Toric Mutations in the dP2 Quiver

Yibo Gao, Zhaoqi Li, Thuy-Duong Vuong, Lisa Yang

July 29, 2016

Overview

1. Introduction and Preliminaries
   - Quiver and cluster mutation
   - The Del Pezzo 2 Quiver (dP2) and its brane tiling
   - Toric mutations
   - Two models of the dP2 quiver

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   - Adjacency between different models
   - ρ-mutations
3. Explicit Formula for Cluster Variables

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4. Subgraph of the Brane Tiling
   - Weighting Scheme and Covering Monomial

Overview

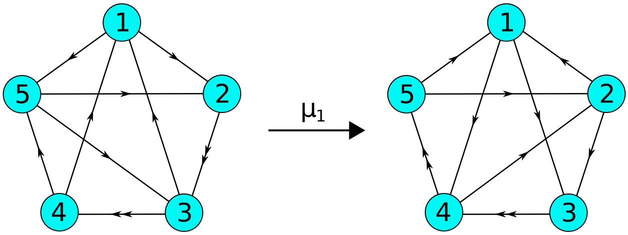

Figure: Example of quiver mutation

Binomial Exchange Relation $$x_1' = \frac{x_2x_5 + x_3x_4}{x_1}.$$

The Del Pezzo 2 Quiver (dP2) and its Brane Tiling

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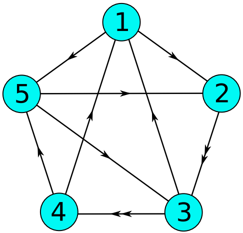
The second Del Pezzo Surface (dP2) is first introduced in the physics literature.



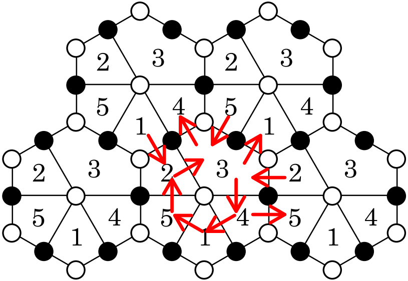


Figure: dP2 quiver and its corresponding brane tiling [HS12]

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A *toric mutation* is a cluster mutation at a vertex with in-degree 2 and out-degree 2.

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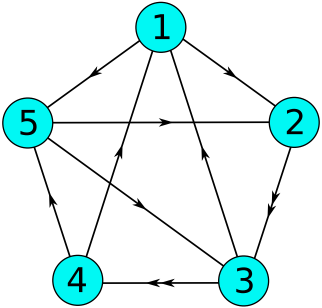

Two Models of the dP2 Quiver

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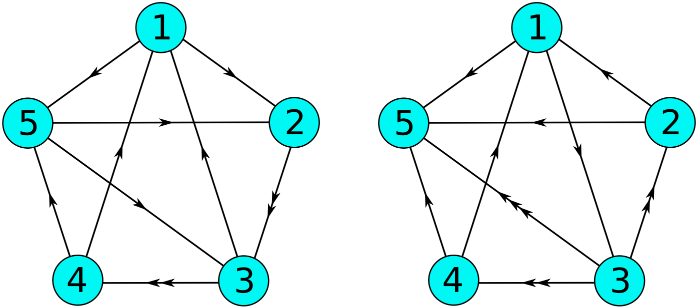

Figure: Model 1 (left) and Model 2 (right) of the dP2 quiver [HS12]

Classification of Toric Mutation Sequences

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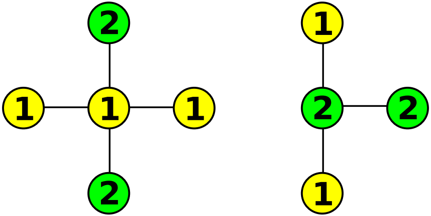

Figure: Adjacency between different models

$\rho-$mutation sequence

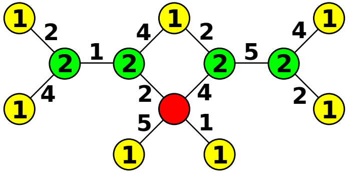

Figure: All possible toric mutation sequences that start from model 1 and return to model 1 the first time.

