1} \leq & \geq & & & \geq & \\
 & \leq & \ddots & & & \geq & \\
 & x_{1,3} & & x_{2,3} & & x_{3,3} & \\
 & & x_{1,2} & & x_{2,2} & & \\
 & & & x_{1,1} & & &
\end{array}
$$

<u>EXAMPLES</u>:

(1) $GT\left(\begin{smallmatrix} 3 \square\square\square \\ \vdots\ \square \\ \square \end{smallmatrix}\right)$

$$
\begin{smallmatrix} 1 & 1 & 3 \\ 1 & 3 & \\ 3 & & \end{smallmatrix} \quad
\begin{smallmatrix} 1 & 1 & 3 \\ 1 & 3 & \\ 2 & & \end{smallmatrix} \quad
\begin{smallmatrix} 1 & 1 & 3 \\ 1 & 2 & \\ 2 & & \end{smallmatrix} \quad
\begin{smallmatrix} 1 & 1 & 3 \\ 1 & 3 & \\ 1 & & \end{smallmatrix} \quad
\begin{smallmatrix} 1 & 1 & 3 \\ 1 & 2 & \\ 1 & & \end{smallmatrix} \quad
\begin{smallmatrix} 1 & 1 & 3 \\ 1 & 1 & \\ 1 & & \end{smallmatrix}
$$

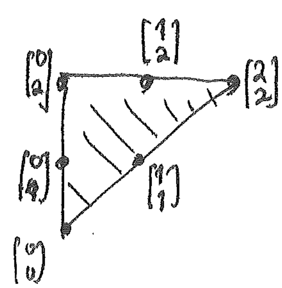
← 6 elements of $GT_{\mathbb{Z}}(\square) \leftrightarrow SST(\square, 3)$



$\cong GT\left(\begin{smallmatrix} 2 \square\square \\ 0 \\ 0 \end{smallmatrix}\right)$

$$
\begin{smallmatrix} 0 & 0 & 2 \\ 0 & 2 & \\ 2 & & \end{smallmatrix} \quad
\begin{smallmatrix} 0 & 0 & 2 \\ 0 & 2 & \\ 1 & & \end{smallmatrix} \quad
\begin{smallmatrix} 0 & 0 & 2 \\ 0 & 1 & \\ 1 & & \end{smallmatrix} \quad
\begin{smallmatrix} 0 & 0 & 2 \\ 0 & 2 & \\ 0 & & \end{smallmatrix} \quad
\begin{smallmatrix} 0 & 0 & 2 \\ 0 & 1 & \\ 0 & & \end{smallmatrix} \quad
\begin{smallmatrix} 0 & 0 & 2 \\ 0 & 0 & \\ 0 & & \end{smallmatrix}
$$

affinely isomorphic (defined below)

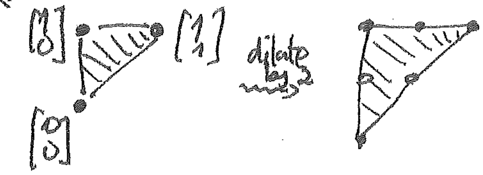


(2) In fact both $GT\left(\begin{smallmatrix}3\\1\\1\end{smallmatrix}\right) \cong GT\left(\begin{smallmatrix}2\\0\\0\end{smallmatrix}\right) \cong GT\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$

still affinely isomorphic, but with some dilation involved

$T=$ $\boxed{1}$ $\boxed{2}$ $\boxed{3}$

$\begin{smallmatrix}0&0&1\\&0\ 1&\\&1&\end{smallmatrix}$ $\begin{smallmatrix}0&0&1\\&0\ 1&\\&0&\end{smallmatrix}$ $\begin{smallmatrix}0&0&1\\&0\ 0&\\&0&\end{smallmatrix}$



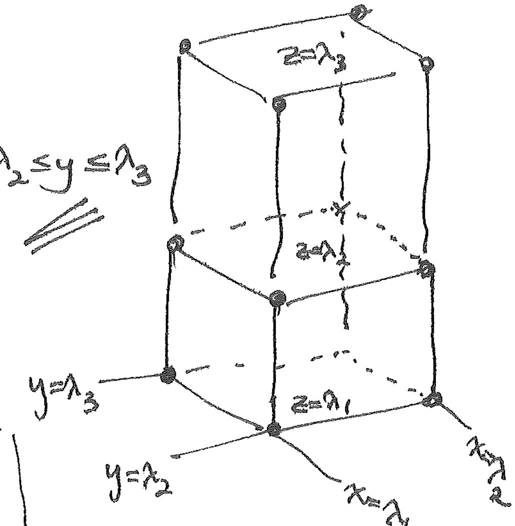
(3) Given $\lambda = (\lambda_1 < \lambda_2 < \lambda_3)$, let's try to draw

