Math. Z. 216, 483–499 (1994)

Mathematische Zeitschrift

Flag f–vectors and the cd–index

Richard P. Stanley *

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA (e-mail: rstan@math.mit.edu, fax: (617) 253-4358)

Received 2 November 1992; in final form 3 May 1993

1 Introduction and elementary properties

The cd–index $\Phi_P(c,d)$ is a noncommutative polynomial in the variables c and d which efficiently encodes the flag f–vector of an Eulerian poset P. The cd–index was first defined by Jonathan Fine (see [B–K, Prop. 2]). In this section we review the relevant background material and give a new proof of the existence of the cd–index. This proof follows from a simple recurrence (Theorem 1.1) satisfied by the cd–index. When we specialize our recurrence to the case of a boolean algebra, we obtain a new generating function for the noncommutative André polynomials of Foata–Schützenberger (Corollary 1.4).

In Sect. 2 we show *via* a shelling argument that $\Phi_P(c,d) \geq 0$ (i.e., all coefficients of $\Phi_P(c,d)$ are nonnegative) when P is the face poset of a shellable regular CW–sphere. Our definition of "shellable" is slightly different from the usual one, but it does include line shellings of convex polytopes. Hence polytopes have nonnegative cd–index, proving a conjecture of Fine (see [B–K, Conj. 3]). We conjecture that $\Phi_P(c,d) \geq 0$ for any Cohen–Macaulay Eulerian poset P (Conjecture 2.1), and we show that this conjecture (if true) gives *all* linear inequalities satisfied by the flag f–vector of a Cohen–Macaulay Eulerian poset (Theorem 2.1).

In Sect. 3 we obtain a formula for the cd–index of a *simplicial* Eulerian poset P. ("Simplicial" means that if $x < \hat{1}$, then the interval $[\hat{0}, x]$ is a boolean algebra.) Using a result in [S$_3$], it follows that $\Phi_P(c,d) \geq 0$ for any Cohen–Macaulay simplicial Eulerian poset P. The two main results in Sects. 2 and 3 are both generalizations of a theorem of M. Purtill [Pu, §8] that $\Phi_P(c,d) \geq 0$ when P is the face lattice of a simplicial convex polytope (and of certain other polytopes). We conclude Sect. 3 with a conjectured refinement (Conjecture 3.1) of noncommutative André polynomials related to our formula for the cd–index of a simplicial Eulerian poset.

* Partially supported by the Goran Gustafsson Foundation at the Royal Institute of Technology (Kungl. Tekniska Högskolan), Stockholm, and by NSF grants #DMS–8401376 and #DMS–9206374

Let us begin with the definition of the cd–index due to Fine. Our general terminology concerning posets (partially ordered sets) is taken from [S$_2$, Ch. 3]. Let P be a finite graded poset of rank $n+1$ with $\hat{0}$ and $\hat{1}$. (We always assume $n \geq 0$, so $\hat{0} < \hat{1}$.) Let ρ denote the rank function of P. Thus $P = P_0 \cup P_1 \cup \cdots \cup P_{n+1}$ (disjoint union), where $x \in P_i$ if and only if $\rho(x) = i$. Every maximal chain of P has the form $\hat{0} = x_0 < x_1 < \cdots < x_{n+1} = \hat{1}$ with $\rho(x_i) = i$. Let $S \subseteq [n] = \{1, 2, \ldots, n\}$, and let P_S denote the S–rank selected subposet of P (with $\hat{0}$ and $\hat{1}$), i.e.,

$$P_S = \{x \in P : \ \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}.$$

Denote by $\alpha(S) = \alpha_P(S)$ the number of maximal chains of P_S. The function $\alpha : 2^{[n]} \to \mathbf{Z}$ is called the *flag f–vector* of P. Define a function $\beta = \beta_P : \ 2^{[n]} \to \mathbf{Z}$ by

$$\beta(S) \ = \ \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha(T), \tag{1}$$

or equivalently,

$$\alpha(S) \ = \ \sum_{T \subseteq S} \beta(T). \tag{2}$$

Then β is called the *flag h–vector* of P. It has many interesting properties, e.g., $\beta_P(S) \geq 0$ when P is a Cohen–Macaulay poset [B–G–S, Thm. 3.3].

We now define a noncommutative polynomial which encodes the flag f–vector (or flag h–vector). If $S \subseteq [n]$, then define a noncommutative monomial $u_S = u_1 u_2 \cdots u_n$ in the variables a and b by

$$u_i \ = \ \begin{cases} a, & i \notin S \\ b, & i \in S. \end{cases} \tag{3}$$

For instance, if $n = 6$ and $S = \{2, 6\}$, then $u_S = abaaab$. Let

$$\Upsilon_P(a,b) = \sum_{S \subseteq [n]} \alpha_P(S) u_S \tag{4}$$

$$\Psi_P(a,b) = \sum_{S \subseteq [n]} \beta_P(S) u_S.$$

It is an immediate consequence of (1) or (2) that

$$\Psi_P(a,b) = \Upsilon_P(a-b, b) \tag{5}$$

$$\Upsilon_P(a,b) = \Psi_P(a+b, b).$$

For instance, if P is the boolean algebra B_3, then

$$\Upsilon_P(a,b) = a^2 + 3ba + 3ab + 6b^2$$

$$\Psi_P(a,b) \ = \ a^2 + 2ba + 2ab + b^2. \tag{6}$$

Suppose now that P is *Eulerian* [S$_2$, Ch. 3.14], i.e., for all $x \leq y$ in P we have $\mu(x,y) = (-1)^{\rho(x,y)}$, where μ denotes the Möbius function of P and where $\rho(x,y) = \rho(y) - \rho(x)$. Bayer and Billera [B–B] show that the flag f–vector α_P then satisfies certain linear relations. Fine [B–K, Thm. 4] observed that the Bayer–Billera relations are equivalent to the following statement.

Proposition 1.1 (Fine) *Let P be a finite graded poset with $\hat{0}$ and $\hat{1}$. Then P satisfies the Bayer–Billera relations if and only if $\Psi_P(a,b)$ can be written as a polynomial $\Phi_P(c,d)$ in $c = a+b$ and $d = ab+ba$.* □

The polynomial $\Phi_P(c,d)$ is called the *cd–index* of P. Thus in particular $\Phi_P(c,d)$ exists if P is Eulerian. The cd–index, if it exists, is unique since $a+b$ and $ab+ba$ are algebraically independent (as noncommutative polynomials) over any field K. For instance, from (6) we see that $\Phi_{B_3}(c,d) = c^2+d$. If we define $\deg(c) = 1$ and $\deg(d) = 2$, then clearly $\Phi_P(c,d)$ is homogeneous of degree n with integer coefficients. For any word w in the letters a,b,c,d we let $\ell(w)$ denote the length (total degree) of w, with $\ell(a) = \ell(b) = \ell(c) = 1$, $\ell(d) = 2$.

Note that if P has a cd–index, i.e., if $\Psi_P(a,b)$ is a polynomial in $a+b$ and $ab+ba$, then $\Psi_P(a,b) = \Psi_P(b,a)$. In other words, $\beta_P(S) = \beta_P(\overline{S})$, where $\overline{S} = [n]-S$. This symmetry condition is not enough to guarantee the existence of the cd–index. Indeed, a noncommutative polynomial $\Psi(a,b)$ satisfies $\Psi(a,b) = \Psi(b,a)$ if and only if $\Psi(a,b)$ is a polynomial (necessarily unique) in the variables $a+b, a^2+b^2, a^3+b^3, \ldots$. On the other hand, $\Psi(a,b) = \Phi(a+b, ab+ba)$ for some polynomial Φ if and only if $\Psi(a,b)$ is a polynomial just in the variables $a+b$ and $a^2+b^2 = (a+b)^2-(ab+ba)$.

Given posets P and Q with $\hat{0}$ and $\hat{1}$, define the *join* $P*Q$ to be the poset on the set $(P-\{\hat{1}\}) \cup (Q-\{\hat{0}\})$ with $x \le y$ in $P*Q$ if either (i) $x \le y$ in $P-\{\hat{1}\}$, (ii) $x \le y$ in $Q-\{\hat{0}\}$, or (iii) $x \in P-\{\hat{1}\}$, $y \in Q-\{\hat{0}\}$. Thus in the notation of [S$_2$, p. 100], $P*Q$ is just the ordinal sum $(P-\{\hat{1}\}) \oplus (Q-\{\hat{0}\})$. It is easy to see that if P and Q are Eulerian, then so is $P*Q$.

