Bijections Between Catalan Objects

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Outline

Preliminary Definitions

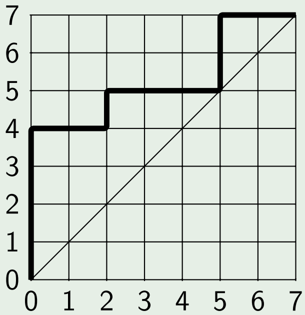
Dyck Paths

Definition

A **Dyck path** in $\mathcal{D}_{2n}$ is a lattice path from $(0,0)$ to (n,n), using only up (U) and right (R) steps, and staying weakly above the main diagonal.

Example ($d \in \mathcal{D}_{14}$)

UUUURRURRRUURR



Dyck Paths

Definition

The **area** of a Dyck path is the number of full boxes between the Dyck path and the main diagonal.

Example ($d \in \mathcal{D}_{14}$)

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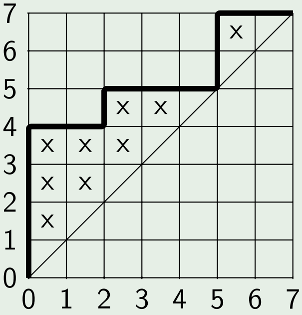


Area $= 9$

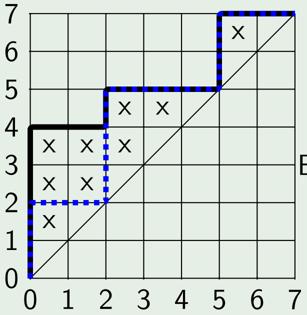
Dyck Paths

Definition

To obtain the **bounce path** d', start at (n, n) and travel left until hitting d, then travel down to the main diagonal. Repeat. The **bounce** of d is the sum of the intermediate x-coordinates where d' touches the main diagonal.

Example ($d \in \mathcal{D}_{14}$)

UUUURRURRRUURR



Area = 9

Bounce = 7

(q,t)-Catalan numbers

Definition ($C_n(q,t)$)

$$C_n(q,t) := \sum_{d \in \mathcal{D}_{2n}} q^{area(d)} t^{bounce(d)},$$

Theorem (Garsia, Haglund, et al)

$$C_n(q,t) = C_n(t,q)$$

This implies that there is a bijection $\mathcal{D}_{2n} \to \mathcal{D}_{2n}$ that exchanges area and bounce. It is an open problem to describe this bijection explicitly.

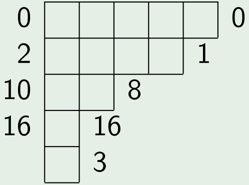
Our main goal in this project was to define area and bounce on Catalan objects other than Dyck paths, because this information might eventually be helpful in constructing this bijection.

Rigged Configurations

Definition (Rigged Configuration)

Roughly speaking, a **rigged configuration** in RC_n is a Young diagram with n boxes, where the rows are labelled with a vacancy number and a rigging.

Example (A rigged configuration in RC_{13})

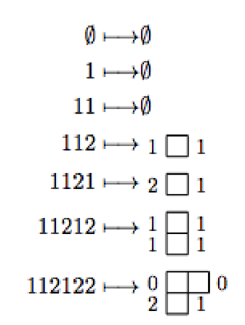


Bijection $\Phi : \mathcal{D}_{2n} \to \mathrm{RC}_n$

Definition (KKR)

There is a bijection $\Phi : \mathcal{D}_{2n} \to \mathrm{RC}_n$, which we describe roughly. Read the Dyck word from left to right. At each right step, add a box to the rigged configuration, according to some rules.

Example (Using Dyck path 112122, where '1' is up, '2' is right)



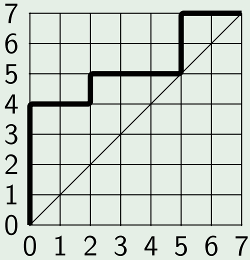
Idea 1: Filled Rigged Configurations

Area as Columns

View the area of a Dyck path as a sum of column areas. This way we can associate to each R step a column area.