$GT(\lambda) \subset \mathbb{R}^3$ coordinates (x,y,z)

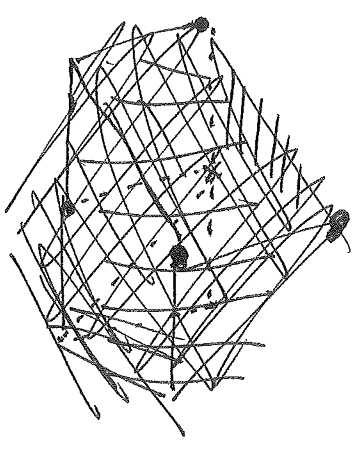
$$\begin{array}{ccccc} \lambda_1 & & \lambda_2 & & \lambda_3 \\ & \leq x \leq & & \leq y \leq & \\ & & \leq z \leq & & \end{array}$$

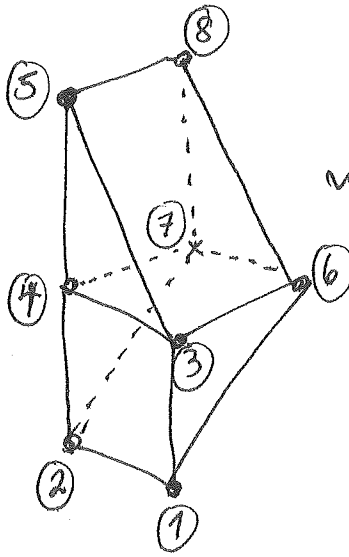
starting with $\lambda_1 \leq x \leq \lambda_2 \leq y \leq \lambda_3$

$\searrow z \swarrow$



Then impose $x \leq z$ and $z \leq y$, slicing off two wedges:

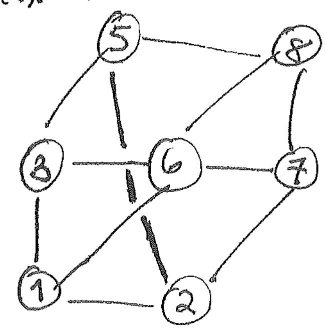




redraw the 1-skeleton = vertices + edges

$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ - symmetric!

$= \langle (13)(78)(25)(6) \rangle$, $= \langle (18)(37)(25)(6) \rangle$



e.g. $\lambda = $ 3 2 1

$s_{\lambda}(x_1,x_2,x_3) = x_1^3x_2^2x_3 + x_1x_2^2x_3^2 + \ldots + \ldots + \ldots + \ldots + \ldots + \ldots$

$SST(\lambda,3) =$	111 / 22 / 3	112 / 22 / 3	113 / 22 / 3	111 / 23 / 3	112 / 23 / 3	113 / 23 / 3	122 / 23 / 3	123 / 23 / 3
GT's =	123 / 23 / 3	123 / 23 / 2	123 / 22 / 2	123 / 13 / 3	123 / 13 / 2	123 / 12 / 2	123 / 13 / 1	123 / 12 / 1
	8	7	6	5	4	3	2	1

not a vertex of the polytope!

Put this into Sage Math Cell:

```
Polyhedron(ieqs=[
    [0,-1,0,1],
    [0,0,1,-1],
    [-1,1,0,0],
    [-2,0,1,0],
    [2,-1,0,0],
    [3,0,-1,0]]).plot()
```

expresses the inequality $3 + 0\cdot x + (-1)y + 0\cdot z \geq 0$

i.e. $y \leq 3$

(4)

② Polytope review & REU PROB8(a),(b)

REU EXERCISE 19

Using multiplicity notation $0^{m_0}1^{m_1}2^{m_2}\cdots\ell^{m_\ell} = \lambda = (\lambda_1 \le \cdots \le \lambda_n)$

(a) Show that $\underset{\text{}}{\dim}\, GT(0^{m_0}1^{m_1}\cdots\ell^{m_\ell}) = \underbrace{\binom{\sum_{i=1}^{\ell} m_i}{2}}_{\binom{n}{2} \text{ since } n=\sum_{i=1}^{\ell} m_i} - \sum_{i=1}^{\ell}\binom{m_i}{2}$