When Q is the boolean algebra B_2 (with cd–index $\Phi_{B_2} = c$) and P is Eulerian then we call the join $P*B_2$ the *suspension* of P, denoted ΣP. There is another construction related to the suspension which we will use. Let us say that a poset P is *near–Eulerian* if it is obtained from an Eulerian poset Q by removing a single coatom (element covered by $\hat{1}$) x. Given P, we can uniquely recover Q by adding a coatom x which covers all $y \in P$ for which the interval $[y,\hat{1}]$ is a three element chain. We call Q the *semisuspension* of P, denoted $Q = \tilde{\Sigma} P$.

Lemma 1.1 *Suppose P and Q are Eulerian. Then*

$$\Phi_{P*Q}(c,d) = \Phi_P(c,d)\Phi_Q(c,d).$$

In particular,

$$\Phi_{\Sigma P} = \Phi_P c.$$

Proof This is an immediate consequence of the obvious fact that

$$\Upsilon_{P*Q}(a,b) = \Upsilon_P(a,b)\Upsilon_Q(a,b),$$

together with equation (5) and Proposition 1.1.

An alternative way to view the cd–index is the following. Let $e = a-b$. Clearly $\Psi_P(a,b)$ can be written uniquely as a polynomial in c and e, viz.,

$$\begin{aligned}\Psi_P(a,b) &= \Psi_P(\tfrac{c+e}{2}, \tfrac{c-e}{2}) \\ &= 2^{-n}\Psi_P(c+e, c-e).\end{aligned}$$

Now $e^2 = c^2-2d$, and it is easy to see that e^{2m+1} is not a polynomial in c and d for any integer $m \ge 0$. Hence for any graded poset P with $\hat{0}$ and $\hat{1}$, the cd–index exists if and only if the monomials $c^{r_1}e^{s_1}c^{r_2}e^{s_2}\cdots$ which appear in $\Psi_P(c+e,c-e)$ with nonzero coefficient satisfy $s_i \equiv 0 \pmod 2$ for all i. To compute $\Phi_P(c,d)$, simply substitute $(c^2-2d)^{s_i/2}$ for e^{s_i} in $2^{-n}\Psi_P(c+e,c-e)$. For instance, when $P = B_3$ we get

$$\begin{aligned}\Psi_P(c+e,c-e) &= (c+e)^2+2(c-e)(c+e)+2(c+e)(c-e)+(c-e)^2\\ &= 6c^2-2e^2\\ &= 6c^2-2(c^2-2d)\\ &= 4(c^2+d) = 4\Phi_P(c,d).\end{aligned}$$

We next give a recursive formula for $\Psi_P = \Psi_P(a,b)$ which shows immediately that $\Phi_P(c,d)$ exists for Eulerian posets. Our result and proof are stated in terms of the incidence algebra $I(P)$ over the ring $\mathbf{Q}\langle a,b\rangle$ of noncommutative rational polynomials in a and b. ($\mathbf{Q}$ denotes the field of rational numbers.) The value of a function $f \in I(P)$ at an interval $[x,y]$ is denoted by f_{xy}. See [S_2, Ch. 3] for background information on incidence algebras. Normally $I(P)$ is defined over a *commutative* ring, but the definition and basic properties of $I(P)$ make sense over noncommutative rings. In particular, the convolution fg is still given by

$$(fg)_{xy} = \sum_{x\le z\le y} f_{xz}g_{zy}.$$

Theorem 1.1 *Let P be Eulerian. Then*

$$\begin{aligned}2\Psi_P = 2\Psi_{\hat{0}\hat{1}} &= \sum_{\substack{\hat{0}<x<\hat{1}\\ \rho(x,\hat{1})=2j-1}} \Psi_{\hat{0}x}c(c^2-2d)^{j-1}\\ &\quad - \sum_{\substack{\hat{0}<x<\hat{1}\\ \rho(x,\hat{1})=2j}} \Psi_{\hat{0}x}(c^2-2d)^j + \begin{cases} 2(c^2-2d)^{k-1}, & \text{if } \rho(\hat{0},\hat{1}) = 2k-1\\ 0, & \text{if } \rho(\hat{0},\hat{1}) = 2k.\end{cases}\end{aligned}$$

Hence $\Phi_P(c,d)$ exists (by induction on the rank of P).

Proof Let $Q = \Sigma P$, the suspension of P. All our computations below take place in the incidence algebra $I(Q)$.

By considering the last element (excluding $\hat{1}$) $x = x_k$ of a chain $\hat{0} = x_0 < \cdots < x_k < \hat{1}$ in Q, it is clear that

$$\Upsilon_{\hat{0}\hat{1}} = \Upsilon_{\hat{0}\hat{1}}(a,b) = \sum_{x<\hat{1}} \Upsilon_{\hat{0}x}ba^{\rho(x,\hat{1})-1},$$

where we define $\Upsilon_{\hat{0}\hat{0}}b = 1$. Hence by (5)

$$\Psi_{\hat{0}\hat{1}} = \Psi_{\hat{0}\hat{1}}(a,b) = \sum_{x<\hat{1}} \Psi_{\hat{0}x}b(a-b)^{\rho(x,\hat{1})-1}, \tag{7}$$

where $\Psi_{\hat{0}\hat{0}}b = 1$. Multiply by $a-b$ on the right and add $\Psi_{\hat{0}\hat{1}}b$ to both sides to obtain

$$\Psi_{\hat{0}\hat{1}}a = \sum_{x} \Psi_{\hat{0}x}b(a-b)^{\rho(x,\hat{1})}. \tag{8}$$

If $f, g, h \in I(Q)$ are defined by

$$f_{xy} = \Psi_{xy}a,\ g_{xy} = \Psi_{xy}b,\ h_{xy} = (a-b)^{\rho(x,y)},$$

then (8) asserts (when applied to all intervals of Q) that $f = gh$ in $I(Q)$.

Since $\mu_{xy} = (-1)^{\rho(x,y)}$ (because Q is Eulerian), it follows easily that

$$h_{xy}^{-1} = (-1)^{\rho(x,y)}(a-b)^{\rho(x,y)}.$$

Then from $fh^{-1} = g$ we get

$$\Psi_{\hat{0}\hat{1}}b = \sum_x \Psi_{\hat{0}x}a(-1)^{\rho(x,\hat{1})}(a-b)^{\rho(x,\hat{1})}, \tag{9}$$

where $\Psi_{\hat{0}\hat{0}}a = 1$. Now add (8) and (9). We get

$$\Psi_{\hat{0}\hat{1}}c = \sum_x \Psi_{\hat{0}x}(a(-1)^{\rho(x,\hat{1})} + b)(a-b)^{\rho(x,\hat{1})}. \tag{10}$$

The term indexed by $x = \hat{1}$ on the right–hand side of (10) is just $\Psi_{\hat{0}\hat{1}}c$ and hence cancels out the left–hand side. The terms for $x = z_1$ and $x = z_2$ (the two elements of Q of rank $n+1$) are each given by $-\Psi_P(a-b)^2$. Hence

$$2\Psi_P = \sum_{\substack{x\in P\\ x\neq \hat{1}_P}} \Psi_{\hat{0}x}(a(-1)^{\rho_P(x,\hat{1})-1} + b)(a-b)^{\rho_P(x,\hat{1})-1}, \tag{11}$$

where ρ_P indicates the rank function of P, not Q.

If $\rho(x,\hat{1}) = 2j+1$ and $x \neq \hat{0}$ in (11), then

$$(a(-1)^{\rho(x,\hat{1})-1} + b)(a-b)^{\rho(x,\hat{1})-1} = c(c^2-2d)^j.$$

If $\rho(x,\hat{1}) = 2j$ and $x \neq \hat{0}$ in (11), then

$$(a(-1)^{\rho(x,\hat{1})-1} + b)(a-b)^{\rho(x,\hat{1})-1} = -(c^2-2d)^j.$$

If $\rho(\hat{0},\hat{1}) = 2k$, then since $\Psi_{\hat{0}\hat{0}}a = \Psi_{\hat{0}\hat{0}}b = 1$, we see that the term indexed by $x = \hat{0}$ in (11) in 0. If $\rho(\hat{0},\hat{1}) = 2k+1$ then this term is $2(a-b)^{2k} = 2(c^2-2d)^k$. Hence (11) is equivalent to the desired recurrence.

