(1)

REU 2017 Day 2 June 13, 2017
V. Reiner "Dihedral actions"

1. q-counts
2. cyclic actions
3. q-Catalan
4. dihedral actions
5. representation theory
6. REU PROB 2.

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1. q-counts

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q\,[n-k]!_q} \quad \text{where } [n]!_q = [n]_q[n-1]_q \cdots [3]_q[2]_q[1]_q \xrightarrow{q=1} n!$$

q-binomial coefficient

$$\text{and } [n]_q = 1+q+q^2+\cdots+q^{n-1} = \frac{1-q^n}{1-q} \xrightarrow{q=1} n$$

$$\frac{[n]!_q}{[k]!_q\,[n-k]!_q} \rightsquigarrow_{q=1} \binom{n}{k}$$

EXAMPLE: $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]_q[3]_q\cancel{[2]_q}\cancel{[1]_q}}{[2]_q[1]_q\cancel{[2]_q}\cancel{[1]_q}} = \frac{[4]_q[3]_q}{[2]_q} = \frac{(1+q+q^2+q^3)(1+q+q^2)}{1+q}$

$$= (1+q^2)(1+q+q^2) = 1+q+2q^2+q^3+q^4$$

$$\rightsquigarrow_{q=1} \quad 6 = \binom{4}{2} \checkmark$$

REU EXERCISE 3:

(b) Show $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum\limits_{\substack{\text{partitions } \lambda \\ \text{with } \lambda_1 \le n-k \\ \text{and at most } k \text{ parts}}} q^{|\lambda|}$ (so $\begin{bmatrix} n \\ k \end{bmatrix}_q \in \mathbb{N}[q]$)

i.e. $\lambda \subset$  $\}k$, $\underbrace{\qquad}_{n-k}$

swap order!

(a) Show $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$

(c) Show that, setting $q = p^d$ a prime power, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \#\{$ k-dimensional $\mathbb{F}_q$-linear subspaces of $\mathbb{F}_q^n \}$

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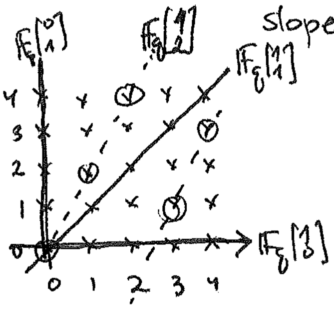
EXAMPLE: $n=2$

$\begin{bmatrix}2\\0\end{bmatrix}_q = 1 \leftarrow$ counts $\{\vec{0}\} \subset \mathbb{F}_q^2$

$\begin{bmatrix}2\\2\end{bmatrix}_q = 1 \leftarrow$ counts $\mathbb{F}_q^2$ itself

$$\begin{bmatrix}2\\1\end{bmatrix}_q = \frac{[2]_q[1]_q}{[1]_q[1]_q} = [2]_q = q+1 = \#\{ \underbrace{\mathbb{F}_q\begin{bmatrix}1\\0\end{bmatrix}, \mathbb{F}_q\begin{bmatrix}1\\1\end{bmatrix}, \mathbb{F}_q\begin{bmatrix}1\\2\end{bmatrix}, \ldots, \mathbb{F}_q\begin{bmatrix}1\\q-1\end{bmatrix}}_{\text{slopes in } \mathbb{F}_q}, \underbrace{\mathbb{F}_q\begin{bmatrix}0\\1\end{bmatrix}}_{\text{"slope} \infty\text{"}} \}$$

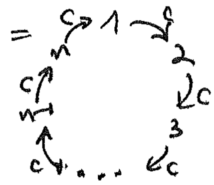
$q=5$



2. Cyclic actions

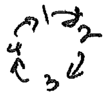
Evaluating $\begin{bmatrix}n\\k\end{bmatrix}_q$ at $q \in \{$powers of $\zeta_n\} = n^{\text{th}}$ roots-of-unity in $\mathbb{C}$, where $\zeta_n = e^{\frac{2\pi i}{n}}$

turns out to count something too, related to an n-cycle $c = (1,2,\ldots,n)$



permuting k-element subsets of $\{1,2,\ldots,n\}$

EXAMPLE: $n=4$, $k=2$ $\quad c=(1,2,3,4)$ permutes



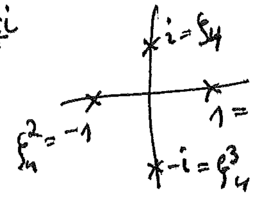
$\{1,2\} \xrightarrow{c} \{2,3\} \xrightarrow{c} \{3,4\} \xrightarrow{c} \{1,4\}=\{4,1\} \xrightarrow{c} \{1,2\}$