(dimension of the smallest affine subspace (= translate of a linear subspace) containing it

e.g. $\dim \underline{GT(113)} = \binom{2+1}{2} - \left(\binom{2}{2}+\binom{1}{2}\right)$

$GT(1^2 3^1)$ $= 3-(1+0) = 2$ ✓

(b) show that $GT(\underbrace{0^1 1^{\ell-1}}_{0\,1\,1\cdots 1,\ \ell-1 \text{ times}})$ is an $(\ell-1)$-dimensional simplex, that is the convex hull of ℓ points and $(\ell-1)$-dimensional

e.g. $GT(01) =$ 

$GT(011) =$ 

$GT(0111) =$ 

$\vdots$

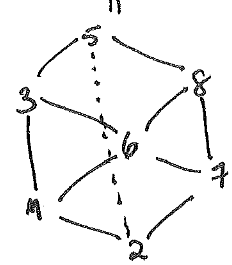
(c) Find an affine isomorphism $GT(0^{m_0}1^{m_1}\cdots\ell^{m_\ell}) \xrightarrow{\sim} GT(0^{m_\ell}1^{m_{\ell-1}}\cdots\ell^{m_0})$,

composition of a linear map and a translation

and show that it gives a nontrivial affine $\mathbb{Z}/2\mathbb{Z}$-symmetry of $GZ(0^{m_0}1^{m_1}\cdots\ell^{m_\ell})$ whenever $m_j = m_{\ell-j}\ \forall j$

e.g. it should give $(18)(25)(37)(6)$ on $GZ(1^1 2^1 3^1)$

$\|$



(5)

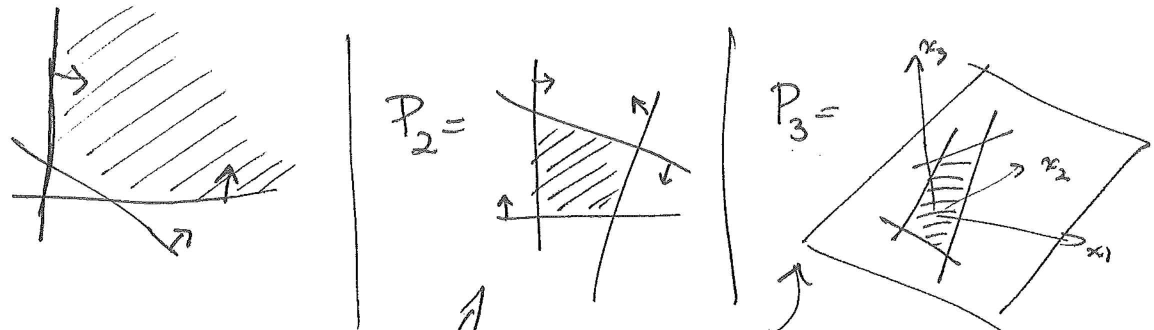
The REU problem will focus on facial structure of $GT(\lambda)$, so recall what faces are...

DEF'N: A <u>polyhedron</u> $P \subset \mathbb{R}^d$ is a <u>finite</u> intersection $P = \bigcap_{i=1}^{t} H_i^+$

where each H_i^+ is a <u>half-space</u> $\{x \in \mathbb{R}^d : a_1x_1 + \dots + a_dx_d \geq b\}$

with (<u>affine</u>) <u>hyperplane</u> $H_i = \{x \in \mathbb{R}^d : a_1x_1 + \dots + a_dx_d = b\}$

e.g. $P_1 =$ $P_2 =$ $P_3 =$



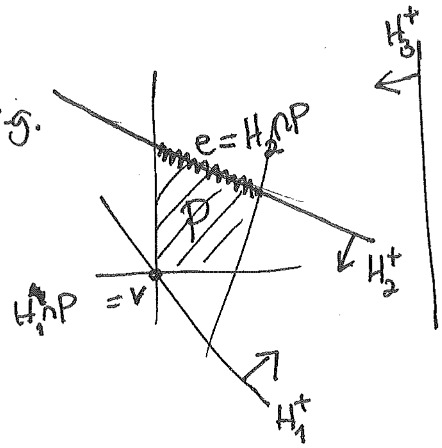
<u>bounded polyhedra</u> or <u>polytopes</u>

A <u>face</u> of a polyhedron P is an intersection $F = H \cap P$

where H is the hyperplane for some <u>half-space $H^+ \supseteq P$</u>

called a supporting half-space

e.g.