By iterating Theorem 1.1 we obtain an expression for $\Phi_P(c,d)$ (or $\Psi_P(a,b)$) in terms of the flag f–vector, as follows.

Corollary 1.1 *For $j \geq 1$ define*

$$\begin{aligned}\omega(2j-1) &= \tfrac{1}{2}c(c^2-2d)^{j-1}\\ \omega(2j) &= -\tfrac{1}{2}(c^2-2d)^j.\end{aligned}$$

If P is Eulerian of rank $n+1$ then

$$\Phi_P(c,d) = \sum_S (c^2-2d)^{\frac{1}{2}(a_1-1)}\omega(a_2-a_1)\omega(a_3-a_2)\cdots\omega(n+1-a_s)\alpha_P(S), \tag{12}$$

where S ranges over all subsets $\{a_1, a_2, \ldots, a_s\}$ of $[n]$ such that $a_1 < a_2 < \cdots < a_s$ and a_1 is odd. □

The expression (12) of $\Phi_P(c,d)$ in terms of the $\alpha_P(S)$'s is of course not unique, since the $\alpha_P(S)$'s satisfy certain linear relations (which are equivalent to the existence of $\Phi_P(c,d)$). We may regard (12) as a kind of canonical way to express Φ_P in terms of the flag f–vector. There may very well be other "canonical" formulas which are advantageous to (12).

Let $\mathcal{F}_n$ be the set of all flag f–vectors $\alpha_P : 2^{[n]} \to \mathbf{R}$, where P ranges over all Eulerian posets of rank $n+1$, and let $\mathbf{R}\mathcal{F}_n$ be the real vector space spanned by $\mathcal{F}_n$. Thus $\mathbf{R}\mathcal{F}_n$ is a subspace of the 2^n–dimensional space consisting of all functions $\alpha : 2^{[n]} \to \mathbf{R}$. Let $\mathcal{C}_n$ be the set of all words w of length n in the variables c and d (where $\ell(c)=1,\ \ell(d)=2$), and let $\mathbf{R}\mathcal{C}_n$ be the real vector space with basis $\mathcal{C}_n$. The dimension of $\mathbf{R}\mathcal{C}_n$ is the $(n+1)$–st Fibonacci number F_{n+1} ($F_1 = F_2 = 1,\ F_{n+1} = F_n + F_{n-1}$). Since the definition of Φ_P depends linearly on α_P, there is a linear transformation $\rho :\ \mathbf{R}\mathcal{F}_n \to \mathbf{R}\mathcal{C}_n$ satisfying $\rho(\alpha_P) = \Phi_P$. Clearly ρ is injective, since we can recover α_P from Φ_P, *via* (4) and (5). The work of Bayer and Billera [B–B] shows that ρ is also surjective, even if we restrict P to range over face lattices of polytopes (i.e., the flag f–vectors α_P, where P is the face lattice of an n–dimensional polytope, span $\mathbf{R}\mathcal{F}_n$). We wish to give a simpler proof than that of [B–B] that ρ is surjective (but we are proving a weaker result, since we allow P to be any Eulerian poset).

Proposition 1.2 *(a) The map $\rho :\ \mathbf{R}\mathcal{F}_n \to \mathbf{R}\mathcal{C}_n$ defined above is surjective. In other words, all homogeneous linear equalities satisfied by the components of flag f–vectors of Eulerian posets of rank $n+1$ are consequences of the existence of the cd–index.*

(b) The affine subspace of $\mathbf{R}\mathcal{C}_n$ spanned by $\rho(\mathcal{F}_n)$ is given by the condition that the coefficient of c^n is one. In other words, all linear equalities (homogeneous or inhomogeneous) satisfied by the components of flag f–vectors of Eulerian posets of rank $n+1$ are consequences of (a) and one additional equality, viz., $\alpha_P(\emptyset) = 1$.

Proof Let $w = w(c,d) \in \mathcal{C}_n$. Let B_2 denote the boolean algebra of rank two, so B_2 is Eulerian with $\Phi_{B_2}(c,d) = c$. For $m \geq 0$, let Q_m denote the face poset of an $(m+2)$-gon. Thus Q_m is Eulerian of rank 3, with $\Phi_{Q_m}(c,d) = c^2 + md$. If $w = w_1 w_2 \cdots w_r$ with $w_i = c$ or d, then define a poset $P_{w,m} = T_1 * T_2 * \cdots * T_r$, where

$$T_i = \begin{cases} B_2, & \text{if } w_i = c \\ Q_m, & \text{if } w_i = d. \end{cases}$$

By Lemma 1.1, we have

$$\Phi_{P_{w,m}}(c,d) = w(c, c^2 + md).$$

Let s denote the number of d's appearing in w, so $r + s = n$. Then, writing $[u]\Omega$ for the coefficient of u in a polynomial Ω, we have $[w]\Phi_{P_{w,m}} = m^s$, and if $w \neq c^n$ and $u \neq w$ then $[u]\Phi_{P_{w,m}} = m^t$, for some $t < s$. Thus for any $w \neq c^n$, we can make the coefficient of w in $\Phi_{P_{w,m}}$ arbitrarily large compared to the other coefficients. From this the proof is immediate.

For a stronger result using the posets $P_{w,m}$ see Theorem 2.1.

Since the *cd*–index Φ_P determines the flag f–vector α_P and the flag h–vector β_P, it is natural to ask for a formula expressing $\alpha_P(S)$ and $\beta_P(S)$ in terms of the coefficients of Φ_P. By Proposition 1.2, $\alpha_P(S)$ and $\beta_P(S)$ can be expressed uniquely as a linear combination of coefficients of Φ_P (valid for all Eulerian P of a fixed rank $n+1$). By (2), we might as well consider only β_P. Given $S \subseteq [n]$, let $\mathcal{W}_S$ be the set of all words $w = v_1 v_2 \cdots v_k$ in c and d of length n (with $\ell(c)=1,\ \ell(d)=2$) such that

$[u_S]w(a+b, ab+ba) \neq 0$ (where u_S is given by (3)), i.e., all words w such that u_S appears in the expansion of $w(a+b, ab+ba)$. Equivalently, if $v_j = d$ then exactly one of the integers $\ell(v_1 v_2 \cdots v_j) - 1$ and $\ell(v_1 v_2 \cdots v_j)$ is in S. For instance, if $n = 6$ and $S = \{1, 3, 5, 6\}$, then

$$\mathcal{W}_S = \{c^6, dc^4, cdc^3, c^2dc^2, c^3dc, d^2c^2, dcdc, cd^2c\}.$$

Proposition 1.3 *Let P be an Eulerian poset of rank $n+1$. Then for any $S \subseteq [n]$ we have*

$$\beta_P(S) = \sum_{w \in \mathcal{W}_S} [w]\Phi_P,$$

where $[w]\Phi_P$ denotes, as in the proof of Proposition 1.2, the coefficient of w in Φ_P.

Proof We have

$$\begin{aligned} \sum_{S \subseteq [n]} \beta_P(S) u_S &= \Phi_P(a+b,\ ab+ba) \\ &= \sum_w ([w]\Phi_P)\, w(a+b,\ ab+ba), \end{aligned} \tag{13}$$

where w ranges over all cd–words of length n.

By definition of $\mathcal{W}_S$ we have

$$w(a+b,\ ab+ba) = \sum_{\substack{S: \\ w \in \mathcal{W}_S}} u_S,$$

so the proof follows by equating coefficients of u_S in (13).

Proposition 1.3 shows that each $\beta_P(S)$ is a sum of certain coefficients (without multiplicities) of Φ_P. In particular, we can interpret the sum $\Phi_P(1,1)$ of *all* the coefficients of Φ_P.

Corollary 1.2 *Let $S_0 = \{1, 3, 5, \ldots\} \cap [n]$ or $S_0 = \{2, 4, 6, \ldots\} \cap [n]$. Then*

$$\Phi_P(1,1) = \beta_P(S_0).$$

Proof Note that $\mathcal{W}_S$ consists of *all* cd–words of length n if (and only if) $S = \{1, 3, 5, \ldots\} \cap [n]$ or $S = \{2, 4, 6, \ldots\} \cap [n]$. Now apply Proposition 1.3.