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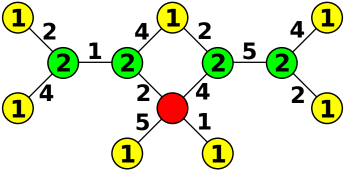

Figure: All possible toric mutation sequences that start from model 1 and return to model 1 the first time.

Definition (ρ-mutations)

$$\rho_1 = \mu_1 \circ (54321), \quad \rho_2 = \mu_5 \circ (12345), \quad \rho_3 = \mu_2 \circ \mu_4 \circ (24),$$

$$\rho_4 = \mu_2 \circ \mu_1 \circ \mu_4 \circ (531), \quad \rho_5 = \mu_4 \circ \mu_5 \circ \mu_2 \circ (351),$$

$$\rho_6 = \mu_2 \circ \mu_1 \circ \mu_2 \circ (531)(24), \quad \rho_7 = \mu_4 \circ \mu_5 \circ \mu_4 \circ (135)(24).$$

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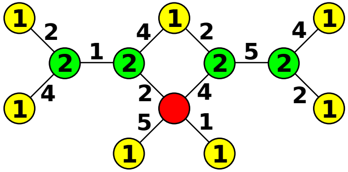

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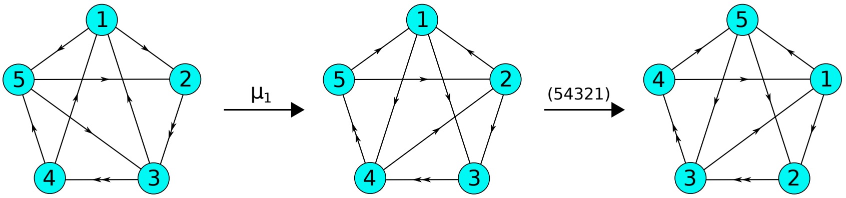
A $\rho-$mutation sequence is a sequence of $\rho-$mutations.

An example: $\rho_1 = \mu_1 \circ (54321)$.



$$(x_1, x_2, x_3, x_4, x_5) \longrightarrow \left(\frac{x_2x_5 + x_3x_4}{x_1} = x_6, x_2, x_3, x_4, x_5\right) \longrightarrow (x_2, x_3, x_4, x_5, x_6)$$

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Proposition (Relations between $\rho-$mutations)

$$\rho_4 = \rho_1^2\rho_3, \quad \rho_5 = \rho_2^2\rho_3, \quad \rho_6 = \rho_1^2, \quad \rho_7 = \rho_2^2.$$

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Proposition (Relations between ρ_1, ρ_2, ρ_3)

$$\rho_1\rho_2 = \rho_2\rho_1 = \rho_3^2 = 1.$$

$$\rho_1^2\rho_3 = \rho_3\rho_1^2, \quad \rho_2^2\rho_3 = \rho_3\rho_2^2, \quad \rho_1\rho_3\rho_2 = \rho_2\rho_3\rho_1.$$

$\rho-$mutation sequence: a visualization

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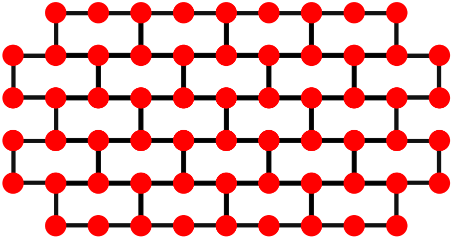

Figure: A visualization for $\rho-$mutation sequence.

$$\rho_1 :\rightarrow, \quad \rho_2 :\leftarrow, \quad \rho_3 :\uparrow / \downarrow .$$

$\rho-$mutation sequence

Theorem

Every toric mutation sequence that starts at Q (the original dP2 quiver) and ends in model 1 can be written as either

$$\rho_1^k(\rho_3\rho_1)^m \quad \text{or} \quad \rho_1^k(\rho_3\rho_1)^m\rho_3,$$

where $k \in \mathbb{Z}$, $m \in \mathbb{Z}_{\geq 0}$ and $\rho_1^{-1} = \rho_2$.

