Factorizations of Coxeter Elements in Complex Reflection Groups

University of Minnesota-Twin Cities 2017 REU

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Complex Reflection Groups

Definition

Let V be a finite dimensional complex vector space of dimension n. A *complex reflection* is an element $r \in \mathrm{GL}(V)$ such that

- r has finite order,
- The fixed space of r is a hyperplane in V, i.e. $\dim_{\mathbb{C}} \ker(r - \mathbf{1}) = n - 1$.

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Familiar Examples:

- The dihedral group $I_2(n)$.
- The group $B = G(2, 1, n)$ of signed $n \times n$ permutation matrices.

Notation and Definitions

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Definition

A ζ-regular element is a $c \in W$ with eigenvalue ζ and corresponding eigenvector not contained in any $H \in \mathcal{R}^*$. A *Coxeter element* is a ζ_h-regular element.

Previous Results

Set

$$f_k = \#\Big\{(r_1, \dots, r_k) \colon c = r_1 \dots r_k,\ r_i \in \mathcal{R}\Big\}$$

Theorem

(Chapuy-Stump, 2014, [5]) For any irreducible, well-generated complex reflection group, W of rank n,

$$\mathrm{FAC}_W(t) = \sum_{k \geq 0} f_k \frac{t^k}{k!} = \left(e^{Nt/n} - e^{-N^* t/n}\right)^n$$

Question Framework

For $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$ a partition of $\mathcal{R}$ with each $\mathcal{R}_i$ a union of conjugacy classes in $\mathcal{R}$, and $\mathcal{C} = (C_1, \ldots, C_m)$ a tuple with $C_i \in \{\mathcal{R}_1, \ldots, \mathcal{R}_\ell\}$. Set

$$g(\mathcal{C}) = \#\Big\{(r_1, \ldots, r_m) \colon c = r_1 \ldots r_m,\ r_i \in C_i\Big\}$$

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Fact

Let $\mathfrak{S}_m$ act on m-tuples $\mathcal{C}$ by permuting its entries. Then, for all $\omega \in \mathfrak{S}_m$ and all tuples $\mathcal{C}$,

$$g(\mathcal{C}) = g(\omega \cdot \mathcal{C})$$

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Let

$$f_{m_1,\ldots,m_\ell} := g(\underbrace{\mathcal{R}_1, \ldots, \mathcal{R}_1}_{m_1 \text{ times}}, \ldots, \underbrace{\mathcal{R}_\ell, \ldots, \mathcal{R}_\ell}_{m_\ell \text{ times}})$$

Question Framework

Consider the generating function

$$\mathrm{FAC}_W(u_1, \ldots, u_\ell) = \sum_{m_1,\ldots,m_\ell \geq 0} f_{m_1,\ldots,m_\ell} \prod_{i=1}^{\ell} \frac{u_i^{m_i}}{m_i!}$$

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Question

For what partitions $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$ does this function have a nice closed form expression?

Hyperplane-Induced Partitions

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Hyperplane-Induced Partitions

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- For $r \in \mathcal{R}$ a reflection, let H_r be the hyperplane fixed by r.
- W acts on $\mathcal{R}^*$ by right multiplication.
- Each conjugacy class $C \subset \mathcal{R}$ determines a unique W-orbit

$$\mathcal{H}_C = \left\{ H_r \subset V \colon r \in C \right\}$$

Hyperplane-Induced Partitions

- Define the equivalence relation on $\mathcal{C}_{\mathcal{R}}(W)$ by

$$C_1 \sim C_2 \iff \mathcal{H}_{C_1} = \mathcal{H}_{C_2}$$

Let $\Theta_1, \ldots, \Theta_\ell$ be the equivalence classes of $\mathcal{C}_{\mathcal{R}}(W)$ under $\sim$ and set

$$\mathcal{R}_i = \#\big\{r \in \mathcal{R} \colon r \in C \text{ for some } C \in \Theta_i\big\}.$$

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- $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$ as above will be called a *hyperplane-induced partition of* $\mathcal{R}$.

The Hurwitz Action and Numbers n_i

Definition

Say $\operatorname{rank}(W) = n$. The *Hurwitz action* of the braid group of type A_{n-1} on factorizations $(t_1, \ldots, t_n)$ of c is given by generators

$$e_i \cdot (t_1, \ldots, t_n) = (t_1, \ldots, t_i t_{i+1} t_i^{-1}, t_i, \ldots, t_n)$$

Theorem (Bessis, 2003 [1])

The Hurwitz action is transitive on the set of minimal-length factorizations $(t_1, \ldots, t_n)$ of any fixed Coxeter element c.