Corollary 1.2 suggests that if $\beta_P(S_0)$ has a combinatorial interpretation, then the coefficients of Φ_P might also have a combinatorial interpretation which refines that of $\beta_P(S_0)$. Similarly if $\beta_P(S_0)$ can be interpreted as the dimension of a certain vector space V, then there might be a "natural" decomposition $V = \coprod_w V_w$ (where $\coprod$ denotes direct sum and w ranges over all cd–words of length n) such that $\dim V_w = [w]\Phi_P$. Various examples of interpreting the coefficients of Φ_P combinatorially are known (due to Purtill [Pu]); see Sect. 3 for the case $P = B_{n+1}$. There is also a wide class of posets P (viz., Cohen–Macaulay posets) for which there is a natural vector space V with $\dim V = \beta_P(S_0)$ (see [B–G–S, Thm. 3.3 and §5]), but we have been unable to find any means of describing a decomposition $V = \coprod V_w$. That such a decomposition may exist is suggested by Conjecture 2.1.

Let us now consider Theorem 1.1 in the special case when P is the boolean algebra B_m. Foata and Schützenberger [F–Sc] defined combinatorially a certain noncommutative polynomial $U_m(c,d)$ in variables c and d (Foata and Schützenberger used s and t instead of c and d), called a (non–commutative) *André polynomial.* Purtill [Pu, Sect. 5] showed that

$$U_m(c,d) = \Phi_{B_m}(c,d).$$

In particular, the combinatorial definition of $U_m(c,d)$ shows that $\Phi_{B_m}(c,d) \geq 0$. We will discuss $U_m(c,d)$ in more detail in Sect. 3. For now we will be content with obtaining a recurrence relation and generating function for $U_m(c,d)$.

Corollary 1.3 *The cd–index $U_m = U_m(c,d) = \Psi_{B_m}(a,b)$ of the boolean algebra B_m satisfies the recurrence*

$$2U_m = \sum_{\substack{0<i<m \\ m-i=2j-1}} \binom{m}{i} U_i c(c^2-2d)^{j-1} - \sum_{\substack{0<i<m \\ m-i=2j}} \binom{m}{i} U_i (c^2-2d)^j + \begin{cases} 2(c^2-2d)^{k-1}, & m = 2k-1 \\ 0, & m = 2k. \end{cases}$$

Proof Immediate from Theorem 1.1, since B_m has $\binom{m}{i}$ elements x of rank i, all of which satisfy $[\hat{0}, x] \cong B_i$.

From Corollary 1.3 we can obtain a generating function for U_m. Define formal power series, whose coefficients are (noncommutative) polynomials in c and d (but whose variable x commutes with everything) as follows:

$$\begin{aligned} \frac{\sinh(a-b)x}{a-b} &= \sum_{j\geq 0} (a-b)^{2j} \frac{x^{2j+1}}{(2j+1)!} \\ &= \sum_{j\geq 0} (c^2-2d)^j \frac{x^{2j+1}}{(2j+1)!} \\ \cosh(a-b)x &= \sum_{j\geq 0} (a-b)^{2j} \frac{x^{2j}}{(2j)!} \\ &= \sum_{j\geq 0} (c^2-2d)^j \frac{x^{2j}}{(2j)!}. \end{aligned}$$

Moreover, if $F(x)$ is any formal power series with noncommutative coefficients such that $F(0) = 0$, then $(1-F(x))^{-1}$ denotes the unique series $G(x)$ satisfying $(1 - F(x))G(x) = G(x)(1-F(x)) = 1$. We also have the formula

$$G(x) = 1 + F(x) + F(x)^2 + \cdots.$$

Corollary 1.4 *With $U_m(c,d)$ as above, we have*

$$\begin{aligned} \sum_{m\geq 1} U_m(c,d)\frac{x^m}{m!} &= \frac{2\sinh(a-b)x}{a-b}\left[1 - \frac{c\cdot\sinh(a-b)x}{a-b} + \cosh(a-b)x\right]^{-1} \\ &= \frac{\sinh(a-b)x}{a-b}\left[1 - \frac{1}{2}\left(\frac{c\cdot\sinh(a-b)x}{a-b} - \cosh(a-b)x + 1\right)\right]^{-1}. \end{aligned} \tag{14}$$

First proof. Multiply the recurrence of Corollary 1.3 by $x^m/m!$ and sum on $m \geq 1$. It is straightforward to obtain the desired generating function. □
Second proof. From (7) there follows (writing Ψ_m for Ψ_{B_m})

$$\Psi_m = \sum_{j=1}^{m-1} \binom{m}{j} \Psi_j b(a-b)^{m-i-1} + (a-b)^{m-1}.$$

Multiplying by $x^m/m!$ and summing on $m \geq 1$ leads to

$$\sum_{m \geq 1} \Psi_m \frac{x^m}{m!} = \frac{e^{(a-b)x}-1}{a-b} \left[1 - b\frac{e^{(a-b)x}-1}{a-b} \right]^{-1}, \qquad (15)$$

where by definition,

$$\frac{e^{(a-b)x}-1}{a-b} = \sum_{j \geq 1} (a-b)^{j-1} \frac{x^j}{j!}.$$

Although (15) is a generating function for $\Psi_m(a,b) = U_m(c,d)$ which looks simpler than (14), it is not as satisfactory since it is unclear how to express the right–hand side of (15) in terms of c and d. The individual factors are not in fact functions of c and d, unlike (14), which only involves *even* powers of $a-b$. However, it is not difficult to show by a direct argument that the right–hand sides of (14) and (15) agree, thereby giving another proof of (14). □

There is a recurrence known for U_m different from Corollary 1.3 (see [F–Sc, Prop. 3.10][Pu, Cor. 6.6]. This other recurrence has the advantage of making it clear that $U_m \geq 0$. On the other hand, it seems difficult to use this recurrence to obtain a generating function for U_m as we have done in Corollary 1.4.

As a final consequence of Theorem 1.1, let us note that it becomes much simpler modulo 2 (though curiously it does not yield a recurrence relation for $\Psi_{\hat{0}\hat{1}}$ modulo 2).

Corollary 1.5 *Let P be Eulerian. Then modulo 2 we have*

$$\sum_{\hat{0}<x<\hat{1}} \Psi_{\hat{0}x} c^{\rho(x,\hat{1})} = 0. \quad \square$$

2 Nonnegativity of the cd–index

Fine [B–K, Conj. 3] conjectured that the cd–index of a convex polytope (i.e., of its face lattice) is nonnegative. Purtill [Pu] proved Fine's conjecture for certain classes of polytopes including simplicial polytopes (and hence also simple polytopes, since if P^* is the dual of P then Φ_{P^*} is obtained from Φ_P by reversing all words [Ba, end of §3]). Bayer and Klapper [B–K, Conj. 5] gave a generalization of Fine's conjecture, viz., the cd–index of a regular CW–sphere is nonnegative. We wish to state an even more general conjecture than that of Bayer and Klapper. Let us call a poset P with $\hat{0}$ and $\hat{1}$ a *Gorenstein* poset* (over a fixed field K) if the order complex $\Delta(P - \{\hat{0},\hat{1}\})$ of $P - \{\hat{0},\hat{1}\}$ (i.e., the set of chains of $P - \{\hat{0},\hat{1}\}$, regarded as a simplicial complex) is nonacyclic and Gorenstein, as defined in [S_1, Ch. II, §5]. By [S_1, Ch. II, Thm. 5.1], it follows that P is Gorenstein* if and only if P is Cohen–Macaulay (over K) and Eulerian.

Conjecture 2.1 If P is Gorenstein* then $\Phi_P(c,d) \geq 0$, i.e., the coefficients of $\Phi_P(c,d)$ are nonnegative. □

Later in this section we will prove a special case of Conjecture 2.1 which includes Fine's conjecture (but only a special case of the conjecture of Bayer and Klapper). First we state a converse to Conjecture 2.1.

Theorem 2.1 *Conjecture 2.1, if true, gives all possible linear inequalities satisfied by flag f–vectors of Gorenstein* posets. More precisely, let $\mathcal{C}_n$ be as in Sect. 1. Suppose that there are real numbers k_v, for $v \in \mathcal{C}_n$, such that*

$$\sum_{v \in \mathcal{C}_n} k_v [v] \Phi_P \geq 0 \tag{16}$$

for all Gorenstein posets P of rank $n+1$, where $[v]\Phi_P$ denotes the coefficient of v in Φ_P. Then $k_v \geq 0$ for all $v \in \mathcal{C}_n$.*

Proof Suppose (16) is valid for all Gorenstein* posets P of rank $n+1$ and that $k_w < 0$ for some $w = w(c,d) = w_1 w_2 \cdots w_r \in \mathcal{W}_n$, where $w_i = c$ or d. Let $P_{w,m}$ be the poset constructed in the proof of Proposition 1.2. If $w \neq c^n$, then for m sufficiently large the left–hand side of (16) is dominated by the term indexed by $v = w$ and hence is negative since $k_w < 0$, a contradiction.