$\{1,3\} \xrightarrow{c} \{2,4\} \xrightarrow{c} \{1,3\}$

THM (R-Stanton-White 2004) When $c=(1,2,\ldots,n)$ permutes k-element subsets of $\{1,2,\ldots,n\}$,

$\begin{bmatrix}n\\k\end{bmatrix}_{q=\zeta_n^d}$ counts the number fixed by c^d.

EXAMPLE: $\begin{bmatrix}4\\2\end{bmatrix}_q = 1+q+2q^2+q^3+q^4 \qquad \zeta_4 = i = e^{\frac{2\pi i}{4}}$



$q=\zeta^0=1$: $6 = \#\{12,13,14,23,24,34\}$

$q=\zeta^2=-1$: $2 = \#\{13,24\}$

$q=\zeta^1=i$ or $q=\zeta^3=-i$: 0 since no 2-subsets $\{i,j\}$ are fixed by c

(3) REU EXERCISE 4: Prove the previous THM by this strategy ---

(a) Show that if ζ is any primitive m^{th} root-of-unity, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=\zeta} = \binom{n'}{k'} \begin{bmatrix} n'' \\ k'' \end{bmatrix}_{q=\zeta} \quad \text{where } n = m \cdot n' + n'', \; 0 \le n'' \le m-1, \quad k = m \cdot k' + k'', \; 0 \le k'' \le m-1$$

i.e. $m \overline{)\,n\,}^{\,n'}$ ... n'' $m \overline{)\,k\,}^{\,k'}$ ... k'' remainders on div. by m

(b) Apply this when $\zeta = \zeta_n^d$,

and compare to brute force count of k-subsets fixed by c^d

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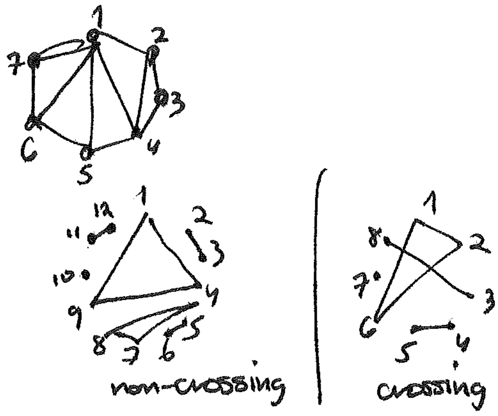
3. q-Catalan

The Catalan ~~number~~ $C_n := \frac{1}{n+1}\binom{2n}{n}$ counts

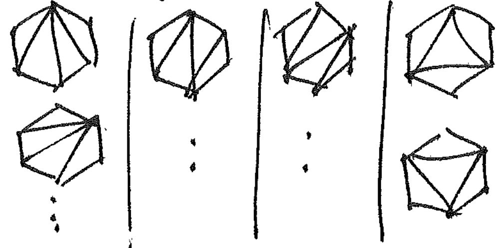
- triangulations of an $(n+2)$-gon
- noncrossing set partitions of $\{1,2,\ldots,n\}$



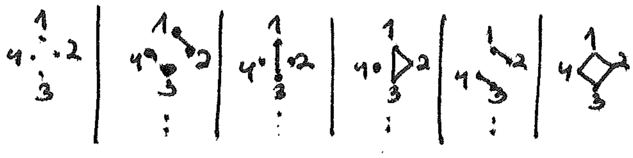
EXAMPLE: $C_4 = \frac{1}{4+1}\binom{2\cdot 4}{4} = \frac{\cancel{8}\cdot 7 \cdot \cancel{6}^{2} \cdot \cancel{5}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}} = 14$

counts both...

