Sandpile Groups of Cubes

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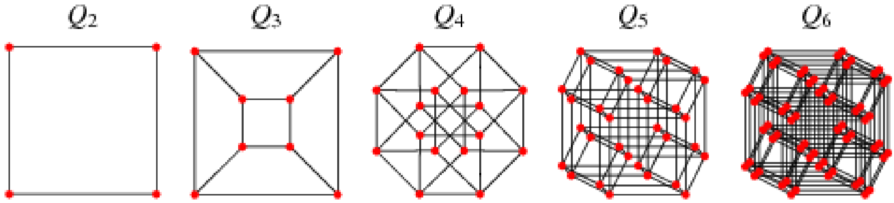
- Analogous Bounds on Other Cayley Graphs
- Higher Critical Groups

Definitions

Definition

The **n-cube** is the graph Q_n with $V(Q_n) = (\mathbf{Z}/2\mathbf{Z})^n$ and an edge between two vertices $v_1, v_2 \in V(Q_n)$ if v_1 and v_2 differ in precisely one place.



Definitions

Definition

The **Laplacian** of a graph G, denoted $L(G)$, is the matrix

$$L(G)_{i,j} = \begin{cases} \deg(v_i) & \text{if } i = j \\ -\#\{\text{edges from } v_i \text{ to } v_j\} & \text{if } i \neq j \end{cases}$$

Example

$$L(Q_1) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad L(Q_2) = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

A Final Definition

Definition

Let G be a graph. Since $L(G)$ is an integer matrix, we may consider it as a $\mathbf{Z}$-linear map $L(G) : \mathbf{Z}^{\#V(G)} \to \mathbf{Z}^{\#V(G)}$. The torsion part of the cokernel of this map is the **critical group** (or **sandpile group**) of G, denoted $K(G)$.

Previous Results I

Theorem [Bai]

For every prime $p > 2$,

$$\mathrm{Syl}_p\left(K(Q_n)\right) \cong \mathrm{Syl}_p\left(\prod_{k=1}^{n} (\mathbf{Z}/k\mathbf{Z})^{\binom{n}{k}}\right).$$

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$$\mathrm{Syl}_p\left(K(Q_n)\right) \cong \mathrm{Syl}_p\left(\prod_{k=1}^{n} (\mathbf{Z}/k\mathbf{Z})^{\binom{n}{k}}\right).$$

Remark

To understand $K(Q_n)$, it then remains to understand $\mathrm{Syl}_2(K(Q_n))$.

Previous Results II

Lemma [Benkart, Klivans, Reiner]

For every $u \in (\mathbf{Z}/2\mathbf{Z})^n$, let $\chi_u \in \mathbf{Z}^{2^n}$ be the vector with entry in position $v \in (\mathbf{Z}/2\mathbf{Z})^n$ equal to $(-1)^{u \cdot v}$. Then χ_u is an eigenvector of $L(Q_n)$ with eigenvalue $2 \cdot \mathrm{wt}(u)$, where $\mathrm{wt}(u)$ is the number of non-zero entries in u.

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Remark

Thus, we understand $L(Q_n)$ entirely as a map $\mathbf{Q}^{2^n} \to \mathbf{Q}^{2^n}$. When considering it as a map $\mathbf{Z}^{2^n} \to \mathbf{Z}^{2^n}$, this leaves us with the task of understanding the $\mathbf{Z}$-torsion.

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Theorem [Benkart, Klivans, Reiner]

There is an isomorphism of $\mathbf{Z}$-modules

$$\mathbf{Z} \oplus K(Q_n) \cong \mathbf{Z}[x_1, \ldots, x_n]/(x_1^2 - 1, \ldots, x_n^2 - 1, n - \sum x_i).$$

Gröbner Basis Background

Definition

Let $R = T[x_1, \ldots, x_n]$, where T is a commutative Noetherian ring. A **monomial order** on R is a total order $<$ on the set of monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ of R. From now on, we implicitly assume a monomial order $<$ on R.

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Notation

Let $I \subseteq [n]$. We write $x_I := \prod_{i \in I} x_i$.

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Definition

Let $f \in R$. Then the **leading term** of f, denoted $\ell\mathrm{t}(f)$, is the term of f greatest with respect to $<$.