Explicit Formula for Cluster Variables

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Definition (Laurent Polynomial for Somos-5 Sequence)

Let x_1, x_2, x_3, x_4, x_5 be our initial variables. Define x_n for each $n \in \mathbb{Z}$ by

$$x_n x_{n-5} = x_{n-1} x_{n-4} + x_{n-2} x_{n-3}.$$

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Notice that $\{x_n\}_{n \geq 1}$ is the somos-5 sequence if $x_1 = x_2 = x_3 = x_4 = x_5 = 1$.

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Definition (Some Constants)

$$A := \frac{x_1x_5 + x_3^2}{x_2x_4}, \qquad B := \frac{x_2x_6 + x_4^2}{x_3x_5}\left(= \frac{x_1x_4^2 + x_2x_3x_4 + x_2^2x_5}{x_1x_3x_5} \right).$$

Explicit Formula for Cluster Variables

Theorem

Define $g(s,k) := \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s+1}{2} \right\rfloor$ *if* k *is even and* $g(s,k) := \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{s}{2} \right\rfloor$ *if* k *is odd. Then we have, for* $k \in \mathbb{Z}$ *and* $s \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \rho_1^k(\rho_3\rho_1)^s\{x_1, x_2, x_3, x_4, x_5\} = \{&A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+1}, \\ &A^{g(s,k)}B^{g(s,k+1)}x_{k+s+2}, \\ &A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+3}, \\ &A^{g(s,k)}B^{g(s,k+1)}x_{k+s+4}, \\ &A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+5}\}. \end{aligned}$$

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Corollary

All cluster variables generated by toric mutations can be written as either

$$A^{n^2}B^{n(n-1)}x_{2m} \quad \text{or} \quad A^{n(n-1)}B^{n^2}x_{2m-1} \quad \text{for some } m, n \in \mathbb{Z}.$$

Subgraph of the Brane Tiling

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Definition (Weighting Scheme)

Associate a weight $w(e) := \frac{1}{x_i x_j}$ to each edge bordering blocks labeled i and j. Let $\mathcal{M}(G)$ be the collection of perfect matchings of G. For each $M \in \mathcal{M}(G)$, define its weight $w(M) = \prod_{e \in M} w(e)$.
Define the weight of the graph G as

$$w(G) := \sum_{M \in \mathcal{M}(G)} w(M).$$

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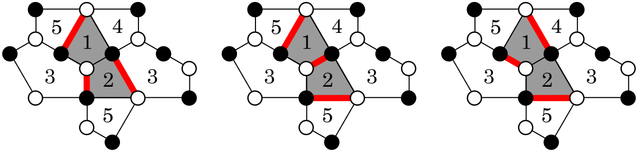

Figure: $w(G) = \frac{1}{x_1 x_5 x_2^2 x_3^2} + \frac{1}{x_1^2 x_5^2 x_2^2} + \frac{1}{x_1^2 x_2 x_3 x_4 x_5}$

Subgraph of the Brane Tiling

Definition (Covering Monomial)

Given a subgraph G, let a_j be the number of blocks labeled j in G. Let b_j be the number of blocks labeled j adjacent to G. Let c_3 be the number of blocks labeled 3 adjacent to G with 4 edges inside G. The covering monomial $m(G)$ is the product $x_1^{a_1+b_1}x_2^{a_2+b_2}x_3^{2a_3+b_3+c_3}x_4^{a_4+b_4}x_5^{a_5+b_5}$.

