Chow Rings of Matroids

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Outline

1. Preliminaries
2. Methods for calculating Hilbert series
3. Uniform matroids and $M_r(\mathbb{F}_q^n)$
4. Future work and other lattices

Motivation

- The Chow ring of a ranked atomic lattice L is a graded ring denoted $A(L)$.
- The proof of the Heron-Rota-Welsh conjecture by Adiprasito-Huh-Katz uses properties of $A(L)$ when L is the lattice of flats of a matroid M.
- We are interested in combinatorial information about the lattice L or the matroid M which can be determined from $A(L)$.

Example

- $L(U_{n,r}) = \{A \subseteq [n] \text{ with } \#A \leq r-1\}$
- $L(M_r(\mathbb{F}_q^n)) = \{A \leq \mathbb{F}_q^n \text{ with } \dim A \leq r-1\}$
- $L(M(K_n)) = \{\text{partitions of } [n]\}$

Definitions

Definition (Feichtner-Yuzvinsky 2004)

Let L be a ranked lattice with atoms $a_1, \ldots, a_k$. The Chow ring of L is

$$A(L) = \mathbb{Z}[\{x_p : p \in L, p \neq \bot\}]/(I + J)$$

where

$$I = (x_p x_q \ : \ p \text{ and } q \text{ are incomparable})$$

$$J = \left(\sum_{q \geq a_i} x_q \ : \ 1 \leq i \leq k \right).$$

Theorem (Adiprasito-Huh-Katz 2015)

The Heron-Rota-Welsh conjecture is true.

Incidence algebra

Theorem (Feichtner-Yuzvinsky 2004)

$$H(A(L),t) = 1 + \sum_{\bot = x_0 < x_1 < \cdots < x_m} \prod_{i=1}^{m} \frac{t - t^{\operatorname{rk} x_i - \operatorname{rk} x_{i-1} - 1}}{1 - t}$$

Proposition

If $\eta, \gamma \in (\mathbb{Q}(t))[L]$ are given by

$$\eta(x,y) = \sum_{i=1}^{\operatorname{rk} y - \operatorname{rk} x - 1} t^i$$

and $\gamma = (1-\eta)^{-1}\zeta$, then $H(A([x,y]),t) = \gamma(x,y)$.

Proposition

$$\gamma_{L \times K} = (1 - t(1-\gamma_L) \otimes (1-\gamma_K))^{-1}(\gamma_L \otimes \gamma_K).$$

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Proposition

$$H(A(L \times B_1), t, s) = (1 + \partial_s) H(A(L), t, s)$$

Applications of AHK results

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Theorem

Let L be a "nicely ranked" atomic lattice with $\operatorname{rk} L = r + 1$ and $\operatorname{rk}(z) = \operatorname{rk}(z') \implies [z, \top] \cong [z', \top]$. Let $z_2, \ldots, z_{r-1} \in L$ with $\operatorname{rk}(z_i) = i$. Then

$$\dim_{\mathbb{Z}} A^q(L) = 1 + \sum_{i=2}^{r} \#L_i \sum_{p=1}^{i-1} \dim_{\mathbb{Z}} A^{q-p}([z_i, \top])$$

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$$H(A(L), t) = [r+1]_t + t \sum_{i=2}^{r} \#L_i [i-1]_t H([z_i, \top], t)$$

Applications of AHK results: examples

Uniform:

$$H(U_{n,r+1},t) = [r+1]_t + t\sum_{i=2}^{r}\binom{n}{i}[i-1]_t\, H(U_{n-i,r+1-i},t).$$

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Subspaces:

$$H\left(A(M_{r+1}(\mathbb{F}_q^n)), t\right) = [r+1]_t + t \sum_{i=2}^{r} [i-1]_t \begin{bmatrix} n \\ i \end{bmatrix}_q H\left(A(M_{r+1-i}(\mathbb{F}_q^n)), t\right)$$

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Theorem

The Hilbert series of $U_{n,n}$ is the Eulerian polynomial

$$H(A(U_{n,n}), t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}.$$

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- Can be used to characterize Hilbert series for $H(A(U_{n,r}), t)$ for all r.

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Theorem

For odd r, the Charney-Davis quantity for the uniform matroid, $U_{n,r}$, of rank r and dimension n is

$$\sum_{k=0}^{\frac{r-1}{2}} \binom{n}{2k} E_{2k}$$

where $E_{2\ell}$ is the ℓ-th secant number, i.e.

$$\operatorname{sech}(t) = \sum_{\ell \geq 0} E_{2\ell} \frac{t^{2\ell}}{(2\ell)!}$$

q-analogs of uniform matroids: $M_r(\mathbb{F}_q^n)$

- The lattice of flats of $M_r(\mathbb{F}_q^n)$ is the lattice of dimension $\leq r$ subspaces in $\mathbb{F}_q^n$.

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Theorem

The Hilbert series of $M(\mathbb{F}_q^n)$ is

$$H(A(M(\mathbb{F}_q^n)), t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)}.$$

q-analogs of uniform matroids: $M_r(\mathbb{F}_q^n)$

- There are again surjective maps $\pi_{n,r} \colon A(M_{r+1}(\mathbb{F}_q^n)) \to A(M_r(\mathbb{F}_q^n))$.

Theorem

The Hilbert series of $K_{n,r} = \ker(\pi_{n,r})$ is

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q-analogs of uniform matroids: $M_r(\mathbb{F}_q^n)$

- Let $\cosh_q(t) = \sum_{n \geq 0} \frac{t^{2n}}{[2n]_q!}$ and $\operatorname{sech}_q(t) = 1/\cosh_q(t)$.

Theorem

For odd r, the Charney Davis quantity of $A(M_r(\mathbb{F}_q^n))$ is

$$\sum_{k=0}^{\frac{r-1}{2}} \binom{n}{2k} E_{2k,q}$$

where $E_{2\ell,q}$ satisfies

$$\operatorname{sech}_q(t) = \sum_{\ell \geq 0} E_{2\ell,q} \frac{t^{2\ell}}{[2\ell]_q!}$$

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Question: Why look at only matroids? Is the Chow ring still nice for more general lattices?

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- Can also generalize some early lemmas needed for hard Leftschetz, etc.

Experimental results

Experimentally, the following have symmetric Hilbert series:

- Polytope face lattices
- Simplicial complexes
- Convex closure lattices
- Various manual examples

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Suggestions for strange families of ranked atomic lattices welcome.

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- $H(A(U_{n,n}), t)$ is the h-polynomial of $\Delta(B_n)$.

Conjecture

$$h\Big(\Delta(L(U_{n,r})), t\Big) = t^2 \sum_{i=1}^{r} \binom{n-i-1}{r-i} H(A(U_{n,i}), t)$$

Further further work

- In what generality do AHK's results hold?
- Investigate Koszulity. No obstructions yet.
- Eigenvalues, normal forms of ample elements?
- More basic operations on matroids and lattices: what happens to the Chow ring?

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