REU 2017 Day 5 P. Pylyavskyy

The rule of three

Let $u_1, u_2, \ldots, u_n$ be non-commuting variables.

$e_3 \overset{?}{=} u_1 u_2 u_3 \neq u_2 u_1 u_3 = \cdots$

Let $e_k := \sum_{i_1 > i_2 > \cdots > i_k} u_{i_1} u_{i_2} \cdots u_{i_k}$

EXAMPLE

$u_1 = a,\ u_2 = b,\ u_3 = c$

$e_1 = a + b + c$

$e_2 = ca + cb + ba$

$e_3 = cba$

$e_k(u_S) = \sum_{\substack{i_1 > \cdots > i_k \\ i_j \in S}} u_{i_1} \cdots u_{i_k}$

EXAMPLE $e_2(u_{\{1,3\}}) = ca$

THEOREM (Kirillov, Blasiak–Fomin)

The following are equivalent:

- For any k, ℓ and $S \subseteq \{1, 2, \dots, N\}$
$$e_k(u_S)e_\ell(u_S) = e_\ell(u_S)e_k(u_S)$$
- These special cases hold:
$$e_1(u_S)e_2(u_S) = e_2(u_S)e_1(u_S) \quad \forall S \text{ with } 2 \le |S| \le 3$$
$$e_1(u_S)e_3(u_S) = e_3(u_S)e_1(u_S) \quad \forall S \text{ with } |S| = 3$$

EXAMPLE $N=3$, a, b, c

$$ba(a+b) = (a+b)ba$$

$$\Leftrightarrow \boxed{baa + bab = aba + bba}$$

$$\overset{?}{\Rightarrow} (ca+ba+cb)(a+b+c) = (a+b+c)(ca+ba+cb)$$

$$\begin{aligned} &\cancel{caa} + cab + \cancel{cac} & & \cancel{aca} + \cancel{aba} + acb \\ &+ \cancel{baa} + \cancel{bab} + bac & \overset{?}{=} \; & + bca + \cancel{bba} + \cancel{bcb} \\ &+ \cancel{cba} + \cancel{cbb} + \cancel{cbc} & & + \cancel{cca} + \cancel{cba} + \cancel{ccb} \end{aligned}$$

$$\Leftrightarrow \boxed{cab + bac = bca + acb}$$

$$(a+b+c)cba = cba(a+b+c)$$

$$\begin{aligned} &acba + bcba + ccba \overset{?}{=} cbaa + cbab + cbac \end{aligned}$$

$$\begin{array}{ccc} acba + (bcb+ccb)a & \overset{?}{=} & cbac + c(baa+bab) \\ \| & & \| \\ acba + (cbb+cbc)a & & cbac + c(aba+bba) \end{array}$$

$$\Rightarrow \boxed{acba + cbca = cbac + caba}$$

$$\boxed{\begin{array}{l} \text{REU EXERCISE 12 Prove these imply} \\ (ba+ca+cb)cba = cba(ba+ca+cb) \end{array}}$$

3 more settings where such a "rule of three" might exist...
(or a "rule of four"...?)

Quasisymmetric functions

α a composition

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, $\alpha_i \in \mathbb{Z}_{>0}$

EXAMPLE $\alpha = (2,3,1)$

Monomial quasisymmetric fn

$$M_\alpha := \sum_{i_1 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$

EXAMPLE

$$M_{(1,1)} = e_2 = x_1^1 x_2^1 + x_1^1 x_3^1 + \cdots$$

$$M_{(2,1)} = x_1^2 x_2^1 + x_1^2 x_3^1 + x_2^2 x_3^1 + \cdots$$

REU PROBLEM 5(a)

Is it true that there is a finite list of initial commutations that makes all of the M_α's commute?

Loop symmetric functions (T. Lam)

Variables come in flavors:

$a, b, c, \quad A, B, C$

$e_1 = a+b+c$

$E_1 = A+B+C$

$e_2 = bA + cA + cB$

$E_2 = Ba + Ca + Cb$

$e_3 = cBa$

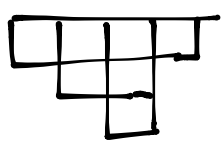
$E_3 = CbA$

REU PROBLEM 5(b)

Is it true that there is a finite list of initial commutations that makes all of the e_k, E_k commute?

Schur Q-functions

Shifted Young diagrams $\longleftrightarrow$ partitions λ with distinct parts

 $\longleftrightarrow$ $\lambda = (4,2,1)$

Work in an alphabet $1' < 1 < 2' < 2 < \dots$

A shifted tableaux of shape λ has

1) labels weakly increasing in rows and columns
2) each row has at most one k
3) each column has at most one k'

$$Q_\lambda := \sum_{\text{shifted tableaux } T \text{ of shape } \lambda} x^T$$

EXAMPLE

$$T = \begin{array}{|c|c|c|c|} \hline 1' & 1' & 3 & 4' \\ \hline \multicolumn{1}{c|}{} & 1 & 3 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}$$

$$x^T = x_1^3 x_3^2 x_4^2$$

It turns out that the Q_λ are symmetric functions, and they linearly span a subring Q' of $\underline{\Lambda}$ which is generated by $p_1, p_3, p_5, \dots$ where $p_k = x_1^k + x_2^k + \dots$

Let's suggest as analogues of e_k here

$$Q_{\square}, \quad Q_{\square\square\square}, \quad Q_{\square\square\square\square\square}, \quad \ldots$$

$$Q_{\square} = 2(x_1 + x_2 + \ldots)$$

$$Q_{\square\square} = 2(x_1^2 + x_2^2 + \ldots) + 4(x_1x_2 + x_1x_3 + \ldots)$$

$$\boxed{i' \mid i} \qquad \qquad \boxed{i \mid j}$$

not needed!

$$\boxed{i' \mid i'} \qquad = Q_{\square}^2$$

CONJ: $Q_{\underbrace{\square\square\square\square\square\square}_{\text{even}}} \in \mathbb{Z}[Q_{\square}, Q_{\square\square\square}, Q_{\square\square\square\square\square}, \ldots]$

(maybe known?)

REU PROBLEM 5(c)

Is it true that there is a finite list of initial commutations that makes all of the $Q_{\overbrace{\square\square\square\square\square}^{\text{odd}}}$ commute?