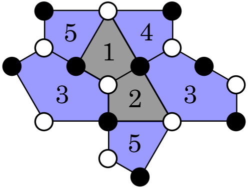

Figure: Example of Covering Monomial: $x_1x_2x_3^2x_4x_5^2$

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For any graph G, denote the product of its weight and its cover monomial as

$$c(G) := w(G)m(G).$$

Contour: Fundamental Shape

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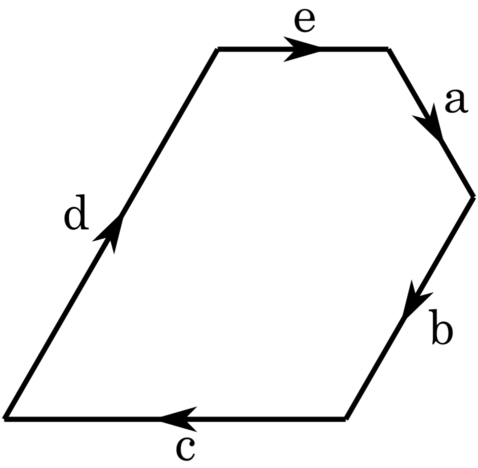


Figure: 5-sided fundamental shape.

Contour: Length

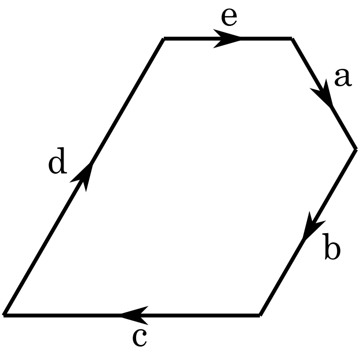
Similar to [LM15], we define the length of our contour.

Definition (Length of Contour)

$$\forall i \in \{a, b, c, d, e\},$$

$$\text{len}(i) = \begin{cases} |i|, & \text{if same direction as the associated side} \\ -|i|, & \text{otherwise} \end{cases}$$

Contour: Length



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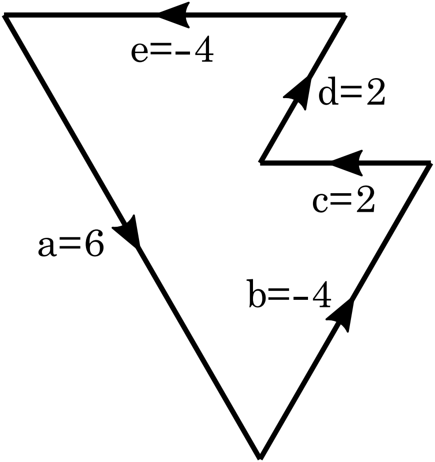
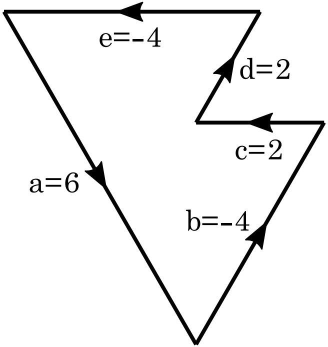


Figure: Length of Contour.

From Contour to Subgraph

Definition (Rules to Get Subgraph)

- positive length $\rightarrow$ keep black points; negative length $\rightarrow$ keep white points.
- $b \equiv d \pmod 2$, keep **special** point; $b \not\equiv d \pmod 2$, remove **special** point.



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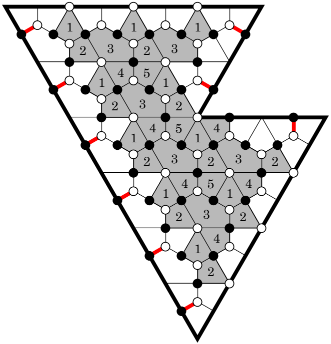

Figure: Example of Subgraph.

