(1)

REU 2017 Day 9 Chow rings of matroids

1. What's a matroid?
2. Characteristic polynomial
3. Chow ring & Adiprasito-Huh-Katz's 2015 work
4. REU Problem 9

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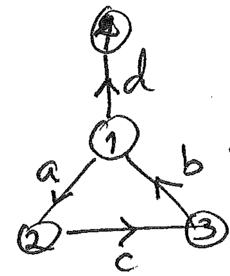
1. What's a matroid?

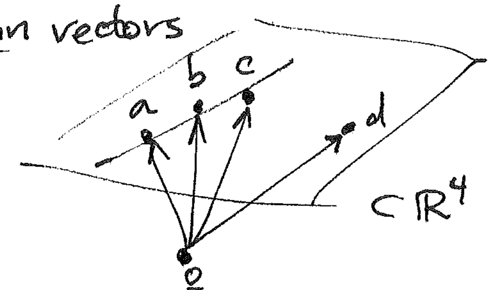
It abstracts a graph $G=(V,E)$, $E = \{a,b,c,d\}$



orientation of edges chosen arbitrarily

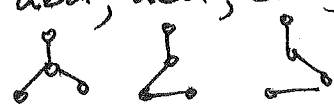
and a matrix $M = M_G$ thought of as column vectors

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} +1 & -1 & 0 & +1 \\ -1 & 0 & +1 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$



by only specifying these equivalent bits of combinatorial data:

- the bases $\mathcal{B}(M) = \{abd, acd, bcd\}$ ($\leftrightarrow$ spanning trees of G)



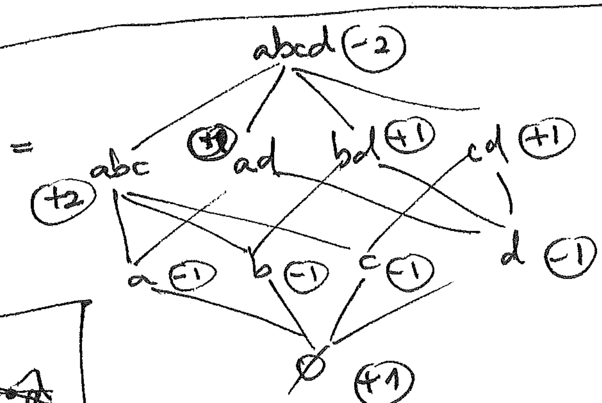
- the independent sets $\mathcal{I}(M) = \{\text{subsets of bases}\} = \left\{\begin{array}{l} abd, acd, bcd, \\ ab, ac, bc, ad, bd, cd, \\ a, b, c, d, \\ \varnothing \end{array}\right\}$ ($\leftrightarrow$ forests in G)
- the circuits $\mathcal{C}(M) = \{\text{minimal dependent sets}\} = \{abc\}$ ($\leftrightarrow$ (minimal) cycles in G)

If M has no zero vectors ($\leftrightarrow$ loops in G), and no parallel vectors ($\leftrightarrow$ parallel edges in G), then one can also specify this via ...

(2)

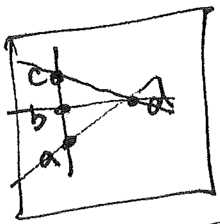
$L(M) :=$ lattice of flats $F \subseteq E$, ordered by $\subseteq$

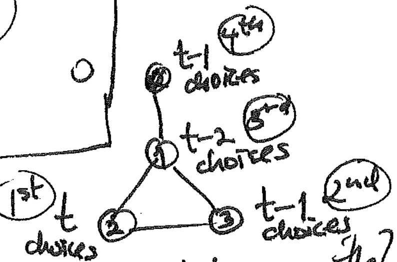
subsets of vectors closed under linear span
i.e. closed under adding vectors that don't increase rank $r(F)$
where $r(A) := \max\{|I| : I \in \mathcal{I}(M), I \subset A\}$