The Hurwitz action preserves the multiset of conjugacy classes $\{C_i \colon t_i \in C_i\}$.

Constants associated to Hyperplane-Induced Partitions

Definition

For any factorization $c = t_1 \cdots t_n$ of a Coxeter element c, set

$$n_i = \#\{j \colon t_j \in \mathcal{R}_i\}$$

This definition is independent of the choice of factorization by the transitivity of the Hurwitz action.

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For any factorization $c = t_1 \cdots t_n$ of a Coxeter element c, set

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This definition is independent of the choice of factorization by the transitivity of the Hurwitz action.

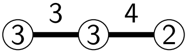
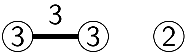
- Let $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$ be a hyperplane-induced partition of $\mathcal{R}$, and let $\mathcal{H}_i$ be the W-orbit of $\mathcal{R}^*$ corresponding to $\mathcal{R}_i$.
- Set

$$N_i := \#\mathcal{R}_i \qquad \text{and} \qquad N_i^* = \#\mathcal{H}_i.$$

Schematic Interpretation of n_i

- The data of the theorem can be read off a Coxeter-Shephard diagram.

Example: $W = G_{26}$ has diagram

 $\Downarrow$ 	Remove edges with even label.
$\Downarrow$ $n_1 = 2$ and $n_2 = 1$	$n_i =$ size of connected component.

Main Result

Theorem

Let W be an irreducible, well-generated complex reflection group with hyperplane-induced partition of reflections $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$. Let n_i, N_i, N_i^* be as before. Then,

$$\mathrm{FAC}_W(u_1, \ldots, u_\ell) = \frac{1}{|W|} \prod_{i=1}^{\ell} \left(e^{\frac{N_i u_i}{n_i}} - e^{-\frac{N_i^* u_i}{n_i}} \right)^{n_i}$$

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Compare with Chapuy-Stump:

$$\mathrm{FAC}_W(t) = \frac{1}{|W|} \left(e^{\frac{Nt}{n}} - e^{-\frac{N^* t}{n}} \right)^{n}$$

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Example: $W = G_{26}$,

$$\begin{aligned} n_1 = 2, \quad N_1 = 24, \quad N_1^* = 12, \\ n_2 = 1, \quad N_2 = 12, \quad N_2^* = 9 \end{aligned}$$

$$\mathrm{FAC}_{G_{26}}(u, t) = \frac{1}{|G_{26}|} \left(e^{12u} - e^{-6u} \right)^2 \left(e^{9t} - e^{-9t} \right)$$

Remarks on the Main Theorem

- The multivariate generating function in our work specializes to that of Chapuy-Stump:

$$\mathrm{FAC}_W(u_1, \ldots, u_\ell)\big|_{u_1=\cdots=u_\ell=t} = \mathrm{FAC}_W(t)$$

- For any $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_\ell$ a hyperplane-induced partition of reflections, ℓ is at most 2.

A Corollary to the Main Theorem

Corollary

Let W be an irreducible, well-generated Coxeter group with Coxeter number h. Set $h_i = \frac{N_i + N_i^*}{n_i}$. Then $h_i = h$ for every i.

A Corollary to the Main Theorem

Corollary

Let W be an irreducible, well-generated Coxeter group with Coxeter number h. Set $h_i = \frac{N_i + N_i^*}{n_i}$. Then $h_i = h$ for every i.

In the real case, this is explained by a proposition of Bourbaki:

Proposition

([3], Ch VI, Section 11, Prop 33) If s_i are reflections corresponding to a basis of an irreducible root system R, the cyclic subgroup $\Gamma = \langle c \rangle$ of order h generated by $c = s_1 s_2 \dots s_l$ acts freely on R and there exist representatives $\theta_1, \dots, \theta_m$ of the Γ-orbits such that each θ_i is in the W-orbit of a simple root.

Sketch of Case-Free Proof of Corollary in Real Case

Let $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ be W-orbits of $\mathcal{R}^*$.

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Because real, $N_i = N_i^*$ and number of roots with hyperplane in $\mathcal{O}_i$ is $2N_i$.

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Because Γ acts freely, get $2N_i/h$ orbits in $\mathcal{O}_i$.

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Remark

In the complex case, Jean-Michel has kindly provided an argument using results of Bessis and Broué-Malle-Rouquier.

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