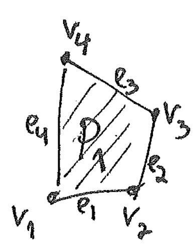


$\emptyset$ = the <u>empty face</u> $= H_3 \cap P$

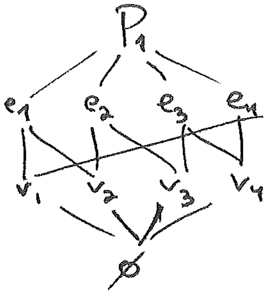
- <u>vertices</u> = 0-dim'l faces
- <u>edges</u> = 1-dim'l faces
- P itself is considered a face of P

The <u>face poset</u> $\mathcal{F}(P) := \{\text{faces of } P\}$ ordered under inclusion turns out to be (not obvious!) a graded lattice of rank $\dim P + 1$ (see e.g. Ziegler's "Lectures on Polytopes")

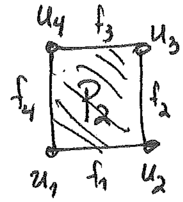
e.g.



$\rightsquigarrow \mathcal{F}(P_1) =$



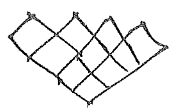
$\cong$ poset isomorphic $\mathcal{F}(P_2)$



Say P_1, P_2 are <u>combinatorially isomorphic</u> if $\mathcal{F}(P_1) \cong \mathcal{F}(P_2)$

Clearly P_1, P_2 <u>affinely isomorphic</u> implies this.

(6)

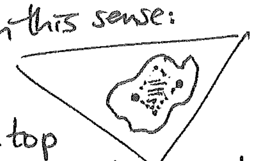
THM (T. McAllister 2006) The map sending a $GT(\lambda)$-pattern $\underline{x}$ to the decomposition  $= \bigsqcup_{i=1}^{r} T_i$ where the T_i are the connected components of the "equal patches" in $\underline{x}$

gives a poset isomorphism

$$\mathcal{F}(GT(\lambda)) \xrightarrow{\sim} \{\underline{GT(\lambda)\text{-tilings}}\} \text{ under refinement}$$

(decompositions  $= \bigsqcup_{i=1}^{r} T_i$ where

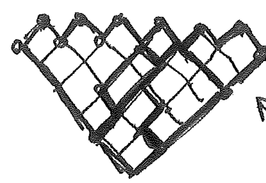
- T_i are connected
- T_i are convex in this sense:



- restricted to the top row, $\bigsqcup_{i=1}^{r} T_i$ partitions like the parts of λ

e.g. $\lambda = 111588$

$$\underline{x} = \begin{matrix} 1\,1\,1\,5\,8\,8 \\ 1\,1\,2\,6\,8 \\ 1\,2\,6\,6 \\ 1\,5\,6 \\ 5\,6 \\ 5 \end{matrix} \longmapsto$$



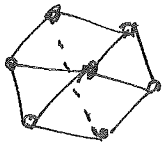
6 tiles total

REU PROBLEM 8(a): What is the diameter of $GT(\lambda)$?

They seem remarkably small.

max distance in the 1-skeleton (=vertices+edges)

e.g. diameter $GT((1,2,3)) = 2$



CONJECTURE: $\operatorname{diam} GT(\lambda_1 \le \dots \le \lambda_n) \le 2(n-2)$ for $n \ge 3$

with equality here $\Leftrightarrow$ $\lambda_1 < \lambda_2 < \dots < \lambda_n$

CONJECTURE: $\operatorname{diam} GT(\underbrace{0^k 1^{n-k}}_{000\cdots011\cdots1}) = 2$

CONJECTURE: $\operatorname{diam} GT(\underbrace{0^1 1^{n-2} 2^1}_{011111\cdots12}) = 2$

RMK: $GT(\lambda)$, $\lambda = \lambda_1 \le \dots \le \lambda_n$, can have nonintegral vertices for $n \ge 5$!

(DeLoera-McAllister 2006)

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REU PROBLEM 8(b): Is there a hidden symmetry in $GZ(\lambda)$?

<u>CONJECTURE</u>: There is a non-trivial $\mathbb{Z}/2\mathbb{Z}$-combinatorial symmetry on $GZ(\lambda)$ $\forall \lambda$

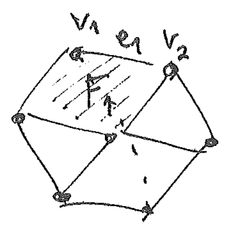
($\overset{?}{\Rightarrow}$ <u>CONJECTURE</u>: $GZ(\lambda_1 < \cdots < \lambda_n)$ has $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ symmetry)

<u>CONJECTURE</u>: $GZ(0^1 1^{n-2} 2^1)$ has a <u>humongous</u> symmetry group, that grows with n.