Example

UUUURRURRRUURR



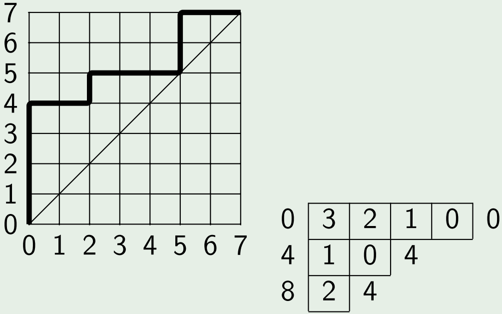
$$Area = 3 + 2 + 2 + 1 + 0 + 1 + 0 = 9$$

Modification of Φ

Recall that Φ draws a box at each right step. We modify the map by filling the box with the column area associated to that right step.

Example

This Dyck Path generates the following filled rigged configuration.



End Result: Partially Successful

0	3	2	1	0	0
4	1	0	4		
8	2	4			

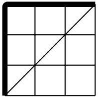
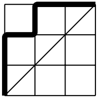
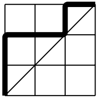
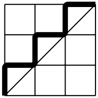
There are a few nice properties that hold:
Each row is rilled with a consecutive decreasing sequence of numbers, for rows with rigging 0 the sequence starts with 0, etc.
However, in general, there is not really a clean formula for which sequence fills a given row.
We were able to derive some (fairly complicated) formulas for particular cases of rigged configurations with a small number of block sizes.

Idea 2: Area and Bounce Shifting Transformation on Dyck Paths

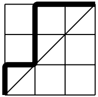
Idea

Define a rule for transforming a Dyck path, so that it decreases area by 1 and increases bounce by 1. Then for any Dyck path, apply this transformation repeatedly, until the area and bounce are exchanged.

Example with $n = 3$

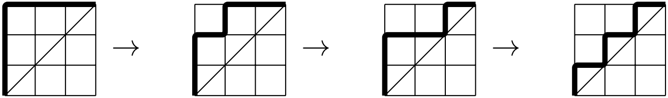
 $\rightarrow$  $\rightarrow$  $\rightarrow$ 

A: 3, B: 0 A: 2, B: 1 A: 1, B: 2 A: 0, B: 3

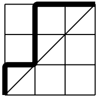


A: 1, B: 1

Example with $n = 3$



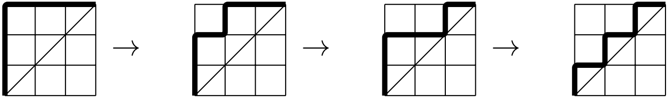
A: 3, B: 0 — A: 2, B: 1 — A: 1, B: 2 — A: 0, B: 3



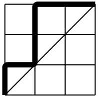
A: 1, B: 1

This approach works for $n \leq 6$.

Example with $n = 3$



A: 3, B: 0 A: 2, B: 1 A: 1, B: 2 A: 0, B: 3



A: 1, B: 1

This approach works for $n \leq 6$.

When $n \geq 7$, the Dyck paths stop breaking down into chains, and the method no longer gives a bijection.

Idea 3: Bijection to 312-Avoiding Permutations

Definitions

Definition (312-avoiding Permutation)

We call σ to be 312-avoiding if σ does not contain a subword $[\sigma_i, \sigma_j, \sigma_k]$ with $\sigma_j < \sigma_k < \sigma_i$. We denote the set of all 312-avoiding permutations on $[n]$ by $\mathcal{S}_n(312)$.

Definition

Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ be a sequence of positive integers. Then for each $i \in [n]$, we call i an **shifted ascent** and σ_i an **ascent top** if $i = 1$ or $\sigma_{i-1} < \sigma_i$.

Properties

Lemma

For a 312-avoiding permutation σ, the ascent tops are increasing from left to right, and they are exactly the left-to-right maxima.

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Lemma

For a 312-avoiding permutation σ, once we fix the positions of the shifted ascents and the corresponding ascent tops, the permutation is uniquely determined.

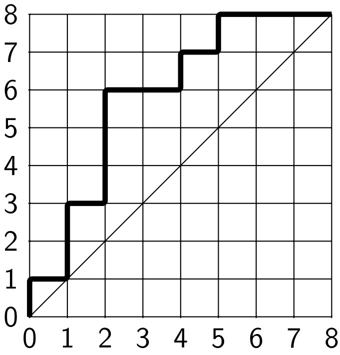
$$\sigma = \underset{\cdot}{\underline{3}}21\underset{\cdot}{\underline{6}}54\underset{\cdot}{\underline{9}}87$$

The Bijection to Dyck Path

Theorem

The algorithm f given below is a bijection from $\mathcal{S}_n(312)$ to $\mathcal{D}_{2n}$.