Main Result

Theorem (Formula of Contours)

Define the contours as follows:

$$A^{n^2}B^{n^2-n}x_{2k} = \left(k-2+n, -\left\lceil \frac{k-4+5n}{2} \right\rceil, 2n-1, \left\lfloor \frac{k-3n}{2} \right\rfloor, 1+n-k\right)$$
$$A^{n^2+n}B^{n^2}x_{2k-1} = \left(k-2+n, -\left\lceil \frac{k-2+5n}{2} \right\rceil, 2n, \left\lfloor \frac{k-2-3n}{2} \right\rfloor, 2+n-k\right)$$

For any such cluster variable, if G is the subgraph of its corresponding contour, then $c(G)$ is the Laurent polynomial of the cluster variable.

Kuo's Condensation Theorems

Kuo's Condensation Theorems [Kuo04] tell us how to write the weight of a large graph in terms of smaller ones.

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Let $G = (V_1, V_2, E)$ be a weighted planar bipartite graph.
Let p_1, p_2, p_3, p_4 be four vertices in a cyclic order on the boundary of G.

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Let $G = (V_1, V_2, E)$ be a weighted planar bipartite graph.
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Theorem (Balanced Kuo Condensation)

Assume $|V_1| = |V_2|$, $p_1, p_3 \in V_1$ *and* $p_2, p_4 \in V_2$. *Then*

$$\begin{aligned} w(G)w(G - \{p_1, p_2, p_3, p_4\}) =& w(G - \{p_1, p_2\})w(G - \{p_3, p_4\}) \\ &+ w(G - \{p_1, p_4\})w(G - \{p_2, p_3\}). \end{aligned}$$

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Theorem (Unbalanced Kuo Condensation)

Assume $|V_1| = |V_2| + 1$, $p_1, p_2, p_3 \in V_1$ and $p_4 \in V_2$. Then

$$\begin{aligned} w(G - \{p_2\})w(G - \{p_1, p_3, p_4\}) =& w(G - \{p_1\})w(G - \{p_2, p_3, p_4\}) \\ &+ w(G - \{p_3\})w(G - \{p_1, p_2, p_4\}). \end{aligned}$$

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Let $G = (V_1, V_2, E)$ be a weighted planar bipartite graph.
Let p_1, p_2, p_3, p_4 be four vertices in a cyclic order on the boundary of G.

Theorem (Non-alternating Kuo Condensation)

Assume $|V_1| = |V_2|$, $p_1, p_2 \in V_1$ and $p_3, p_4 \in V_2$. Then

$$\begin{aligned} w(G - \{p_1, p_4\})w(G - \{p_2, p_3\}) =& w(G)w(G - \{p_1, p_2, p_3, p_4\}) \\ &+ w(G - \{p_1, p_3\})w(G - \{p_2, p_4\}). \end{aligned}$$

Proof Sketch

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We use induction on n.
Base case: $x_k \rightarrow$ Somos-5 Sequence.
Inductive Step:

$$\begin{aligned}(A^{(n+1)^2}B^{n^2+n}x_{2k})(A^{n^2}B^{n^2-n}x_{2k+2}) =& (A^{n^2+n}B^{n^2}x_{2k+3})(A^{n^2+n}B^{n^2}x_{2k-1}) \\ &+ (A^{n^2+n}B^{n^2}x_{2k+1})^2\end{aligned}$$

$$\begin{aligned}w(G-\{p_1,p_2\})w(G-\{p_3,p_4\}) =& w(G)w(G-\{p_1,p_2,p_3,p_4\}) \\ &+ w(G-\{p_1,p_3\})w(G-\{p_2,p_4\}).\end{aligned}$$

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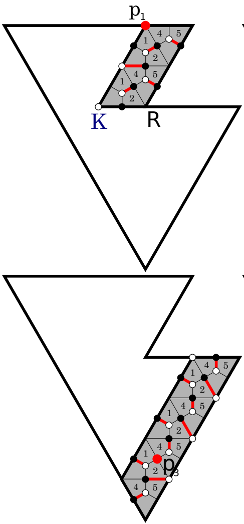

Figure: Position of p_1 through p_4 when $(a, b, c, d, e) = (+, -, +, +, -) - R$

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Thank you!!