$r(F)$: 3, 2, 1, 0

$\mu(\phi, F)$ values circled





EXAMPLE: This helps abstract the chromatic polynomial $P_G(t) = \#\{$proper vertex-colorings with t colors$\}$

$$= t(t-1)(t-2)(t-1)$$
$$= t^4 - 4t^3 + 5t^2 - 2t$$

(unsigned) coefficients $(1, 4, 5, 2)$ ← conjectured unimodal by Read 1968, log-concave $a_k^2 \geq a_{k-1}a_{k+1}$ by Hoggar 1974

log-concave: $4^2 \geq 1\cdot 5$, $5^2 \geq 4\cdot 2$

2. Characteristic polynomial

Turns out $P_G(t) = t^{c(G)} \underbrace{\chi_{M_G}(t)}$, $c(G) = \#$ connected components of G

characteristic polynomial $\chi_M(t) = \sum_{\text{flats } F \in L(M)} \mu(\phi, F)\, t^{r(E)-r(F)}$

Möbius function
$\mu(\phi,\phi) := +1$
$\mu(\phi,F) := -\sum_{G<F} \mu(\phi,G)$

$$= t^3 - 4t^2 + 5t^1 - 2t^0$$
$$= (t-1)\underbrace{\overline{\chi}_M(t)}$$

=: reduced characteristic polynomial

$$= (t-1)(t^2 - 3t + 2)$$

unsigned coefficient sequence $(1, 3, 2)$
$\shortparallel$
$(\mu_0, \mu_1, \ldots, \mu_{r-1})$
where $r = r(M)$

conjectured unimodal, log-concave by Rota 1971 – Heron 1972 – Welsh 1976 – Brylawski 1977

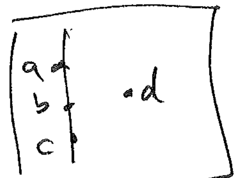
log-concave: $3^2 \geq 1\cdot 2$

Almost no progress for 35 years! Until....

3. The Chow ring of $\underline{M}$ (Feichtner–Yuzvinsky 2004)

$$A(M) := \mathbb{Z}[x_F]_{F \text{ a nonempty proper flat of } M} \Big/ \left(\left(x_F x_G : \begin{matrix} F \not\subset G, \\ G \not\subset F \end{matrix} \right) + \left(\alpha_i - \alpha_j : i \neq j \text{ in } E \right) \right)$$

where $\alpha_i := \sum_{\substack{\text{non-}\emptyset \\ \text{proper flats} \\ F \ni i}} x_F$

EXAMPLE: $M =$  has

$$A(M) = \mathbb{Z}[\overbrace{x_a, x_b, x_c, x_d, x_{abc}, x_{ad}, x_{bd}, x_{cd}}^{\text{degree 1}}] \Big/ \left(\begin{matrix} x_a x_b, & x_{ad} x_{bd}, & x_a x_{bd}, & \overbrace{\alpha_a - \alpha_d,}^{\text{degree 1}} \\ x_a x_c, & x_{ad} x_{cd}, & x_c x_{bd} & \alpha_b - \alpha_d, \\ \vdots & x_{bd} x_{cd}, & \vdots & \underbrace{\alpha_c - \alpha_d}_{\substack{\| \\ x_c + x_{abc} + \cancel{x_{cd}} \\ - x_d - x_{ad} - x_{bd} - \cancel{x_{cd}}}} \\ x_c x_d, & x_{ad} x_{abc} & & \\ & x_{bd} x_{abc} & & \\ & x_{cd} x_{abc} & & \end{matrix} \right)$$

Since the relations $x_F x_G = 0$, $\alpha_i = \alpha_j$ are homogeneous,

$$A(M) = \underbrace{A^0(M)}_{\substack{\| \\ \mathbb{Z}}} \oplus A^1(M) \oplus A^2(M) \oplus \dots$$ is a graded ring

i.e. $A^i(M) \cdot A^j(M) \subset A^{i+j}(M)$

REU EXERCISE 20: In the above example, show

(a) • $A(M) = \underset{\substack{\| \\ \mathbb{Z}}}{A^0(M)} \oplus \underset{\substack{\| \\ \mathbb{Z}^5}}{A^1(M)} \oplus \underset{\substack{\| \\ \mathbb{Z}}}{A^2(M)}$, i.e. $A^3(M) = A^4(M) = \dots = 0$

• with an isomorphism $A^2(M) \xrightarrow[\sim]{\deg} \mathbb{Z}$

sending $x_{F_1} x_{F_2} \mapsto +1$ $\forall$ flats $\underset{\text{rank 1}}{F_1} \subsetneq \underset{\text{rank 2}}{F_2}$

e.g. $x_a x_{ad} \mapsto +1$

but
$x_a^2 \mapsto -1$
$x_d^2 \mapsto -2$
$x_{abc}^2 \mapsto -1$
$x_{ad}^2 \mapsto -1$