There remains only the case $w = c^n$. But $\Phi_{P_{c^n,m}} = c^n$ (for any m), so the left–hand side of (16) is just k_{c^n}. Hence $k_{c^n} \geq 0$, as desired.

Note 2.1 Equation (16) is a *homogeneous* linear inequality, and one may wonder whether there are *inhomogeneous* linear inequalities not implied by (16). But since $[c^n]\Phi_P = 1$, any inhomogeneous inequality can be converted to a homogeneous one, so we gain nothing new by considering the inhomogeneous case.

Our next goal is to prove Conjecture 2.1 for a wide class of Gorenstein* posets P, viz., face posets (with $\hat{1}$ adjoined) of shellable regular CW–spheres. (All terms will be discussed below. In particular, our notion of shellability is slightly different from the standard one but still includes face lattices of convex polytopes, confirming the conjecture of Fine mentioned earlier.)

We begin with a lemma concerning the *cd*–index of subdivisions of regular Eulerian CW–complexes. Following Björner [Bj], define a (finite) *CW–poset* to be a finite poset P with $\hat{0}$, such that for all $x > \hat{0}$ in P, the geometric realization $|(\hat{0}, x)|$ of the open interval $(\hat{0}, x)$ is homeomorphic to a sphere. By [Bj], a CW–poset is the same as the face poset $P(\Omega)$ of a regular CW–complex Ω. We will be concerned with the case when $P_1(\Omega)$ is Eulerian of rank $n+1$, where $P_1(\Omega) = P(\Omega) \cup \{\hat{1}\}$ (the face poset of Ω with a $\hat{1}$ adjoined). Thus $P_1(\Omega)$ has a *cd*–index $\Phi_{P_1(\Omega)} := \Phi_\Omega$.

Let σ be a facet (n–cell) of Ω. (All cells τ of Ω are taken to be *open*. The closure of τ is denoted $\bar{\tau}$, so $\partial\tau = \bar{\tau} - \tau$.) Let Ω' be obtained from Ω by subdividing $\bar{\sigma}$ into a regular CW-complex with two facets σ_1 and σ_2, such that $\partial\sigma$ is unchanged and $\overline{\sigma_1} \cap \overline{\sigma_2}$ is a regular $(n-1)$–dimensional CW–ball Γ. Thus $\partial\sigma \cap \Gamma = \partial\Gamma$. Let Γ' be the regular CW–complex obtained from Γ by adjoining a single new facet ($(n-1)$–cell) τ attached to $\partial\Gamma$, so $\partial\tau = \partial\Gamma$.

We have that $|\Omega| \approx |\Omega'|$ so $P_1(\Omega')$ is Eulerian of rank $n+1$ and has a *cd*–index $\Phi_{\Omega'}$. If Λ is a regular CW–complex and $|\Lambda|$ is a sphere or ball, then by slight abuse of terminology we also say that Λ is a sphere or ball. Thus Γ' is an $(n-1)$–sphere and $\partial\Gamma$ is an $(n-2)$–sphere, so $P_1(\Gamma')$ and $P_1(\partial\Gamma)$ have *cd*–indices $\Phi_{\Gamma'}$ and $\Phi_{\partial\Gamma}$. In terms of the face posets $P = P_1(\Gamma)$ and $P' = P_1(\Gamma')$, we have that P is near–Eulerian and that P' is the semisuspension $\tilde{\Sigma}P$. We also call Γ' the *semisuspension* of Γ and write $\Gamma' = \tilde{\Sigma}\Gamma$. In a similar fashion, if Λ is a regular CW–sphere, then define the *suspension* $\Sigma\Lambda$ to be the regular CW–sphere obtained from Λ by adjoining two facets σ_1 and σ_2 attached to all of Λ. Thus $P_1(\Sigma\Lambda) = \Sigma P_1(\Lambda)$.

With Ω and Ω' as above, define

$$\check{\Phi}(c, d) = \Phi_{\Omega'}(c, d) - \Phi_{\Omega}(c, d),$$

the change in the cd–index when we subdivide Ω to get Ω'. The next lemma expresses $\check{\Phi}$ in terms of $\Phi_{\Gamma'}$ and $\Phi_{\partial\Gamma}$.

Lemma 2.1 *With notation as above, we have*

$$\check{\Phi} \quad = \quad \Phi_{\Gamma'}c - \Phi_{\partial\Gamma}(c^2 - d). \tag{17}$$

Proof The new flags of faces obtained in adjoining τ to Γ to get Γ' are just flags in $P(\partial\Gamma)$ with τ adjoined at the top. Hence

$$\Upsilon_{\Gamma'} \quad = \quad \Upsilon_{\Gamma} + \Upsilon_{\partial\Gamma}b. \tag{18}$$

Now consider the flags in Ω' which are not flags in Ω. There are two kinds: (i) Flags containing a face in $\Gamma - \partial\Gamma$. Since either σ_1 or σ_2 (or neither) can be at the top of such a flag, these new flags contribute $\Upsilon_{\Gamma}(a+2b) - \Upsilon_{\partial\Gamma}a(a+2b)$ to $\Upsilon_{\Omega'} - \Upsilon_{\Omega}$. (ii) Flags containing σ_1 or σ_2, but no face of $\Gamma - \partial\Gamma$. Let $x_1 < x_2 < \cdots < x_i < \sigma_j$ be such a flag φ. If $x_i \notin \partial\Gamma$ then φ simply replaces the flag $x_1 < x_2 < \cdots < x_i < \sigma$ of $P(\Omega)$ and hence makes no contribution to $\Upsilon_{\Omega'} - \Upsilon_{\Omega}$. If however $x_i \in \partial\Gamma$, then both $x_1 < x_2 < \cdots < x_i < \sigma_1$ and $x_1 < x_2 < \cdots < x_i < \sigma_2$ are flags in $P(\Omega')$ which replace the flag $x_1 < x_2 < \cdots < x_i < \sigma$ of $P(\Omega)$. Hence the total contribution of such flags to $\Upsilon_{\Omega'} - \Upsilon_{\Omega}$ is $\Upsilon_{\partial\Gamma}ab$.

It follows that

$$\Upsilon_{\Omega'} - \Upsilon_{\Omega} = (\Upsilon_{\Gamma} - \Upsilon_{\partial\Gamma}a)(a + 2b) + \Upsilon_{\partial\Gamma}ab.$$

Substituting $\Upsilon_{\Gamma'} - \Upsilon_{\partial\Gamma}b$ for Υ_{Γ} (which we can do by (18)) yields

$$\Upsilon_{\Omega'} - \Upsilon_{\Omega} = \Upsilon_{\Gamma'}(a + 2b) - \Upsilon_{\partial\Gamma}(a^2 + ab + ba + 2b^2).$$

Substituting $a - b$ for a yields by (5)

$$\begin{aligned} \Psi_{\Omega'} - \Psi_{\Omega} &= \Psi_{\Gamma'}(a + b) - \Psi_{\partial\Gamma}(a^2 + b^2) \\ &= \Psi_{\Gamma'}c - \Psi_{\partial\Gamma}(c^2 - d), \end{aligned}$$

and the proof follows.

Corollary 2.1 *With notation as above, we have*

$$[wdc]\check{\Phi} \quad = \quad [wd]\Phi_{\Gamma'} \tag{19}$$

$$[wd]\check{\Phi} \quad = \quad [w]\Phi_{\partial\Gamma} \tag{20}$$

$$[wc^2]\check{\Phi} \quad = \quad [wc](\Phi_{\Gamma'} - \Phi_{\Sigma(\partial\Gamma)}), \tag{21}$$

where $\Sigma(\partial\Gamma)$ denotes the suspension of $\partial\Gamma$.

Proof The first two formulas are immediate from (17). Also immediate is the formula

$$[wc^2]\check{\Phi} = [wc]\Phi_{\Gamma'} - [w]\Phi_{\partial\Gamma}.$$

Now by Lemma 1.1 we have $\Phi_{\Sigma(\partial\Gamma)} = \Phi_{\partial\Gamma}c$, and the proof follows.