$\sigma = \underline{136}5\underline{78}42$ is a 312-avoiding permutation. The height sequence for it is $h = 13667888$ and the Dyck path d is the following:



Nice Properties

Theorem (Area & Bounce & First return)

Let $\sigma \in \mathcal{S}_n(312)$, form the height sequence h of σ, and let $d = f(\sigma)$. Then

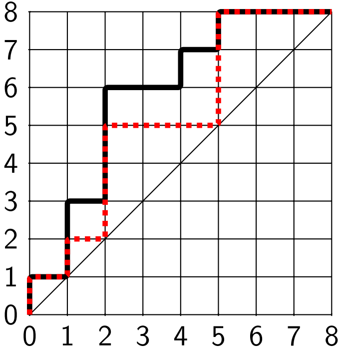
(a) *For each $i \in [n]$, the numbers of $i < j \leq n$ with $\sigma_j < \sigma_i$ equals the number of full boxes between d and the main diagonal in column i. Consequently, the number of inversions of σ equals the area of d: $inv(\sigma) = area(d)$.*

(b) *For σ, we form a sequence b: set $b_n = h_n = n$, and for each $1 \leq i < n$, set $b_i = b_{i+1}$ if $b_{i+1} \leq h_i$ and set $b_i = i$ otherwise. Then this corresponds to the bounce path of d, and $bounce(d) = \sum_{i<n, b_i = i} b_i$.*

(c) *The position of 1 in σ equals the y-coordinate of the first return to the diagonal in the Dyck path d.*

The Bijection to Dyck Path

$\sigma = \underline{136}5\underline{78}42$. $\quad$ $\text{inv}(\sigma) = 11$.
$h = 13667888$.
$b = 12555888$. $\quad$ $\text{bounce} = 1 + 2 + 5 = 8$.
The Dyck path $d = f(\sigma)$ is the following:



$\text{area}(d) = 11$
$\text{bounce}(d) = 8$

Involution

Definition

Define $\Omega : \mathcal{S}_n(312) \to \mathcal{S}_n(312)$ as follows: For $\sigma \in \mathcal{S}_n(312)$, let the shifted ascents be $\{i_1, i_2, \cdots, i_r\}$ and let the ascent tops be $\{\sigma_{i_1}, \sigma_{i_2}, \cdots, \sigma_{i_r}\}$. Then $\Omega(\sigma)$ is defined to be the unique 312-avoiding permutation given by shifted ascents $\{n+1-\sigma_{i_r}, n+1-\sigma_{i_{r-1}}, \cdots, n+1-\sigma_{i_1}\}$ and ascent tops $\{n+1-i_r, n+1-i_{r-1}, \cdots, n+1-i_1\}$. It is clear that Ω is an involution.

Example

Let $\sigma = \underline{136}5\underline{78}42$. The set of shifted ascents is $\{1, 2, 3, 5, 6\}$ and the set of ascent tops is $\{1, 3, 6, 7, 8\}$. So for $\Omega(\sigma)$, the set of shifted ascents is $\{1, 2, 3, 6, 8\}$, the set of ascent tops is $\{3, 4, 6, 7, 8\}$, and $\Omega(\sigma) = \underline{346}52\underline{7}1\underline{8}$.

The Bijection to Dyck Path

Theorem

f intertwines the maps $*$ and Ω. That is, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{S}_n(312) & \xrightarrow{\Omega} & \mathcal{S}_n(312) \\ \downarrow f & & \downarrow f \\ \mathcal{D}_{2n} & \xrightarrow{*} & \mathcal{D}_{2n} \end{array}$$

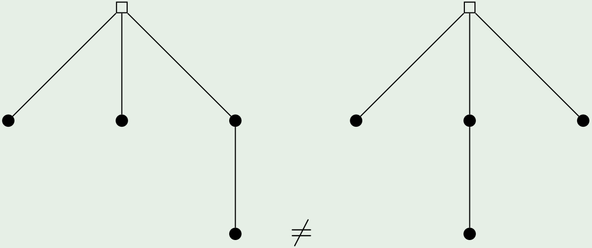
where $*$ is the map that reverses the Dyck path.