6-gon triangulations || noncrossing partitions of $\{1,2,3,4\}$



$6 + 3 + 3 + 2 = 14$



$1 + 4 + 2 + 4 + 2 + 1 = 14$

THM (RSW 2004) MacMahon's q-Catalan number $C_n(q) := \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q$ has

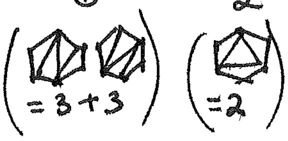
(a) $C_n(q)\big|_{q=\zeta_{n+2}^d}$ counting $(n+2)$-gon triangulations fixed by c_{n+2}^d

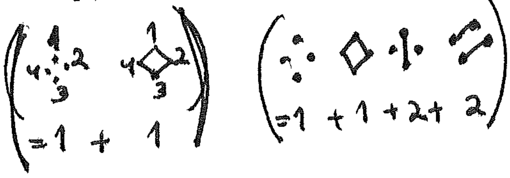
(b) $C_n(q)\big|_{q=\zeta_n^d}$ counting noncrossing partitions of $\{1,2,\ldots,n\}$ fixed by c_n^d

EXAMPLE: $C_4(q) = \frac{1}{[5]_q}\begin{bmatrix}8\\4\end{bmatrix}_q = \frac{[8]_q[7]_q[6]_q[5]_q}{[5]_q[4]_q[3]_q[2]_q}$

$$= 1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12} \quad (\in \mathbb{N}[q]\,?)$$

$q=1$	$q=-1$	$q=e^{\frac{2\pi i}{3}}$	$q=e^{\frac{2\pi i}{6}}$	$q=e^{\frac{2\pi i}{4}}=i$	$q=-1$	$q=1$
14 ✓	6	2	0	2	6	14 ✓


$(= 3+3)$ $(=2)$



$(=1 + 1)$ $(=1+1+2+2)$

REMARK: We only know a similar brute force proof!

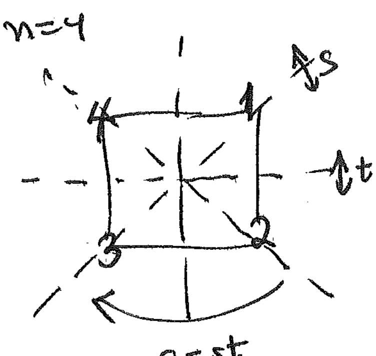
4. Dihedral actions

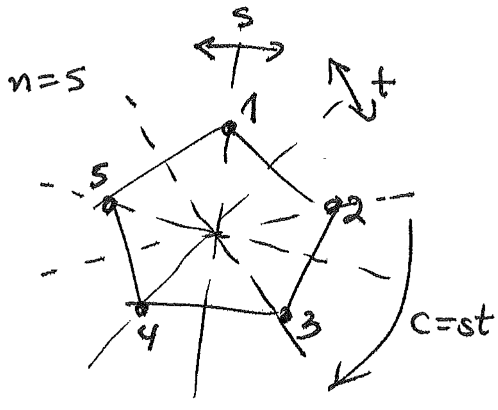
In fact, $\{1,2,\ldots,n\}$ isn't just cycled by $c=(1,2,\ldots,n)$ and powers c^d,

it carries an action of the dihedral group $I_2(n)$ of order $2n$

$:=$ symmetries of a regular n-gon

EXAMPLES:





Abstractly, $I_2(n) = \langle \underbrace{s,t}_{\text{generators}} \mid \overbrace{s^2=t^2=e=(st)^n}^{\text{relations}} \rangle$

$$= \langle s,c \mid s^2=c^n=e,\; \underset{(=scs)}{scs^{-1}} = c^{-1} \rangle$$

$$= \{\underbrace{e,c,c^2,\ldots,c^{n-1}}_{n \text{ rotations}}, \underbrace{s,\overset{=t}{sc},sc^2,\ldots,sc^{n-1}}_{n \text{ reflections}}\}$$