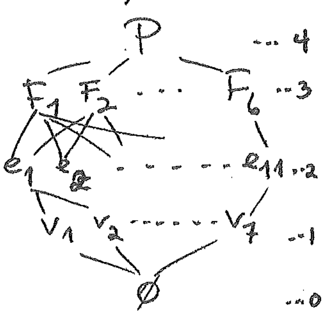
③ <u>Flag numbers & cd-index</u>

For a d-dim'l polytope P, the <u>flag</u> <u>f-vector</u> $(f_S : S \subset \{1,2,\ldots,d\})$ records each flag number $f_S := \#\left\{\begin{array}{l}\text{flags of faces in } P \text{ passing through} \\ \text{ranks } S \text{ in the face poset}\end{array}\right\}$, but is <u>highly redundant</u>, and has values <u>unnecessarily large</u> ...

e.g. $P = GZ(\lambda_1<\lambda_2<\lambda_3)$ has



$\mathcal{F}(P)=$



S	f_S	h_S	ab-monomial
$\emptyset$	1	1	$1 \cdot aaa$
1	7	$6 = 7-1$	$6baa$
2	11	10	$10aba$
3	6	5	$5aab$
12	22	5	$5bba$
13	22	$10 = 22-7-6+1$	$10bab$
23	22	6	$6abb$
123	44	1	$1bbb$

Better is the <u>flag h-vector</u> $(h_S : S \subset \{1,2,\ldots,d\})$ defined via
$$h_S := \sum_{T \subseteq S} (-1)^{|S-T|} f_T$$
(or equivalently via inclusion-exclusion $f_S = \sum_{T \subseteq S} h_T$)

FACTS: (not obvious)
- $h_S \geq 0$
- $h_S = h_{\{1,\ldots,d\} \setminus S}$

ab-index of $P = a^3 + 6ba^2 + 10aba + 5a^2b + 5b^2a + 10bab + 6ab^2 + b^3$
$= (a+b)^3 + 5ba^2 + 9aba + \underline{4a^2b} + \underline{4b^2a} + 9bab + 5ab^2$
$= c^3 + 4(a+b)(ab+ba) + 5ba^2 + 5aba + 5bab + 5ab^2$
$= c^3 + 4cd + 5dc$ = cd-index of P

Even better is the <u>cd-index</u> of P defined by 1st creating the <u>ab-polynomial</u>
$$\sum_{S \subseteq \{1,\ldots,d\}} h_S \, \underline{ab(S)}$$
a noncommutative degree d monomial $aababba\cdots ba$, b's in positions S

and then writing it in terms of $c = a+b$, $d = ab+ba$. It can always be done! (Bayer-Billera-Fine)

(8) THM (Stanley 1994) cd-indices of polytopes P have nonnegative coefficients.

REU PROBLEM 8(c):

Compute the cd-index for some families of $GT(\lambda)$,

e.g. $GT(0^k 1^{n-k})$

$GT(0^1 1^{n-2} 2)$

$GT(\lambda_1 < \lambda_2 < \dots < \lambda_n)$ i.e. $GT(1^1 2^1 3^1 \cdots n^1)$

vertex counts studied by Gusev-Kiritchenko-Timorin 2013

RMK: SAGE has the face poset $\mathcal{F}(P)$
and its flag f-vector
flag h-vector
but not cd-index (that I could find).

REU EXERCISE 20

(a) Show that the ring map defined by

$$\mathbb{Z}\langle c,d\rangle \longrightarrow \mathbb{Z}\langle a,b\rangle$$

noncommutative polynomials in a,b with $\mathbb{Z}$ coefficients

$$c \longmapsto a+b$$
$$d \longmapsto ab+ba$$

is injective, but not surjective.

In particular, cd-indices are unique.

(b) Show that as a subset of $\mathbb{R}^{2^d}$,

the set $\left\{ \begin{array}{c} \text{flag f-vectors } (f_S) \\ \text{of } d\text{-dim'l polytopes} \end{array} \right\}$ affinely spans a

space of dimension at most $F_d - 1$ where $\left\{ \begin{array}{l} F_0 = F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \end{array} \right\}$

are Fibonacci numbers

It turns out that equality holds (Bayer-Billera)