(b) The elements $\alpha := \alpha_a = \alpha_b = \alpha_c = \alpha_d$ and $\beta = \sum_{\text{all flats } F} x_F - \alpha$

satisfy $\deg(\alpha^2) = +1$ $(= \mu_0)$
$\deg(\alpha\beta) = \pm 3$ $(= \mu_1)$
$\deg(\beta^2) = +2$ $(= \mu_2)$

THEOREM (5) (Adiprasito–Huh–Katz, 2015) For a matroid M of rank r,

not so hard

(i) One always has $A(M) = \underset{\mathbb{Z}}{\underset{\shortparallel}{A^0(M)}} \oplus A^1(M) \oplus \cdots \oplus A^{r-2}(M) \oplus A^{r-1}(M)$

$$A^{r-1}(M) \xrightarrow[\cong]{\deg} \mathbb{Z}, \qquad \forall F_1 \subsetneq \cdots \subsetneq F_{r-1} \text{ maximal flags of flats: } \quad x_{F_1} x_{F_2} \cdots x_{F_{r-1}} \longmapsto +1$$

(ii) The elements $\alpha := \alpha_i \;\; \forall i \in E$

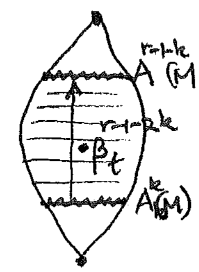
$$\beta := \sum_{\substack{\text{all non-}\emptyset \\ \text{proper flats } F}}' x_F \; - \; \alpha$$

satisfy $\mu_k = \deg(\alpha^{r-1-k}\beta^k)$ where $(\mu_0, \ldots, \mu_{r-2})$ = unsigned coefficients of $\overline{\chi}_M(t)$

hard!

(iii) One has isomorphisms $A^k(M) \longrightarrow A^{r-1-k}(M) \;\; \forall k \leq \frac{r-1}{2}$ and working with $\mathbb{R}$ coefficients, they can be realized explicitly via certain ample elements $\beta_t \in A^1(M)$ as

$$x \longmapsto \beta_t^{r-1-2k} \cdot x$$



(iv) The quadratic forms $A^k(M) \xrightarrow{Q_k} \mathbb{R}$

$$x \longmapsto Q_k(x) := \deg(x \cdot \beta_t^{r-1-2k} \cdot x)$$

have easily predictable signature $\forall$ ample β_t

(v) β is a limit of ample β_t's, leading to $\mu_k^2 \geq \mu_{k-1}\mu_{k+1}$, and the log-concavity conjectures!

not obvious why this should be!

REU PROBLEM 9:

For various "favorite" matroids, e.g.

- $M=M_G$ where G = complete graph K_n

- uniform (=generic) matroids $U_{r,n}$ (rank; size of E)
- $M=\{\text{all vectors in } \mathbb{F}_q^n\}$

(a) compute explicit formulas for the Hilbert series

$$\underbrace{1}_{h_0=1}+\underbrace{\dim A^1(M)}_{h_1}t^1+\underbrace{\dim A^2(M)}_{h_2}t^2+\dots+\underbrace{\dim A^{r-2}(M)}_{h_{r-2}}t^{r-2}+\underbrace{t^{r-1}}_{h_{r-1}=1}$$

(a not very ~~explicit~~ pretty formula for M_{K_n} appears in Feichtner-Yuzvinsky!)

(b) compute γ-vectors & Charney-Davis quantities $\boxed{\text{keywords!}}$ for $(h_0, h_1, \dots, h_{r-1})$

(c) compute the eigenvalues of the quadratic forms $Q_k(x) = x\cdot\beta^{r-1-2k}x$ on $A^k(M)$

(d) what about the Smith normal forms for their associated symmetric matrices $A=A^T$ with $Q_k(x)=x^TAx$

(e) When is $A(M)$ a $\boxed{\text{Koszul}}$ algebra?

Try this in Sage Math Cell Server:

```
G = Graph( [(1,2,'a'), (3,1,'b'), (2,3,'c'), (1,4,'d')] );
M = Matroid(G);
print(list(M.bases()));
for r in [0,1,2,3]:
    print( list(M.flats(r)));
AM = M.chow_ring();
AM.gens();
```