We now give the inductive definition of shellability which we need here. Our definition will be given only for regular CW–complexes Ω which are Eulerian (i.e., $P_1(\Omega)$ is Eulerian), since we need all complexes under consideration to have a *cd*–index.

Definition 2.1 *Let Ω be an Eulerian regular CW–complex of dimension n. We say that Ω or $P_1(\Omega)$ is* S–shellable *(short for "spherically shellable") if either $\Omega = \{\phi\}$ (so $P_1(\Omega)$ is a two–element chain with cd–index 1), or else we can linearly order the facets (open n–cells) of Ω, say $\sigma_1, \sigma_2, \ldots, \sigma_r$, such that for all $1 \leq i \leq r$ the following conditions hold (where both $^-$ and cl denote closure).*

(a) $\partial\overline{\sigma_1}$ is S–shellable (of dimension $n-1$),

(b) For $2 \leq i \leq r-1$, let

$$\Gamma_i \quad = \quad cl[\partial\overline{\sigma_i} - ((\overline{\sigma_1} \cup \cdots \cup \overline{\sigma_{i-1}}) \cap \overline{\sigma_i})]. \tag{22}$$

Thus Γ_i is the subcomplex of $\partial\overline{\sigma_i}$ generated by all $(n-1)$–cells of $\partial\overline{\sigma_i}$ which are not contained in the complex $\overline{\sigma_1} \cup \cdots \cup \overline{\sigma_{i-1}}$ generated by the previous cells. Then we require that Γ_i is near–Eulerian (i.e., $P_1(\Gamma_i)$ is near–Eulerian) of dimension $n-1$, and that the semisuspension $\tilde{\Sigma}\Gamma_i$ is S–shellable, with the first facet of the shelling being the facet $\tau = \tau_i$ adjoined to Γ_i to obtain $\tilde{\Sigma}\Gamma_i$.

Note the following consequence of the previous definition. The complex Γ_i is near–Eulerian for $2 \leq i \leq r-1$ and thus has an $(n-2)$–dimensional Eulerian boundary $\partial\Gamma_i$ (where by definition $\partial\Gamma_i$ is generated by all $(n-2)$–cells contained in the closure of exactly one $(n-1)$–cell of Γ_i). Moreover, $\partial\Gamma_i = \partial\tau_i$. Since by (a) and (b) we have that $\partial\tau_i$ is S–shellable, there follows:

$$\partial\Gamma_i \text{ is } S\text{–shellable for } 2 \leq i \leq r-1. \tag{23}$$

The essential difference between S–shellability and the usual definition [Bj, Def. 4.1] of shellability (which we call here *C–shellability*) of a regular CW–complex is the following: In C–shellability, it is required that $\partial\overline{\sigma_i}$ is C–shellable, with the shelling beginning with the facets of $\partial\overline{\sigma_i}$ contained in $\overline{\sigma_1} \cup \cdots \cup \overline{\sigma_{i-1}}$. For S–shellability, we are in essence "merging" all the facets of $\partial\overline{\sigma_i}$ contained in $\overline{\sigma_1} \cup \cdots \cup \overline{\sigma_{i-1}}$ into a single facet τ_i before commencing with the shelling of $\partial\overline{\sigma_i}$ (beginning with τ_i). One can give an example of an S–shelling which is not a C–shelling, based on the existence of nonshellable balls whose boundaries are shellable spheres (e.g., [Ne][R]). Similarly we can find a C–shelling which is not an S–shelling, based on the existence of shellable balls whose boundaries are nonshellable spheres [Pa]. However, a Bruggesser–Mani line shelling of the boundary complex of a convex polytope [B–M] is an S–shelling (as well as a C–shelling), because in such a shelling the complex $\partial\Gamma_i$ is itself polytopal. Thus Theorem 2.2 applies to this situation. Let us also mention that it is easy to see (analogous to [D–K, p. 35, (1)]) that an S–shellable Eulerian regular CW–complex Ω is in fact a sphere (and that the subcomplexes Γ_i of equation (22) are balls), so from now on we might as well assume that Ω is a sphere.

Theorem 2.2 *Let Ω be an S–shellable regular CW–sphere of dimension n, so that the augmented face poset $P_1(\Omega)$ is Eulerian. Then*

$$\Phi_\Omega(c,d) := \Phi_{P_1(\Omega)}(c,d) \geq 0.$$

Proof Let $\sigma_1, \sigma_2, \ldots, \sigma_r$ be an S–shelling of Ω. For $1 \leq i \leq r-1$, let $\Omega_i = \tilde{\Sigma}(\overline{\sigma_1} \cup \cdots \cup \overline{\sigma_i})$. Thus $\Omega_{r-1} \cong \Omega$. We prove by induction on $n = \dim \Omega$ and on i that $\check{\Phi}_i := \Phi_{\Omega_i} - \Phi_{\Omega_{i-1}} \geq 0$, whence $\Phi_{\Omega_i} \geq 0$ so in particular $\Phi_{\Omega_{r-1}} = \Phi_\Omega \geq 0$. When $n = -1$ we have $\Omega = \{\phi\}$ and $\Phi_\Omega = 1$. Hence assume $n \geq 0$. By Definition 2.1(a), we have that $\partial\overline{\sigma_1}$ is S–shellable of dimension $n-1$, so by induction $\Phi_{\partial\overline{\sigma_1}} \geq 0$. Now $\Omega_1 \cong \Sigma(\partial\overline{\sigma_1})$, so by Lemma 1.1 we have $\Phi_{\Omega_1} = \Phi_{\partial\overline{\sigma_1}} c \geq 0$. This establishes the base $i = 1$ of the induction.

Now assume that $i > 1$ and that $\Phi_{\Omega_{i-1}} \geq 0$. Let Γ_i be as in equation (22). Let σ be the facet of Ω_{i-1} which was adjoined to $\overline{\sigma_1} \cup \cdots \cup \overline{\sigma_{i-1}}$ to obtain Ω_{i-1}. Thus Ω_i is obtained from Ω_{i-1} by subdividing σ into two facets σ_1 and σ_2, with $\overline{\sigma_1} \cap \overline{\sigma_2} = \Gamma_i$. We are precisely in the situation of Lemma 2.1, so setting $\check{\Phi} = \Phi_{\Omega_i} - \Phi_{\Omega_{i-1}}$, $\Gamma = \Gamma_i$ and $\Gamma' = \Gamma_i' = \tilde{\Sigma}\Gamma_i$, we know that (19), (20), and (21) hold. By Definition 2.1(b), we have that Γ' is S–shellable of dimension $n-1$, so by induction and (19) we have $[wdc]\check{\Phi} \geq 0$. Similarly equation (23) implies that $\partial\Gamma$ is S–shellable of dimension $n-2$, so by induction and (20) we have $[wd]\check{\Phi} \geq 0$.

Now consider equation (21). By Definition 2.1 there is an S–shelling $\zeta_1, \zeta_2, \ldots$ of Γ' such that the first facet ζ_1 has boundary $\partial\zeta_1 = \partial\Gamma$. Thus $\tilde{\Sigma}\overline{\zeta_1} = \Sigma(\partial\Gamma)$, so by induction $\Phi_{\Gamma'} - \Phi_{\Sigma(\partial\Gamma)} \geq 0$. Hence by (21) we have $[wc^2]\check{\Phi} \geq 0$, and the proof is complete.

Since as mentioned above boundary complexes of convex polytopes are S–shellable, we obtain the conjecture of Fine mentioned in the introduction:

Corollary 2.2 *Let P be the face lattice of a convex polytope. Then $\Phi_P \geq 0$.* □

Let us note that since the posets $P_{w,m}$ used to prove Theorem 2.1 are face posets of S–shellable Eulerian regular CW–complexes, there follows from Theorem 2.2 the following corollary.