Rooted Planar Trees

Definition (Rooted Planar Trees)

A **rooted planar tree** T is a tree with a distinguished root and a fixed ordering on the children of every vertex. Denote the set of all rooted planar trees on $n + 1$ vertices by $\mathrm{RPT}(n)$

Example (Two different trees in RPT(4))

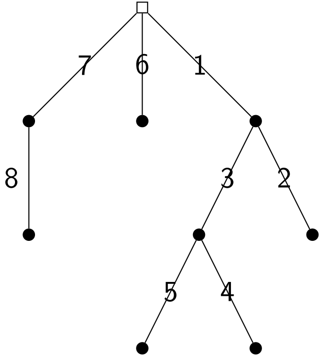


The Bijection to Rooted Planar Trees

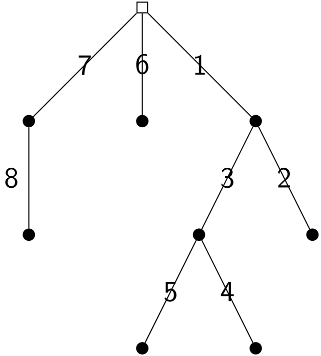
Theorem

Define an algorithm $g : RPT(n) \to \mathcal{S}_n(312)$*: For* $T \in RPT(n)$*, we start from the root and walk around the tree on the right side. Number the edges in the order that we first encounter them, and write down a sequence* σ *of edge numbers in the order of our second encounter of them. Then* g *is a bijection.*

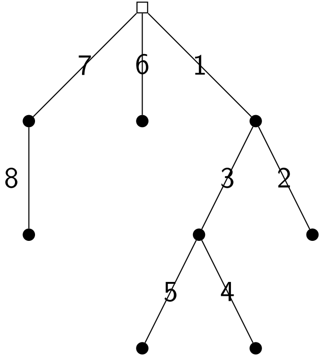
The Bijection to Rooted Planar Trees



The Bijection to Rooted Planar Trees

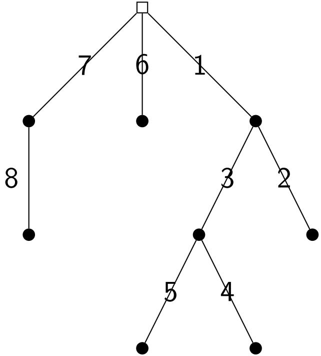


The corresponding permutation is $\sigma =$



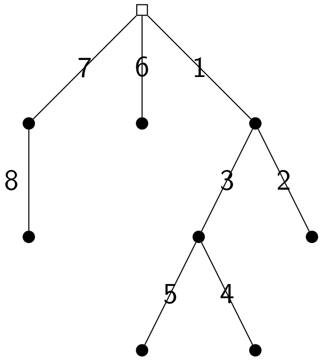
The corresponding permutation is $\sigma = 2$

The Bijection to Rooted Planar Trees



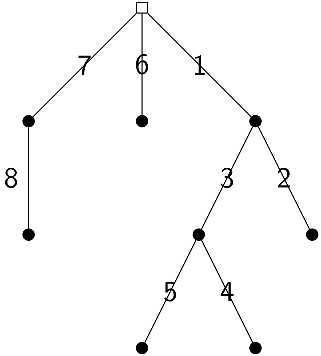
The corresponding permutation is $\sigma = 24$

The Bijection to Rooted Planar Trees



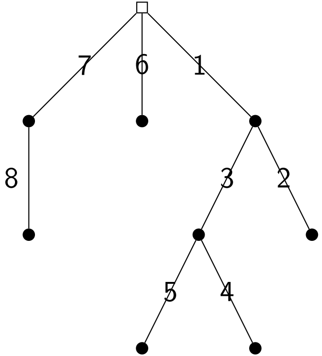
The corresponding permutation is $\sigma = 245$