Gröbner Basis Background

Definition

Let $I \triangleleft R$ be an ideal. Then the **leading term ideal** of I is

$$\mathrm{LT}(I) = (\{\ell\mathrm{t}(f) \mid f \in I\}).$$

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Definition

Let $I \lhd R$ an ideal. A **Gröbner basis** of I is a generating set $S = \{g_1, \ldots, g_k\}$ of I satisfying either of the following two properties:

- For every $f \in I$, we can write $\ell\mathrm{t}(f) = c_1\,\ell\mathrm{t}(g_1) + \cdots + c_k\,\ell\mathrm{t}(g_k)$ for some $c_i \in R$.
- $LT(I) = (\ell\mathrm{t}(g_1), \ldots, \ell\mathrm{t}(g_k))$.

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- $LT(I) = (\ell\mathrm{t}(g_1), \ldots, \ell\mathrm{t}(g_k))$.

Theorem

When T is a PID, every ideal $I \lhd R$ has a Gröbner basis.

Relevance to Our Situation

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Let $I \triangleleft R$ be an ideal. Then, as T-modules,

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Remark

By the isomorphism mentioned previously, to understand $K(Q_n)$ it suffices to understand a Gröbner basis for the ideal

$$I_n := (x_1^2 - 1, \ldots, x_n^2 - 1, n - \sum x_i)$$

in $\mathbf{Z}[x_1, \ldots, x_n]$.

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in $\mathbf{Z}[x_1, \ldots, x_n]$. However, the Gröbner basis is very complicated.

Relevance to Our Situation

Lemma

Let J_n denote the ideal $(x_1^2 - 1, \ldots, x_n^2 - 1, n - \sum x_i)$ in $\mathbf{Z}/2^i\mathbf{Z}[x_1, \ldots, x_n]$. Then the factors of $\mathbf{Z}/2\mathbf{Z}, \ldots, \mathbf{Z}/2^{i-1}\mathbf{Z}$ in $\mathbf{Z}[x_1, \ldots, x_n]/I_n$ and $\mathbf{Z}/2^i\mathbf{Z}[x_1, \ldots, x_n]/J_n$ are the same.

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Goal

Understand a Gröbner basis of J_n for $i = 2$, and thus understand the number of $\mathbf{Z}/2\mathbf{Z}$-factors in $\mathrm{Syl}_2\, K(Q_n)$.

The Case $i = 2$

Conjecture

For every odd integer m, let

$$W_m = \{(2 + \epsilon_2, 4 + \epsilon_4, \ldots, m - 3 + \epsilon_{m-3}, m - 1, m) \mid \epsilon_i \in \{0, 1\}\}.$$

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Then

$$LT(J_n) = (x_1) + (x_2^2, \ldots, x_n^2) + \sum_{\substack{m \leq n \\ m \text{ odd}}} \sum_{I \in W_m} (2x_I).$$

An Observation

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The highest cyclic factor has size equal to the highest additive order of an element in

$$K(Q_n) \cong \mathbf{Z}[x_1, x_2, \ldots, x_n]/(x_1^2 - 1, \ldots, x_n^2 - 1, n - x_1 - x_2 - \ldots - x_n)$$

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Lemma

The elements $x_i - 1$ have highest additive order in $K(Q_n)$ for all $i \in \{1, \ldots, n\}$.

An Observation

Proof Outline

- Show a polynomial has a multiple in I_n only if it has the form

$$f(x_1, \ldots, x_n) = \sum_{I \subseteq [n]} c_I (x_I - 1)$$

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Proof Outline

- Show a polynomial has a multiple in I_n only if it has the form

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- Show $x_I - 1$ has a multiple in I_n for every $I \subseteq [n]$.
- Show $\mathrm{ord}(x_i - 1) \geq \mathrm{ord}(x_I - 1)$ for any $I \subseteq [n]$.

The Order of $x_1 - 1$

We switch back to $\mathbf{Q}^{2^n}$:

$$x_1 - 1 \sim \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The Order of $x_1 - 1$

We want to find the smallest C such that $\exists \mathbf{v} \in \mathbf{Z}^{2^n}$ satisfying

$$L(Q_n) \cdot \mathbf{v} = \begin{pmatrix} -C \\ C \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

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Idea: Work in the χ_u-basis!

The Order of $x_1 - 1$

Both terms are nice:

$$L(Q_n) \sim \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & 2 & & & & \vdots \\ \vdots & & 2 & & & \vdots \\ \vdots & & & 4 & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 2n \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ \frac{1}{2^{n-1}} \\ 0 \\ \frac{1}{2^{n-1}} \\ \vdots \\ \frac{1}{2^{n-1}} \end{pmatrix}$$

The Order of $x_1 - 1$

We now have the χ_u-coordinates of **v**:

$$\mathbf{v} \sim \begin{pmatrix} 0 & \frac{1}{2^n} & 0 & \frac{1}{2^{n+1}} & \cdots & \frac{1}{n2^n} \end{pmatrix}^T$$

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The order of $x_1 - 1$ is $\leq 2^n \cdot \mathrm{LCM}(1, 2, \dots, n)$

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Corollary

The size of the largest cyclic factor in $\mathrm{Syl}_2(K(Q_n))$ is $\leq 2^{n+\log_2 n}$

Other Cayley Graphs

Goal

Generalize the technique used for the cube graph to other Cayley graphs.