Corollary 2.3 *The inequalities $\Phi_\Omega \geq 0$ yield all linear inequalities satisfied by flag f–vectors of face posets of S–shellable Eulerian regular CW–complexes Ω.* □

As an interesting special case of Conjecture 2.1 and Theorem 2.2, we can ask for which pairs (S,T) of subsets of $[n]$ do we have $\beta_P(S) \geq \beta_P(T)$ for all Gorenstein* posets P of rank $n+1$. For this end, given $S \subseteq [n]$ define $\omega(S) \subseteq [n-1]$ by the condition $i \in \omega(S)$ if and only if exactly one of i and $i+1$ belongs to S. For instance if $n = 8$ and $S = \{2,4,5,8\}$, then $\omega(S) = \{1,2,3,5,7\}$. Thus by definition of $\mathcal{W}_S$, we see that a cd–word $w = v_1v_2\cdots v_j \in \mathcal{C}_n$ satisfies $w \in \mathcal{W}_S$ if and only if $\ell(v_1v_2\cdots v_i) + 1 \in \omega(S)$ whenever $v_{i+1} = d$. There follows:

Conjecture 2.2 The following two conditions on subsets S, T of $[n]$ are equivalent:
(i) For every Gorenstein* poset P of rank $n+1$, we have $\beta_P(S) \geq \beta_P(T)$.
(ii) $\omega(S) \supseteq \omega(T)$. □

Theorem 2.3 *(a) Conjecture 2.2 follows from Conjecture 2.1 (and Theorem 2.1).*
(b) Conjecture 2.2 is true if the phrase "Gorenstein poset P" in (i) is replaced with "face poset P of an S–shellable Eulerian regular CW–complex."* □

In particular, for a fixed Gorenstein* poset P, we have that $\beta_P(S)$ is conjecturally maximized for $S = \{1,3,5,\ldots\}$ and $S = \{2,4,6,\ldots\}$. This conjecture is true for face posets of S–shellable Eulerian regular CW–complexes.

Let us consider Theorem 2.3(b) in the case when P is the boolean algebra B_m. It is well-known [S2, Cor. 3.12.2] that $\beta_{B_m}(S)$ is the number of permutations $\pi = \pi_1\pi_2\cdots\pi_m$ of $1,2,\ldots,m$ whose descent set $D(\pi) := \{1 \le i \le m-1 : \pi_i > \pi_{i+1}\}$ is equal to S. The inequalities on $\beta_{B_m}(S)$ given by Theorem 2.3(b) are equivalent to a result of Niven [Ni], later given a simpler proof by de Bruijn [dB]. Of course the fact that $\Phi_{B_m} \ge 0$ imposes further inequalities on the $\Phi_{B_m}(S)$'s, though not of the form $\beta(S) \ge \beta(T)$. As mentioned in Sect. 1, the inequality $\Phi_{B_m} \ge 0$ follows from the theory of André polynomials and does not require Theorem 2.2.

An interesting application of Theorem 2.2 appears in [C–D, §7], where it is used to prove a special case of an intriguing conjecture (Conjecture D) concerning the h-vector of certain triangulated spheres.

3 Eulerian simplicial posets

In this section we obtain a formula for the cd–index of a (very) special class of posets. A finite poset P is *simplicial* if P has a $\hat{0}$ and every interval $[\hat{0},x]$ is a boolean algebra. Thus a simplicial meet–semilattice is just the face poset of a (finite) simplicial complex. By slight abuse of terminology, we say that an Eulerian poset P is *simplicial* if $P - \{\hat{1}\}$ is a simplicial poset. Let f_{i-1} denote the number of elements x in a simplicial poset P for which $[\hat{0},x] \cong B_i$. In particular, $f_{-1} = 1$. Let $n-1 = \max\{i : f_{i-1} \ne 0\}$. Thus for an Eulerian simplicial poset P, we have rank$(P) = n+1$. The vector $f(P) = (f_0,\ldots,f_{n-1})$ is called the *f–vector* of P. Let $S = \{a_1,a_2,\ldots,a_s\} \subseteq \{1,2,\ldots,n\}$, with $a_1 < a_2 < \cdots < a_s = j$. Since f_{j-1} elements x of P satisfy $[\hat{0},x] \cong B_j$, there follows

$$\begin{aligned}\alpha_P(S) &= f_{j-1}\alpha_{B_j}(S-\{j\}) \\ &= f_{j-1}\binom{j}{a_1,a_2-a_1,\ldots,j-a_{s-1}}.\end{aligned} \tag{24}$$

Hence $\alpha_P(S)$ (and thus also $\beta_P(S)$) is completely determined by $f(P)$ and S.

If P is a simplicial poset with f–vector $f(P) = (f_0,\ldots,f_{d-1})$, then write

$$\sum_{i=0}^{n-1} f_i(x-1)^{n-i} = \sum_{i=0}^{n} h_i x^{n-i}.$$

The vector $h(P) = (h_0,h_1,\ldots,h_n)$ is called the *h–vector* of P and is often easier to work with than the f–vector. In particular, we have the following two results.

Proposition 3.1 (S3, Thm. 3.10) *Let P be a Cohen–Macaulay simplicial poset. Then $h_i \ge 0$ for all i.* □

Proposition 3.2 *(equivalent to [S2, Ch. 3, (40)]). Let P be an Eulerian simplicial poset of rank $n+1$. Then $h_i = h_{n-i}$ for all i.* □

Thus for Eulerian simplicial posets, the flag f–vector (and hence the cd–index) is determined entirely by the numbers $h_0,h_1,\ldots,h_{\lfloor n/2\rfloor}$. Hence the space spanned by flag f–vectors of Eulerian simplicial posets of rank $n+1$ has dimension at most (in fact, exactly) $1 + \lfloor n/2 \rfloor$. This is much smaller than the dimension F_{n+1} of the space spanned by flag f–vectors of all Eulerian posets of rank $n+1$ (see Proposition 1.2 and the discussion preceding it). Thus simplicial Eulerian posets are an extremely special (though nonetheless interesting) class of Eulerian posets. We will give an explicit formula for the cd–index of a simplicial Eulerian poset P which, together with Proposition 3.1, will make it obvious that $\Phi_P \geq 0$ when P is Cohen–Macaulay, thereby proving Conjecture 2.1 in the simplicial case.

Let Λ^n denote the boundary complex of an n–dimensional simplex. Thus $P_1(\Lambda^n) \cong B_{n+1}$, and Φ_{Λ^n} is the polynomial U_{n+1} of Corollary 1.3. Any ordering of the facets of Λ^n is an S–shelling, and all such shellings are equivalent *via* an automorphism of Λ^n. Fix an S–shelling $\sigma_0, \sigma_1, \ldots, \sigma_n$. Let $\Lambda_i^n = \tilde{\Sigma}(\overline{\sigma_0} \cup \cdots \cup \overline{\sigma_i})$, so $\Lambda_{n-1}^n \cong \Lambda^n$. Define $\check{\Phi}_i = \check{\Phi}_i^n = \Phi_{\Lambda_i^n} - \Phi_{\Lambda_{i-1}^n}$ (with $\check{\Phi}_0 = \Phi_{\Lambda_0^n}$). Hence

$$\check{\Phi}_0 + \check{\Phi}_1 + \cdots + \check{\Phi}_{n-1} = U_{n+1},$$

and by the proof of Theorem 2.2 we have

$$\check{\Phi}_i \geq 0, \qquad 0 \leq i \leq n-1. \tag{25}$$

Theorem 3.1 *Let P be a simplicial Eulerian poset of rank $n+1$, with h–vector $h(P) = (h_0, h_1, \ldots, h_n)$. Then*

$$\Phi_P = \sum_{i=0}^{n-1} h_i \check{\Phi}_i^n. \tag{26}$$

Proof If the simplicial poset Q is the face poset of a regular CW–complex Γ, then we write

$$h(Q) = h(\Gamma) = (h_0(\Gamma), h_1(\Gamma), \ldots, h_n(\Gamma)).$$

Suppose first that P is the face poset of an S-shellable Eulerian regular CW-complex, say with shelling $\sigma_1, \ldots, \sigma_r$. If the simplex σ_j intersects $\bar{\sigma}_1 \cup \cdots \cup \bar{\sigma}_{j-1}$ in a union of i faces of σ_j, then

$$h_i(\tilde{\Sigma}(\bar{\sigma}_1 \cup \cdots \cup \bar{\sigma}_j)) - h_i(\tilde{\Sigma}(\bar{\sigma}_1 \cup \cdots \cup \bar{\sigma}_{j-1})) = \begin{cases} \delta_{ij}, & j > 0, \\ \delta_{i0} + \delta_{in}, & j = 0, \end{cases}$$

while

$$\Phi(\tilde{\Sigma}(\bar{\sigma}_1 \cup \cdots \cup \bar{\sigma}_j)) - \Phi(\tilde{\Sigma}(\bar{\sigma}_1 \cup \cdots \cup \bar{\sigma}_{j-1})) = \check{\Phi}_j^n,$$

where δ_{ij} denotes the Kronecker delta. Hence (26) is true for such P. It is easily seen that we can find such P whose h-vectors span the linear span of all h-vectors of simplicial Eulerian posets of rank $n+1$. Since $h_0(P) = h_n(P) = 1$ and since Φ_P depends linearly on $h(P)$ (by (24)), the proof follows.