(5) The symmetries picture gives a representation (= group homomorphism into some $GL_n(\mathbb{F})$ ← field)

$$I_2(m) \xrightarrow{\rho_{def}} GL_2(\mathbb{R})$$

$$c \longmapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{where } \theta = \frac{2\pi}{n}$$

$$s \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{if we make } s \text{ swap } x,y \text{ axes}$$

$\vdots$

The permutation actions on vertices $\{1,2,3,\ldots,n\}$
or k-subsets of $\{1,2,\ldots,n\}$
or triangulations
or non crossing partitions $\quad I_2(m) \xrightarrow{\rho} GL_N(\mathbb{R})$

give various other representations
sending group elements to permutation matrices.

GENERAL PROBLEM: Decompose these other representations into direct sums of irreducible representations, that is, those with no non-zero, proper subspaces stabilized (even working over $\mathbb{C}$, instead of

See Steinberg's book, Ch. 2-4

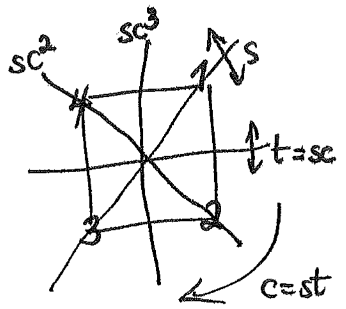
THM (Maschke) This can always be done for finite group reps $G \xrightarrow{\rho} GL_N(\mathbb{C})$

THM (Frobenius) One can do it by computing the character $\chi_\rho : G \to \mathbb{C}$, $g \mapsto$ Trace(

and decomposing it uniquely as a sum of the irreducible rep's characters

There are various tricks (e.g. orthogonality relations) that make it easier.

EXAMPLE: $G = I_2(4) = \{e, c, c^2, c^3, s, \overset{t}{\overset{\shortparallel}{sc}}, sc^2, sc^3\}$



has 5 conjugacy classes
$e \mid c, c^3 \mid c^2 \mid s, sc^2 \mid sc, sc^3$

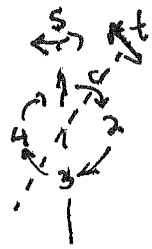
and 5 irreducible reps & characters:

(6)

character table → (conjugacy classes)

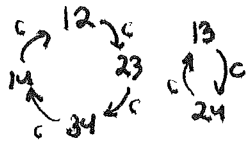
irreducible characters:

	e	c, c^3	c^2	s, sc^2	$\overset{t}{sc}, sc^3$
$\mathbb{1}$	1	1	1	1	1
χ_s	1	-1	1	-1	$+1$
χ_t	1	-1	1	$+1$	-1
$\chi_s\chi_t = \chi_{\det}$	1	$+1$	1	-1	-1
χ_{def}	2	0	-2	0	0



	e	c, c^3	c^2	s, sc^2	sc, sc^3
$\chi_{\{1,2,3,4\}}$	4	0	0	2	0
$\chi_{\{2\text{-subsets}\}}$	6	0	2	2	2

s fixes $\{1,3\}, \{2,4\}$ t fixes $\{1,2\}, \{3,4\}$



Can expand
$$\chi_{\{1,2,3,4\}} = \mathbb{1} + \chi_t + \chi_{\mathrm{def}}$$
$$\chi_{\{2\text{-subsets}\}} = \mathbb{1} + \mathbb{1} + \chi_t + \chi_s + \chi_{\mathrm{def}}$$
uniquely

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6. REU PROBLEM 2

Expand into irreducibles (a) $\mathbb{Z}_2(n)$ permuting {k-subsets of $\{1,2,\ldots,n\}$} (war (-u

(b) —" ——— {noncrossing partitions of $\{1,2,\ldots,n\}$}

(c) $\mathbb{Z}_2(n+2)$ —" — {triangulations of $(n+2)$-gon}

- What binomial, Catalan identities ensue?
- Does $C_n(q)$ help describe the characters at all?
- What about Garsia & Haiman's (q,t)-Catalan number $C_n(q,t)$?