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Key Theorem [Benkart, Klivans, Reiner]

Let G be the n-th power of a directed cycle of size k. Then

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Generalize $x_1 - 1$

Lemma

As before, $x_i - 1$ has maximal order in $K(G)$ for all $i \in \{1, \ldots, n\}$.

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Remark

However, $x_i - 1$ does not have a nice form in the χ_u-basis. So we must find another high-order term with a nice form. One such element is $(k-1) - x_i - x_i^2 - \cdots - x_i^{k-1}$.

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Lemma

$k \cdot \text{ord}\left((k-1) - x_i - x_i^2 - \cdots - x_i^{k-1}\right) = \text{ord}(x_i - 1).$

Form in χ_u-basis

The form for $(k-1) - x_i - x_i^2 - \cdots - x_i^{k-1}$ in the χ_u-basis is as follows:

$$\begin{pmatrix} k-1 \\ -1 \\ -1 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ \frac{1}{k^n} \\ \vdots \\ \frac{1}{k^n} \\ 0 \\ \frac{1}{k^n} \\ \vdots \\ \frac{1}{k^n} \\ \vdots \end{pmatrix}.$$

Bounds for $k = 3, 4$

Theorem ($k = 3$)

Let $k = 3$. Then the size of the largest cyclic factor of $\mathrm{Syl}_3(K(G))$ is $\leq 3^{n+1+\lfloor \log_3(n) \rfloor}$.

Theorem ($k = 4$)

Let $k = 4$. Then the size of the largest cyclic factor of $\mathrm{Syl}_2(K(G))$ is $\leq 4^{n+1+\lfloor \log_4(n) \rfloor}$.

A Different Viewpoint

Set $C_1(G)$, $C_0(G)$ to be formal groups of $\mathbf{Z}$-linear combinations of the edges and vertices of G respectively.

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There is a chain complex

$$0 \to C_1(G) \xrightarrow{E} C_0(G) \xrightarrow{\epsilon} \mathbf{Z} \to 0$$

where E is the **incidence matrix** of G and $\epsilon(\sum n_i v_i) = \sum n_i$ is the **augmentation map**.

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Lemma

$L(G) = EE^T$ and $K(G) = \ker(\epsilon)/\mathrm{Im}(L(G)) = \ker(\epsilon)/\mathrm{Im}(EE^T)$

Extension to Cell Complexes

Fix a cell complex X. There is a cellular chain complex

$$\ldots \to C_i(X) \xrightarrow{\partial_i} C_{i-1}(X) \to \ldots \to C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbf{Z} \to 0$$

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The ***i*-th critical group** of X is $K_i(X) = \ker(\partial_i)/\mathrm{Im}(\partial_{i+1}\partial_{i+1}^T)$

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Related to cellular spannng trees, higher-dimensional dynamical systems on X.

Initial Results

We have an extension of Bai's Theorem:

Theorem

For any prime $p > 2$,

$$\mathrm{Syl}_p\left(K_i(Q_n)\right) \simeq \mathrm{Syl}_p\left(\bigoplus_{j=i+1}^{n} (\mathbf{Z}/j\mathbf{Z})^{\binom{n}{j}\binom{j-1}{i}}\right)$$

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- Can show $\partial_{i+1}\partial_{i+1}^T + \partial_i^T\partial_i = L(Q_{n-i})^{\oplus\binom{n}{i}}$.

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- Can show $\partial_{i+1}\partial_{i+1}^T + \partial_i^T\partial_i = L(Q_{n-i})^{\oplus\binom{n}{i}}$.
- $\partial_{i+1}\partial_{i+1}^T$ and $\partial_i^T\partial_i$ are diagonalizable and commute, so they have the same eigenvectors.

Further Directions

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- A lower bound on the top cyclic factor: Examine minors of $L(Q_n)$?
- Top cyclic factor bounds on $K_{s_1} \times K_{s_2} \times \ldots \times K_{s_n}$.
- Extend the top cyclic factor bound to higher critical groups.

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Questions?