Corollary 3.1 *Let P be a Gorenstein* simplicial poset. Then $\Phi_P \geq 0$.*

Proof Combine Proposition 3.1, equation (25), and Theorem 3.1.

Corollary 3.1 establishes Conjecture 2.1 when P is simplicial. A special case of Corollary 3.1 was proved earlier by M. Purtill [Pu, Cor. 8.4].

We conclude with a conjecture (which we suspect will not be too hard to prove) concerning a combinatorial interpretation of the polynomials $\check{\Phi}_i^n$. We assume familiarity with the theory of *André permutations* [F–Sc][F–St][Pu]. Let $\mathcal{A}_n^I$ denote the set of augmented André permutations in the symmetric group $\mathcal{S}_n$. Let $\mathcal{A}_n^{II}$ denote the set of André permutations of the second kind in $\mathcal{S}_n$. Finally we consider a class of permutations related to those in $\mathcal{A}_n^{II}$ suggested by S. Sundaram. An *augmented Sundaram permutation* (or *André permutation of the third kind*) in $\mathcal{S}_n$ is a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ such that (a) $\pi_n = n$, and (b) if for any $j \geq 0$ the elements $n-1, n-2, \ldots, n-j$ are removed from the word $\pi = \pi_1\pi_2\cdots\pi_n$, then the resulting word $\rho_1\rho_2\cdots\rho_{n-j}$ has no *double descents*, i.e., no $\rho_{i-1} > \rho_i > \rho_{i+1}$. (In particular, the case $j = 0$ says that w itself has no double descents.) Let $\mathcal{A}_n^{III}$ denote the set of all Sundaram permutations in $\mathcal{S}_n$. For instance, $\mathcal{A}_4^{III} = \{1234, 1324, 2134, 2314, 3124\}$.

Given a permutation $\pi \in \mathcal{S}_n$ with no double descents, let $U_\pi = U_\pi(c, d)$ denote the *reduced variation* of π. (U_π is a noncommutative monomial in c and d of length $n-1$, with $\ell(c) = 1$, $\ell(d) = 2$.) Using the theory of André permutations developed by Foata and Schützenberger [F–Sc], Purtill [Pu, Thm. 7.1] gave the following combinatorial interpretation of $U_n = \Phi_{B_n}$:

$$U_n = \sum_{\pi \in \mathcal{A}_{n-1}^X} U_\pi, \qquad (27)$$

where $X = I$ or II. By similar reasoning one can establish (27) for $X = III$. Conjecture 3.1 below gives a refinement of this result. Define

$$\begin{aligned} \mathcal{A}_{n,i}^X &= \{\pi \in \mathcal{A}_n^X : \pi_{n-1} = n - 1 - i\}, \text{ if } X = I \text{ or } III \\ \mathcal{A}_{n,i}^{II} &= \{\pi \in \mathcal{A}_n^{II} : \pi_n = n - i\}. \end{aligned}$$

Conjecture 3.1 We have

$$\check{\Phi}_i^n = \sum_{\pi \in \mathcal{A}_{n+1,i}^X} U_\pi,$$

where $X = I,\ II$, or III. □

Some values of the polynomials $\check{\Phi}_i^n$ are:

$$
\begin{aligned}
n = 1: \quad & \check{\Phi}_0 = c \\
n = 2: \quad & \check{\Phi}_0 = c^2 \\
& \check{\Phi}_1 = d \\
n = 3: \quad & \check{\Phi}_0 = c^3 + dc \\
& \check{\Phi}_1 = dc + cd \\
& \check{\Phi}_2 = cd \\
n = 4: \quad & \check{\Phi}_0 = c^4 + 2dc^2 + 2cdc \\
& \check{\Phi}_1 = dc^2 + 2cdc + c^2 d + d^2 \\
& \check{\Phi}_2 = cdc + c^2 d + 2d^2 \\
& \check{\Phi}_3 = c^2 d + d^2. \\
n = 5: \quad & \check{\Phi}_0 = c^5 + 3dc^3 + 5cdc^2 + 3c^2 dc + 4d^2 c \\
& \check{\Phi}_1 = dc^3 + 3cdc^2 + 3c^2 dc + c^3 d + 4d^2 c + 2dcd + 2cd^2 \\
& \check{\Phi}_2 = cdc^2 + 2c^2 dc + c^3 d + 3d^2 c + 3dcd + 4cd^2 \\
& \check{\Phi}_3 = c^2 dc + c^3 d + d^2 c + 3dcd + 4cd^2 \\
& \check{\Phi}_4 = c^3 d + 2dcd + 2cd^2.
\end{aligned}
$$

Acknowledgement. I am grateful to the referee for many helpful suggestions.

References

[Ba] Bayer, M.M.: The Dehn–Sommerville equations revisted. Preprint.

[Bj] Björner, A.: Posets, regular CW complexes and Bruhat order. European J. Combinatorics **5**, 7–16 (1984).

[B–B] Bayer, M.M., Billera, L.J.: Generalized Dehn–Sommerville relations for polytopes, spheres and Eulerian partially ordered sets. Invent. math. **79**, 143–157 (1985).

[B–G–S] Björner, A., Garsia, A., Stanley, R.: An introduction to the theory of Cohen–Macaulay partially ordered sets. In: I. Rival: Ordered sets (pp. 583–615) Dordrecht Boston London: Reidel 1982,

[B–K] Bayer, M.M., Klapper, A.: A new index for polytopes. Discrete Comput. Geom. **6**, 33–47 (1991).

[B–M] Bruggesser, H., Mani, P.: Shellable decompositions of cells and spheres. Math. Scand. **29**, 197–205 (1971).

[C–D] Charney, R., Davis, M.: The Euler characteristic of a nonpositively curved, piecewise linear Euclidean manifold. Preprint.

[dB] de Bruijn, N.G.: Permutations with given ups and downs. Nieuw Arch. Wisk. (3) **18**, 61–65 (1970).

[D–K] Danaraj, G., Klee, V.: Which spheres are shellable?. Ann. Discrete Math. **2**, 33–52 (1978).

[F–Sc] Foata, D., Schützenberger, M.P.: Nombres d'Euler et permutations alternantes. In: J.N. Srivastava et al.: A Survey of Combinatorial Theory (pp. 173–187), Amsterdam: North-Holland 1973.

[F–St] Foata, D. Strehl, V.: Euler numbers and variations of permutations. In: Colloquio Internazionale sulle Teorie Combinatorie 1973, Tome I (Atti Dei Convegni Lincei 17, pp. 119–131), Academia Nazionale dei Lincei 1976.

[Ne] Newman, M.H.A.: A property of 2–dimensional elements. Proc. Akad. Wet. **29**, 1401–1405 (1926).

[Ni] Niven, I.: A combinatorial problem of finite sequences. Nieuw Arch. Wisk. (3) **16**, 116–123 (1968).

[Pa] Pachner, U.: Konstruktionsmethoden und das kombinatorische Homöomorphieproblem für Triangulationen kompakter semilinearer Mannigfaltigkeiten. Abh. Math. Sem. Hamb. **57**, 69–85 (1987).

[Pu] Purtill, M.: André permutations, lexicographic shellability and the cd-index of a convex polytope, Trans. Amer. Math. Soc. **338**, 77–104 (1993).

[R] Rudin, M.E.: An unshellable triangulation of a tetrahedron. Bull. Amer. Math. Soc. **64**, 90–91 (1958).

[S_1] Stanley, R.: Combinatorics and Commutative Algebra (Progress in Mathematics, vol. 41) Boston Basel Stuttgart: Birkhäuser 1983.

[S_2] Stanley, R.: Enumerative Combinatorics, vol. 1. Pacific Grove, Wadsworth and Brooks/Cole 1986.

[S_3] Stanley, R.: f–vectors and h–vectors of simplicial posets. J. Pure Appl. Alg. **71**, 319–331 (1991).

This article was processed by using the LaTeX style file *pljour1* from Springer-Verlag.