The Bijection to Rooted Planar Trees



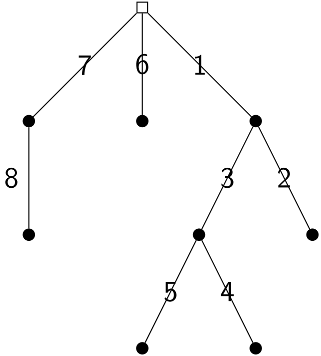
The corresponding permutation is $\sigma = 2453$

The Bijection to Rooted Planar Trees



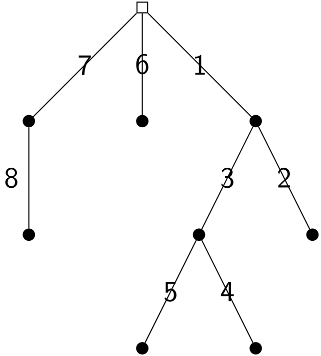
The corresponding permutation is $\sigma = 24531$

The Bijection to Rooted Planar Trees



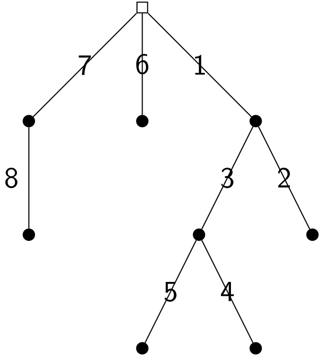
The corresponding permutation is $\sigma = 245316$

The Bijection to Rooted Planar Trees



The corresponding permutation is $\sigma = 2453168$

The Bijection to Rooted Planar Trees



The corresponding permutation is $\sigma = 24531687$.

The Bijection to Rooted Planar Trees

Corollary

We have the commutative diagram:

$$\begin{array}{ccc} \mathcal{S}_n(312) & \xrightarrow{f} & RPT(n) \\ & \nwarrow^{g} & \downarrow \pi \\ & & \mathcal{D}_{2n} \end{array}$$

where π is the planar code for the rooted planar trees.

To Rigged Configuration

We have some partial result. We have an algorithm to get the partition shape of the rigged configuration:

$$\sigma = \ 2\ 4\ 5\ 3\ 1\ 6\ 8\ 7$$

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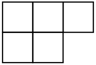

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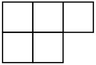
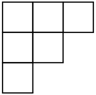


$$\to \sigma = 2\ 4\ \not{5}\ \not{3}\ \not{1}\ \not{6}\ \not{8}\ \not{7}$$



To Rigged Configuration

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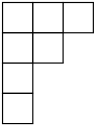
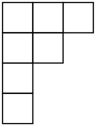
$$\rightarrow \sigma = 2\ \cancel{4}\ \cancel{5}\ \cancel{3}\ \cancel{1}\ \cancel{6}\ \cancel{8}\ \cancel{7}$$



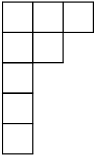
To Rigged Configuration

(cont'd)

$$\rightarrow \sigma = 2\ \cancel{4}\ \cancel{5}\ \cancel{3}\ \cancel{1}\ \cancel{6}\ \cancel{8}\ \cancel{7}$$



$$\rightarrow \sigma = \cancel{2}\ \cancel{4}\ \cancel{5}\ \cancel{3}\ \cancel{1}\ \cancel{6}\ \cancel{8}\ \cancel{7}$$



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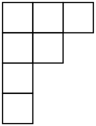
$$\rightarrow \sigma = 2\ \not{4}\ \not{5}\ \not{3}\ \not{1}\ \not{6}\ \not{8}\ \not{7}$$



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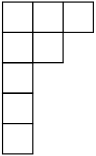

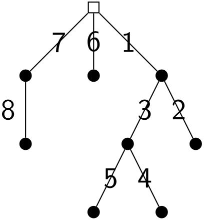

To Rigged Configuration

Problem

How to get the riggings directly from the 312-permutation?

Still unclear how to do that.

References

A. M. Garsia and J. Haglund, A positivity result in the theory of Macdonald polynomials, Proc. Nat. Acad. Sci. U.S.A. 98 (2001), 43134316.

A. M. Garsia and J. Haglund , A proof of the q, t-Catalan positivity conjecture, Discrete Math. 256 (2002), 677717.

R. Reynolds. Rigged Configurations and Catalan Objects: Completing a Commutative Diagram with Dyck Paths and Rooted Planar Trees.

Thank you for listening!!!!!!